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Eigenvectors in the Context of the Schur-Szegö Composition of Polynomials

Vladimir Petrov Kostov

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Any monic degree n polynomial P in one variable such that $P(-1) = 0$ can be represented as a composition of Schur-Szegö of $n-1$ polynomials of the form $(x+1)^{n-1}(x+a_i)$. The mapping expressing the symmetric functions of the numbers a_i via the coefficients of the polynomial $P/(x+1)$ is affine and bijective (see [1]). We show that the eigenvalues of this mapping equal 1, $n/(n-1)$, $n^2/(n-1)(n-2)$, …, $n^{n-2}/(n-1)!$, and we prove the hyperbolicity, self-reciprocity and other properties of the polynomials defined by the eigenvectors of its linearization.

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1. Introduction

In the present paper we consider polynomials in one variable. All definitions are the same in the case of complex and in the case of real polynomials. The *composition of Schur-Szegö (CSS)* of two degree n polynomials $P = \sum_{j=0}^{n} p_j x^j$ and $Q = \sum_{j=0}^{n} q_j x^j$ is defined by the formula $P \underset{n}{*} Q = \sum_{j=0}^{n} p_j q_j x^j / C_n^j$, $C_n^j = n!/j!(n-j)!$. The index n under the asterisk is put because one can also compose P and Q when considered as degree $n+k$ polynomials with k leading coefficients equal to 0. In this case we set $P \underset{n+k}{*} Q = \sum_{j=0}^{n} p_j q_j x^j / C_{n+k}^j$.

The CSS is commutative and associative. For the composition of s degree n polynomials $P_\nu = \sum_{j=0}^{n} p_{\nu,j} x^j$ (called *composition factors*) one has the formula

$$P_1 \underset{n+k}{*} \cdots \underset{n+k}{*} P_s = \sum_{j=0}^{n} p_{1,j} \cdots p_{s,j} x^j / (C_{n+k}^j)^{s-1} \ .$$

It can be checked straightforwardly that the polynomial $(x+1)^n$ plays the role of unity in the sense that

$$Q \underset{n}{*} (x+1)^n = Q \qquad (1)$$

for any degree n polynomial Q. More properties of the CSS see in [4] or [5].

In paper [2] the following result was announced (see its proof in [1]):

Theorem 1. *For each degree n monic polynomial P having one of its roots at (-1) there exist unique up to permutation numbers a_j, $j = 1, \ldots, n-1$, such that*

$$P = K_{a_1} \underset{n}{*} \cdots \underset{n}{*} K_{a_{n-1}} \qquad (2)$$

where the composition factor K_{a_i} equals $(x+1)^{n-1}(x+a_i)$.

Remark 2.

1) When the Schur-Szegö composition of polynomials is applied one often prefers the less commonly used form of polynomials $\sum_{j=0}^{n} C_n^j h_j x^j$. When a polynomial K_a is presented in this form, its coefficients h_j equal $(aC_{n-1}^j + C_{n-1}^{j-1})/C_n^j = ((n-j)a + j)/n$, i.e. they are consecutive terms of an arithmetic progression.

2) By convention we set $K_\infty = (x+1)^{n-1}$. When among the composition factors there is at least one K_∞, one has to modify formula (2) by putting a constant factor in the right-hand side; in such a case P is not monic (its leading coefficient is 0).

3) If a degree $n-k$ polynomial P is considered as a degree n one with k leading coefficients equal to 0, then k of the numbers a_i equal respectively ∞ and $(-\nu/(n-\nu))$, $\nu = n-1, \ldots, n-k+1$. If a polynomial P is divisible by x^s, then s of the numbers a_i equal respectively $(-\nu/(n-\nu))$, $\nu = 0, \ldots, s-1$. Indeed, the coefficient of x^ν in K_a is equal to 0 exactly when $a = -\nu/(n-\nu)$. On the other hand, if this coefficient equals 0 in P, then it must be 0 in at least one of the composition factors K_{a_i}.

Notation 3. *For arbitrary $n \geq 2$ set $P = (x+1)P_1$ where $P_1 = c_0 x^{n-1} + c_1 x^{n-2} + \cdots + c_{n-1}$. Most often we set $c_0 = 1$, but sometimes we have to consider the case when P and P_1 are not necessarily monic.*

Set $\sigma_\nu := \sum_{1\leq j_1<j_2<\cdots<j_\nu\leq n-1} a_{j_1}\cdots a_{j_\nu}$. When P is monic (resp. non-monic) and there is no K_∞ among the composition factors in (2), then we set $\sigma_0 = 1$ (resp. we put a factor $\sigma_0 = c_0$ in the right-hand side of (2)). The mapping $\Phi : (c_1,\ldots,c_{n-1}) \mapsto (\sigma_1,\ldots,\sigma_{n-1})$ is affine and bijective (see [1]). In the present paper we explicit the eigenvalues of this mapping and we prove properties of the polynomials defined by its eigenvectors, see Theorem 12..

Consider first as example the case $n = 3$. (The case $n = 2$ is trivial – one has $P = (x+1)(x+a_1) = K_{a_1}$.) To simplify the notation we set

$$(3)\qquad \begin{aligned} P = (x+1)(x^2+px+q) &= x^3+(p+1)x^2+(q+p)x+q \\ &= (x+1)^2(x+a_1) \ {\textstyle\mathop{*}_3}\ (x+1)^2(x+a_2) \end{aligned}$$

Set $a_1 + a_2 = A$, $a_1a_2 = B$. The right-hand side of (3) equals

$$\begin{aligned} x^3+\frac{(2+a_1)(2+a_2)x^2}{3}+\frac{(1+2a_1)(1+2a_2)x}{3}+a_1a_2 \\ = (x+1)\left(x^2+\frac{(1+2A+B)x}{3}+B\right) \ . \end{aligned}$$

Thus for $n = 3$ the mapping Φ is equivalent to the linear system

$$p=\frac{1+2A+B}{3}\ ,\ q=B \ \text{ or equivalently }\ A-1=\frac{3(p-1)-q}{2}\ ,\ B=q\ .$$

The mapping Φ can be considered also as a linear mapping $(p-1,q) \mapsto (A-1,B)$. It has two eigenvalues – $\lambda_1 = 1$ and $\lambda_2 = 3/2$. To these two eigenvalues there correspond the eigenvectors $\vec{v}_1 = (1,1)$ and $\vec{v}_2 = (1,0)$. They define the polynomials $V_1 = (x+1)(x+1)$ and $V_2 = (x+1)x$ in the sense that the coefficients of the polynomials $V_1/(x+1)$ and $V_2/(x+1)$ are the components of $\vec{v}_1$ and $\vec{v}_2$. Sometimes we say that (conversely) the polynomial V_1 or $V_1/(x+1)$ defines the vector $\vec{v}_1$. The matrices of the linearizations of the affine mappings Φ^{-1} and Φ equal respectively $\begin{pmatrix} 2/3 & 1/3 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3/2 & -1/2 \\ 0 & 1 \end{pmatrix}$.

R e m a r k 4. Notice that the eigenvectors of the linearization of the mapping Φ are defined by degree $n-2$ polynomials, because we consider in fact products of monic degree $n-1$ polynomials with $(x+1)$. Such polynomials have $n-1$ coefficients which play the role of parameters. The tangent space to their set is the space $\mathbf{C}_{n-2}[x]$ of all degree $n-2$ polynomials. The eigenvalues of the mapping Φ are the $(n-1)$ eigenvalues of its linearization which is a mapping $\mathbf{C}_{n-2}[x] \to \mathbf{C}_{n-2}[x]$.

Consider next the case $n=4$. Set

$$P = (x+1)(x^3+px^2+qx+r) = (x+1)^3(x+a_1) \, {}^*_4 \, (x+1)^3(x+a_2) \, {}^*_4 \, (x+1)^3(x+a_3) \, .$$

Set $a_1+a_2+a_3 = A$, $a_1a_2+a_1a_3+a_2a_3 = B$, $a_1a_2a_3 = C$. As in the case $n=3$ one finds

$$P = x^4 + \frac{(3+a_1)(3+a_2)(3+a_3)x^3}{4^2} + \frac{(3+3a_1)(3+3a_2)(3+3a_3)x^2}{6^2} +$$

$$+ \frac{(1+3a_1)(1+3a_2)(1+3a_3)x}{4^2} + a_1a_2a_3 \ , \quad \text{i.e.}$$

$$P = x^4 + \frac{(27+9A+3B+C)x^3}{16} + \frac{27(1+A+B+C)x^2}{36} + \frac{(1+3A+9B+27C)x}{16} + C$$

$$= (x+1)\left(x^3 + \frac{(11+9A+3B+C)x^2}{16} + \frac{(1+3A+9B+11C)x}{16} + C\right) \, .$$

The matrices of the linearizations of Φ and Φ^{-1} equal respectively

$$\begin{pmatrix} 2 & -2/3 & 1/3 \\ -2/3 & 2 & -4/3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 9/16 & 3/16 & 1/16 \\ 3/16 & 9/16 & 11/16 \\ 0 & 0 & 1 \end{pmatrix} \, .$$

Their eigenvalues and the corresponding eigenvectors are $8/3$, $4/3$, 1 (of Φ), $3/8$, $3/4$, 1 (of Φ^{-1}) and $(1,-1,0)$, $(1,1,0)$, $(1,2,1)$. The eigenvectors define the polynomials $(x+1)(x-1)x$, $(x+1)(x+1)x$ and $(x+1)(x+1)^2$.

R e m a r k 5. It is clear from formula (2) that one has (see Notation 3.) $c_{n-1} = a_1 \cdots a_{n-1} = \sigma_{n-1}$. Set $\tilde{c} = (c_1, \ldots, c_{n-2})^t$, $\tilde{\sigma} = (\sigma_1, \ldots, \sigma_{n-2})^t$. The mappings Φ and Φ^{-1} can be presented in the form

(4)
$$\Phi : \begin{pmatrix} \sigma_0 \\ \tilde{\sigma} \\ \sigma_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ D & M & H \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ \tilde{c} \\ c_{n-1} \end{pmatrix} ,$$

$$\Phi^{-1} : \begin{pmatrix} c_0 \\ \tilde{c} \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ F & M^{-1} & G \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \tilde{\sigma} \\ \sigma_{n-1} \end{pmatrix}$$

where D, H, F and G are vector-columns of size $(n-2)$. When we set $c_0 = 1$, then we consider the mappings Φ and Φ^{-1} as affine ones between spaces of dimension $n-1$. When c_0 is considered as a parameter, then Φ and Φ^{-1} are linear mappings between spaces of dimension n. In this case they have one more eigenvalue which is equal to 1.

It is clear from the structure of the matrices that the space $\mathcal{S}$ of degree $n-2$ polynomials divisible by x is invariant for Φ and Φ^{-1}.

In what follows we set $c_0 = 1$ and consider the mappings Φ and Φ^{-1} as affine ones. The matrices of their linearizations equal respectively $\begin{pmatrix} M & H \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} M^{-1} & G \\ 0 & 1 \end{pmatrix}$. In Subsection 5.1. we explain how to compute explicitly the matrix M^{-1} which describes the action of Φ^{-1} on the subspace $\mathcal{S}$.

2. The new results

In the rest of the paper all polynomials are presumed real.

Definition 6. If P is a degree n polynomial in one variable, then its *reverted* one is the polynomial $P^R = x^n P(1/x)$. A polynomial P is *self-reciprocal* if up to a sign it is equal to its reverted one, i.e. $P = \pm P^R$. Hence if $P(x_0) = 0$ and $P = \pm P^R$, then $P(1/x_0) = 0$.

Remark 7. 1) Notice that P^R might be of degree smaller than n. We set $1/0 = \infty$, $1/\infty = 0$.

2) If $P = -P^R$ and deg$P$ is even, then the middle coefficient of P is 0. One has $P = -P^R$ (resp. $P = P^R$) exactly if 1 is a root of P of odd multiplicity (resp. of even multiplicity or a nonroot).

3) It is clear that if P is self-reciprocal and *hyperbolic*, i.e. with all roots real, then its roots different from 1 and (-1) constitute couples of the kind $(x_0, 1/x_0)$. (In the case of a real, but not necessarily hyperbolic polynomial, they constitute such couples and/or quadruples of the form $(z_0, \overline{z_0}, 1/z_0, 1/\overline{z_0})$.)

Definition 8. An $m \times l$-matrix $R = (r_{i,j})_{i=1}^{m}{}_{j=1}^{l}$ is *centre-symmetric* if $r_{i,j} = r_{m+1-i,l+1-j}$ for all (i, j).

Remark 9. It is clear that if R is centre-symmetric, square and invertible, then R^{-1} is also centre-symmetric. This follows from $R = JRJ$

where $J = J^{-1}$ is the permutation matrix having units on the anti-diagonal and zeros elsewhere.

Proposition 10. *The matrices M, M^{-1}, $(D\ M\ H)$ and $(F\ M^{-1}\ G)$, see (4), are centre-symmetric. Their entries are rational numbers.*

Proof.

It is to be checked directly that if $\Phi : P_1 \mapsto T_1$, then $\Phi : P_1^R \mapsto T_1^R$ (by observing that $K_a^R = a(K_{1/a})$ etc.). The same applies to Φ^{-1} as well. Set $D = (d_1, \ldots, d_{n-2})^t$, $H = (h_1, \ldots, h_{n-2})^t$. The coefficient σ_ν of $x^{n-\nu}$ $(1 \leq \nu \leq n-1)$ of the polynomial T_1 equals $d_\nu + \sum_{l=1}^{n-2} M_{\nu,l} c_l + h_\nu c_{n-1}$. The one of x^ν equals $\sigma_{n-\nu} = d_{n-\nu} + \sum_{l=1}^{n-2} M_{n-\nu,l} c_l + h_{n-\nu} c_{n-1}$ $(*)$ and this is the coefficient of $x^{n-\nu}$ in T_1^R. The latter is obtained by replacing in formula $(*)$ the quantity c_l by c_{n-l}, $l = 1, \ldots, n-1$ (recall that $c_0 = 1$). This must be true for any values of the coefficients c_l, therefore one has $M_{n-\nu,l} = M_{\nu,n-l}$ and $d_\nu = h_{n-\nu}$ for all $l = 1, \ldots, n-1$, $\nu = 1, \ldots, n-1$.

That the entries of the matrix M^{-1} are rational (hence the ones of M as well) follows from an explicit computation in Subsection 5.1.. For the matrices $(D\ M\ H)$ and $(F\ M^{-1}\ G)$ one can use the same type of computation. ∎

Proposition 11. *The mappings Φ and Φ^{-1} have an eigenvalue $\lambda_1 = 1$ with eigenvector $\vec{w} = (C_{n-2}^0, C_{n-2}^1, \ldots, C_{n-2}^{n-2})$. The latter defines the polynomials $P_1 = (x+1)^{n-2}$ and $P = (x+1)P_1 = (x+1)^{n-1}$.*

Indeed, it is clear that the vector $\vec{w}$ defines the polynomials P_1 and P. Using $(n-2)$ times equality (1), one can write

$$P = (x+1)^{n-1} = \underbrace{(x+1)^{n-1}(x+1) \underset{n}{*} \cdots \underset{n}{*} (x+1)^{n-1}(x+1)}_{(n-2)\ \text{factors}} \underset{n}{*} (x+1)^{n-1} .$$

One can interpret the polynomial P as a degree n polynomial having one of its roots at ∞. Hence for the polynomial P the numbers a_j from formula (2) equal 1 ($(n-2)$ times) and ∞ which coincide with the roots of P_1. This means that the polynomial P defines an eigenvector of Φ and of Φ^{-1} corresponding to the eigenvalue 1. ∎

The main results of the present paper are contained in the following theorem.

Theorem 12.

1) The mapping Φ has $n-1$ distinct real eigenvalues $\lambda_1 = 1$, $\lambda_2 = n/(n-1)$, $\lambda_3 = n^2/(n-1)(n-2)$, …, $\lambda_{n-1} = n^{n-2}/(n-1)!$.

2) The corresponding eigenvectors are defined by monic polynomials of the form $(x+1)^{n-1}$, $x(x+1)^{n-2}$, $x(x+1)^{n-3}Q_1(x)$, $\ldots$, $x(x+1)Q_{n-3}(x)$ *where* $degQ_j = j$, $j = 1, \ldots, n-3$, $Q_j(-1) \neq 0$. *The coefficients of the polynomials* Q_j *are rational.*

3) The polynomials Q_j *are self-reciprocal.*

4) The roots of every polynomial Q_j, $j \geq 1$, *are positive and distinct.*

5) One has $(Q_j)^R = (-1)^j Q_j$. *For* j *odd (resp. for* j *even) one has* $Q_j(1) = 0$ *(resp.* $Q_j(1) \neq 0$*). The middle coefficient of* $(x+1)^{n-j-2}Q_j$ *is* 0 *when* n *is even and* j *is odd.*

6) For j *fixed and* $n \to \infty$ *the polynomial* Q_j *has a limit which is a hyperbolic monic degree* j *polynomial* Q_j^* *with all roots positive, with rational coefficients, satisfying the equality* $(Q_j^*)^R = (-1)^j Q_j^*$ *and the condition* $Q_j^*(1) = 0$ *for* j *odd.*

Remark 13. With the exception of the eigenvalue 1 and its eigenvector $\vec{w}$ (see Proposition 11.) all eigenvalues of Φ correspond to the invariant subspace $\mathcal{S}$, see Remark 5.. In part 5) of the theorem we formulate the property about $(x+1)^{n-j-2}Q_j$, not about $x(x+1)^{n-j-2}Q_j$, because the latter is of odd degree and has no middle coefficient.

Example 14. It follows from parts 2), 3) and 5) of the theorem that for all n the polynomials Q_1 (resp. Q_2) are of the form $x-1$ (resp. x^2+vx+1, $v \in \mathbf{Q}$). Hence $Q_1^* = x-1$, see part 6) of the theorem. Compare the coefficients of x of the polynomials

$$x(x+1)^{n-4}Q_2 = x(x+1)^{n-4}(x^2+vx+1) = \cdots + ((n-4)+v)x^2 + x \quad \text{and}$$

$$D := x(x+1)^{n-1} \underset{n}{*} (x+a_1)(x+1)^{n-1} \underset{n}{*} (x+a_2)(x+1)^{n-1} \underset{n}{*} (x+1)^{n-1}$$

the second of them being up to a constant factor the representation (2) of the former in which we omit the composition factors K_1 (see (1)). Notice that $a_1a_2 = 1$, $a_i \in \mathbf{R}$ (see parts 3) and 4) of the Theorem). As $x(x+1)^{n-4}Q_2$ defines an eigenvector of Φ, the numbers $(-a_i)$ must be the roots of Q_2, i.e. $a_1 + a_2 = v$. One has

$$\begin{aligned} D = \cdots \quad &+ \quad \frac{C_{n-1}^1(C_{n-1}^2 a_1 + C_{n-1}^1)(C_{n-1}^2 a_2 + C_{n-1}^1)C_{n-1}^2 x^2}{(C_n^2)^3} \\ &+ \quad \frac{(C_{n-1}^1 a_1 + 1)(C_{n-1}^1 a_2 + 1)C_{n-1}^1 x}{(C_n^1)^3} \ . \end{aligned}$$

The eigenvector defined by the polynomial $x(x+1)^{n-4}Q_2$ corresponds to the eigenvalue $\lambda_4 = n^3/(n-1)(n-2)(n-3)$ of Φ. Therefore the coefficient of x in D equals $1/\lambda_4$. Solving the equation

$$\frac{((n-1)a_1+1)((n-1)a_2+1)(n-1)}{n^3} = \frac{(n-1)(n-2)(n-3)}{n^3}$$

and using the equalities $a_1 a_2 = 1$, $a_1 + a_2 = v$, one finds $v = (-3n+4)/(n-1)$. Hence $Q_2^* = x^2 - 3x + 1$.

In Section 3. we recall some known results to prepare the proof of the theorem. The latter is given in Section 4.. Some open questions concerning the coefficients and the zeros of the polynomials Q_j and Q_j^* (in particular – the possible interlacing of the zeros) are given in Subsection 5.2..

3. Some known results and their applications

Recall first a result from paper [3] (see Proposition 1.4 there):

Proposition 15. *If the degree n polynomials P and Q have roots $x_P \neq 0$ and $x_Q \neq 0$ of multiplicities respectively m_P and m_Q where $m_P + m_Q \geq n$, then the polynomial $P \ {}^*_n\ Q$ has a root $(-x_P x_Q)$ of multiplicity $m_P + m_Q - n$.*

(In [3] the condition $x_P \neq 0$, $x_Q \neq 0$ is omitted which is not correct.) Applying this proposition to equality (2) one obtains the following result:

Corollary 16. *If the polynomial P has a root (-1) of multiplicity $\mu \geq 1$, then among the numbers a_i defined by equality (2) there are exactly $\mu - 1$ which equal 1.*

Indeed, applying $n-2$ times the above proposition to the right-hand side of equality (2), one obtains that the multiplicity of (-1) as a root of P is equal to $n - \nu$ where ν is the quantity of numbers a_i which are different from 1 (we use also equality (1) here). ■

Recall some algebraic equalities connected with the CSS. For two degree n polynomials P and $Q = xS$, $\deg S = n-1$, one has the formula

$$P \ {}^*_n\ Q = \frac{x}{n}(P' \ {}^*_{n-1}\ S) \tag{5}$$

Differentiation in the context of the CSS obeys the rule

$$(P \ {}^*_n\ Q)' = \frac{1}{n}(P' \ {}^*_{n-1}\ Q') \tag{6}$$

For the proofs of these formulas see Theorem 1.3 and Example 1 of [2]. Also in [2] (see Remark 6 there) is to be found the proof of the following fact. For a degree n polynomial P set $P^{[1]} = nP - xP'$. Suppose that $\deg T = n-1$. Then

$$P \mathop{*}_{n} T = \frac{1}{n}(P^{[1]} \mathop{*}_{n-1} T) \tag{7}$$

The following example shows how the above formulas can give nontrivial information about the representation of polynomials in form (2).

E x a m p l e 17. Consider the polynomial xP where $\deg P = n$. For this polynomial its representation in form (2) (as for degree $n+1$, not for degree n polynomials) contains a composition factor $(x+1)^n x$, see part 3) of Remark 2.. Hence the representation is of the form

$$(x{+}1)^n x \mathop{*}_{n+1} R \text{ where } R = \tilde{K}_{a_2} \mathop{*}_{n+1} \cdots \mathop{*}_{n+1} \tilde{K}_{a_n}\ ,\ \tilde{K}_{a_i} = (x{+}1)^n (x{+}a_i)\ ,\ a_1{=}0\ .$$

On the other hand, formulas (5) and (1) imply that $xP = \frac{x}{n+1}((x+1)^n \mathop{*}_{n} R') = \frac{x}{n+1}R'$, i.e. $P = \frac{R'}{n+1}$. Apply $n-2$ times formula (6) to find R':

$$R' = \frac{1}{(n+1)^{n-2}}((\tilde{K}_{a_2})' \mathop{*}_{n} \cdots \mathop{*}_{n} (\tilde{K}_{a_n})')$$

where $(\tilde{K}_{a_i})' = (n{+}1)(x{+}1)^{n-1}(x{+}b_i)\ ,\ b_i = \frac{na_i+1}{n+1}$. Hence $P = K_{b_2} \mathop{*}_{n} \cdots \mathop{*}_{n} K_{b_n}$, i.e. formulas (1), (5) and (6) allow to find the link between the representations (2) of xP and P.

As next application of the above formulas we prove the following proposition.

Proposition 18. *For the polynomial* $L = x(x+1)^{n-2} = (x+1)x(x+1)^{n-3}$ *one has*

$$\frac{n-1}{n}L = x(x{+}1)^{n-1} \mathop{*}_{n} \underbrace{(x+1)^{n-1}(x+1) \mathop{*}_{n} \cdots \mathop{*}_{n} (x+1)^{n-1}(x+1)}_{n-3 \text{ factors}} \mathop{*}_{n} (x{+}1)^{n-1} \tag{8}$$

Hence the vector $(C^0_{n-3}, C^1_{n-3}, \ldots, C^{n-3}_{n-3}, 0)$ *defined by the polynomial* L *is an eigenvector of the mappings* Φ *and* Φ^{-1} *corresponding to the eigenvalues respectively* $\lambda_2 = n/(n-1)$ *and* $(\lambda_2)^{-1} = (n-1)/n$.

P r o o f.

Equality (1) allows one to reduce the right-hand side of (8) to

$$L_1 := x(x{+}1)^{n-1} \mathop{*}_{n} (x{+}1)^{n-1} = \frac{x}{n}((x{+}1)^{n-1} \mathop{*}_{n-1} ((x{+}1)^{n-1})') = \frac{(n-1)x(x+1)^{n-2}}{n}$$

(we use (5) and (1) here). Hence $L_1 = ((n-1)/n)L$. This means that $\sigma_j = (n/(n-1))c_j$, $j = 1, \ldots, n-1$. ■

We explain now how to use equality (7) to obtain the next eigenvalue of Φ and the eigenvector corresponding to it.

Proposition 19. *The mapping Φ has an eigenvalue equal to $n^2/(n-1)(n-2)$ with eigenvector defined by the polynomial $x(x+1)^{n-3}(x-1)$.*

Proof.

Consider the composition of polynomials

$$W_1 := x(x{+}1)^{n-1} \mathop{*}_{n} \underbrace{(x{+}1)^{n-1}(x{+}1) \mathop{*}_{n} \cdots \mathop{*}_{n} (x{+}1)^{n-1}(x{+}1)}_{n-4 \text{ factors}} \mathop{*}_{n} (x{+}1)^{n-1} \mathop{*}_{n} (x{+}1)^{n-1} .$$

By formula (2) it defines a polynomial for which the numbers a_j equal 0, 1 ($(n-4)$ times), ∞ and ∞. Equality (1) allows to reduce this composition to

$$W_1 = x(x+1)^{n-1} \mathop{*}_{n} (x+1)^{n-1} \mathop{*}_{n} (x+1)^{n-1} .$$

Observe that by equalities (7) and (1) one has

$$(x+1)^{n-1} \mathop{*}_{n} (x+1)^{n-1} = \frac{1}{n}((x+1)^{n-1} \mathop{*}_{n-1} ((x+1)^{n-1})^{[1]}) = \frac{1}{n}((x+1)^{n-1})^{[1]}$$

and

$$\begin{aligned} ((x+1)^{n-1})^{[1]} = n(x+1)^{n-1} - x((x+1)^{n-1})' &= (x{+}1)^{n-2}(nx + n - (n-1)x) \\ &= (x+1)^{n-2}(x+n) . \end{aligned}$$

We consider here $(x+1)^{n-1}$ as a degree n (not $(n-1)$) polynomial. Applying equalities (5) and (1) to W_1 we transform it as follows:

$$W_1 = \frac{x}{n}((x{+}1)^{n-1} \mathop{*}_{n-1} ((x{+}1)^{n-1} \mathop{*}_{n} (x{+}1)^{n-1})') = \frac{x}{n^2}((x{+}1)^{n-1} \mathop{*}_{n-1} ((x{+}1)^{n-2}(x{+}n))') =$$

$$= \frac{x}{n^2}((x+1)^{n-2}(x+n))' = \frac{(n-1)x(x+1)^{n-3}(x+n-1)}{n^2} .$$

Look for a monic degree $(n-1)$ polynomial of the form $V := \kappa(\alpha L_1 + W_1)$ (where $L_1 = x(x+1)^{n-1} \ {}^*_n\ (x+1)^{n-1} \ {}^*_n\ (x+1)^{n-1}$ was defined in the proof of Proposition 8 and $\kappa \in \mathbf{R}$) for which the $(n-1)$-tuple of quantities σ_j is proportional to the $(n-1)$-tuple of coefficients c_j. Obviously one has

$$\alpha L_1 + W_1 = x(x+1)^{n-1} \ {}^*_n\ (x+1)^{n-1} \ {}^*_n\ ((x+1)^{n-1}(\alpha x + \alpha + 1)) \ .$$

This means that for the polynomial V the numbers a_j (see (2)) equal 0, ∞, $\rho' := (\alpha+1)/\alpha$ and 1 ($(n-4)$ times), and one has $\kappa = 1/(n-1)(\alpha+1+(n-1)\alpha)$. On the other hand, the polynomial V/κ equals (see Proposition 18.)

$$\frac{(n-1)\alpha x(x+1)^{n-2}}{n} + \frac{(n-1)x(x+1)^{n-3}(x+n-1)}{n^2} =$$

$$= \frac{(n-1)x(x+1)^{n-3}((\alpha + \frac{1}{n})x + \alpha + \frac{(n-1)}{n})}{n} \ .$$

Hence the quantities opposite to the roots of $V/(x+1)$ equal 0, ∞, $\rho'' := (\alpha + (n-1)/n)/(\alpha + 1/n)$ and 1 ($(n-4)$ times). The polynomial V defines an eigenvector of the mapping Φ if and only if $\rho' = \rho''$, i.e. $(\alpha+1)/\alpha = (\alpha + (n-1)/n)/(\alpha + 1/n)$ which implies $\alpha = -1/2$. The corresponding eigenvalue equals the ratio of the leading coefficients of the polynomials $(x+1)^{n-1}(\alpha x + \alpha + 1)$ and $(n-1)x(x+1)^{n-3}((\alpha + 1/n)x + \alpha + (n-1)/n)/n$ computed for $\alpha = -1/2$, i.e. $n^2/(n-1)(n-2)$. One can say that the corresponding eigenvector is defined by the polynomial $x(x+1)^{n-3}(x-1)$. ■

4. Proof of Theorem 12.

4.1. Proof of parts 1) and 2)

Consider the degree $(n-1)$ polynomials $T_s := (x+1)^{n-1} \ {}^*_n \cdots {}^*_n\ (x+1)^{n-1}$ (s composition factors, $1 \leq s \leq n-2$).

Lemma 20. *1) The polynomials T_s are representable in the form $T_s = \frac{1}{n^{s-1}}(x+1)^{n-s}T_s^*$ where T_s^* are monic degree $(s-1)$ polynomials.*

2) For $s > 1$ one has $T_s^*(-1) = (n-1)(n-2)\cdots(n-s+1)$.

Indeed, for $s = 1$ one has $T_s^* = 1$. For $s = 2$ as in the proof of Proposition 19. one finds $T_2 = (x+1)^{n-1} \ {}^*_n\ (x+1)^{n-1} = (x+1)^{n-2}(x+n)/n$ so one can set $T_2^* = x + n$; one has $T_2(-1) = n - 1$.

Prove the lemma by induction on s. For $s > 2$ one has

$$T_s = T_{s-1} \; {}^{*}_{n} \; (x+1)^{n-1} = \frac{1}{n}\left((x+1)^{n-1} \; {}^{*}_{n-1} \; (T_{s-1})^{[1]}\right) = \frac{1}{n}(T_{s-1})^{[1]}$$

$$= \frac{1}{n^{s-1}}((x+1)^{n-s+1}T^{*}_{s-1})^{[1]}$$

(we use again equalities (7) and (1) here). It is to be checked directly that for a monic degree $(n-1)$ polynomial R (this is the case of $(x+1)^{n-s+1}T^{*}_{s-1}$) the polynomial $R^{[1]} = nR - xR'$ is also monic and degree $(n-1)$. Hence, such is

$$((x+1)^{n-s+1}T^{*}_{s-1})^{[1]} = (x+1)^{n-s}(n(x+1)T^{*}_{s-1} - x(n-s+1)T^{*}_{s-1} - x(x+1)(T^{*}_{s-1})') \, .$$

Set $T^{*}_{s} = n(x+1)T^{*}_{s-1} - x(n-s+1)T^{*}_{s-1} - x(x+1)(T^{*}_{s-1})'$. This is a monic degree $(s-1)$ polynomial – the polynomial $(T^{*}_{s-1})'$ is degree $(s-3)$ with leading coefficient equal to $(s-2)$. As $T^{*}_{s-1}(-1) = (n-1)(n-2)\cdots(n-s+2)$, one has $T^{*}_{s}(-1) = (n-1)(n-2)\cdots(n-s+1)$. ■

Denote by $\mathcal{I}$ the ideal generated by the polynomial $x+1$. Consider the polynomials

$$V_s = x(x+1)^{n-1} \; {}^{*}_{n} \; \underbrace{(x+1)^n \; {}^{*}_{n} \; \cdots \; {}^{*}_{n} \; (x+1)^n}_{n-s-2 \quad \text{times}} \; {}^{*}_{n} \; T_s \tag{9}$$

The numbers a_j computed for the polynomial V_s equal 0, 1 ($n-s-2$ times) and ∞ (s times). They are opposite to the roots of the polynomial $x(x+1)^{n-s-2}$. Add to this polynomial a factor $x+1$, i.e. consider instead of it the polynomial $x(x+1)^{n-s-1}$ so that it could play the role of the polynomial P from (2). We say then that the numbers a_j define the polynomial $x(x+1)^{n-s-1}$.

On the other hand, $V_s = x(T_s)'/n$ (we apply equalities (5) and (1) here). The polynomial T_s is of the form $(n-1)(n-2)\cdots(n-s+1)(x+1)^{n-s}/n^{s-1} + R_s$, $R_s \in \mathcal{I}^{n-s+1}$. Hence V_s is of the form

$$(n-1)(n-2)\cdots(n-s)x((x+1)^{n-s-1}/n^s + G_s) \;\; , \;\; G_s \in \mathcal{I}^{n-s} \tag{10}$$

In the basis defined by the polynomials $(x+1)^{n-1}$, V_1, …, V_{n-2} the matrix of Φ is triangular and its diagonal entries equal 1, $n/(n-1)$, …, $n^{n-2}/(n-1)(n-2)\ldots 2 = n^{n-2}/(n-1)!$. Hence these are its eigenvalues. They are distinct, hence the matrix can be diagonalized by conjugating with a suitable triangular matrix with units on the diagonal, i.e. as a result of a triangular unipotent change of the basis. After such a change the new basis consists of eigenvectors and they are of form (10).

Recall that the polynomials Q_j are monic. Then the polynomials $x(x+1)^{n-j-2}Q_j$ define eigenvectors of a matrix with rational entries (see Proposition 10.) whose eigenvalues are simple and rational. Hence all coefficients of the polynomials $x(x+1)^{n-j-2}Q_j$ and Q_j are rational.

4.2. Proof of part 3)

Suppose that $\Phi(x(x+1)^{n-2-j}Q_j) = \lambda_{j+2}x(x+1)^{n-2-j}Q_j$, $\lambda_{j+2} \in \mathbf{R}$. Proposition 10. implies that

$$\Phi((x(x+1)^{n-2-j}Q_j)^R) = \lambda_{j+2}(x(x+1)^{n-2-j}Q_j)^R \ .$$

When defining $(x(x+1)^{n-2-j}Q_j)^R$ we consider $x(x+1)^{n-2-j}Q_j$ as a degree n (not $n-1$) polynomial. Hence the polynomial $(x(x+1)^{n-2-j}Q_j)^R$ defines another eigenvector corresponding to the eigenvalue λ_{j+2}. The eigenvalues of Φ are distinct. Therefore one has

$$(x(x+1)^{n-2-j}Q_j)^R = \sigma x(x+1)^{n-2-j}Q_j \quad , \quad \sigma \in \mathbf{R} \ .$$

Revert both sides of this equality:

$$x(x+1)^{n-2-j}Q_j = ((x(x+1)^{n-2-j}Q_j)^R)^R = \sigma(x(x+1)^{n-2-j}Q_j)^R$$
$$= \sigma^2 x(x+1)^{n-2-j}Q_j \ .$$

Hence $\sigma^2 = 1$, i.e. $\sigma = \pm 1$. This means that $Q_j^1 := x(x+1)^{n-2-j}Q_j$ is self-reciprocal. One has

$$(Q_j^1)^R = x^n \left(\frac{1}{x}\left(\frac{1}{x}+1\right)^{n-2-j} Q_j\left(\frac{1}{x}\right)\right) = x(x+1)^{n-2-j}x^jQ_j\left(\frac{1}{x}\right) = \pm Q_j^1 = \pm x(x+1)^{n-2-j}Q_j$$

which implies $x^jQ_j(1/x) = \pm Q_j(x)$, i.e. Q_j is also self-reciprocal.

4.3. Proof of part 4)

Lemma 21. *1) Suppose that a polynomial P has all roots positive. Then the numbers a_i defined after it in Theorem 1. are all negative and distinct.*

2) Consider a polynomial $x(x+1)^{n-j-2}Q_j$ (where Q_j is defined in part 2) of the theorem) as a polynomial P in the left-hand side of (2) taking into account part 2) of Remark 2.. (We consider $x(x+1)^{n-j-2}Q_j$ as a degree n

polynomial with zero leading coefficient.) Then exactly one of the numbers a_i *defined after it equals* 0*, exactly* $n-j-3$ *of them equal* 1*, and exactly one equals* ∞ *(see part 3) of Remark 2.). The remaining* j *numbers* a_i *are negative and distinct.*

P r o o f.

1^0. Prove part 1). By the Descartes rule the sequence of coefficients of the polynomial P has n sign changes, i.e. the maximal possible number. The composition of each couple K_a, $K_{\bar{a}}$ in equality (2) is a polynomial with all coefficients positive. The same is true about every couple of equal composition factors K_a with $a \in \mathbf{R}$. On the other hand, part 1) of Remark 2. implies that the sequence of coefficients of each K_a with $a < 0$ has exactly one sign change. Hence the only possibility for the sequence of coefficients of P to have n sign changes is realizable when all numbers a_j are negative and distinct. (To be negative and distinct is necessary, but not sufficient – it is necessary and sufficient the sign changes in the sequences of coefficients of the different composition factors K_{a_i} to occur at different positions.)

2^0. To prove part 2) one can apply Corollary 16. to deduce the presence of exactly $n-j-3$ units among the numbers a_i. The presence of one 0 (resp. of one ∞) is clear from the fact that the right-hand side of (2) must be divisible by x (resp. a polynomial of degree $n-1$, not n). The remaining j ones must be negative and distinct (by the Descartes rule the sequence of coefficients of the polynomial $x(x+1)^{n-j-2}Q_j$ has at least j sign changes, hence there are at least j negative and distinct among the numbers a_i, see the proof of part 1)). ∎

Consider the mapping Φ as a mapping from the open "simplicial angle" $\Delta_j = \{(x_1, \ldots, x_j) \in \mathbf{R}^j | 0 < x_1 < \cdots < x_j\}$ into itself where x_i are the roots of Q_j and their images are the numbers $(-a_i)$ (excluding 0, ∞ and $n-j-3$ times (-1), see 2^0 of the proof of the lemma) in the increasing order. The mapping Φ can be extended by continuity on $\overline{\Delta_j}$ where the closure includes the infinite points as well, i.e. $\overline{\Delta_j} = \{(x_1, \ldots, x_j) \in (\mathbf{R} \cup \infty)^j | 0 \leq x_1 \leq \cdots \leq x_j \leq \infty\}$. If a root x_i tends to $+\infty$, then we consider the corresponding factor in P to equal not $x - x_i$ but $1 - y_i x$ where $y_i = 1/x_i$. In this way one can find in a neighbourhood of any point of $\overline{\Delta_j}$ a local family of polynomials continuously depending on their roots or on their inverse quantities. The set $\overline{\Delta_j}$ is homeomorphic to a j-dimensional simplex (the mapping $x_i \mapsto x_i/(1+x_i)$, $i = 1, \ldots, j$, defines a homeomorphism between $\overline{\Delta_j}$ and the set $\{(x_1, \ldots, x_j) \in \mathbf{R}^j | 0 \leq x_1 \leq \cdots \leq x_j \leq 1\}$).

Part 3) of Remark 2. implies that the images under Φ of all points from the boundary of $\overline{\Delta_j}$ with at least one coordinate equal to 0 or ∞ are points from the interior of $\overline{\Delta_j}$. (Indeed, if P is divisible by x^k, $k > 1$, (resp. its k leading coefficients equal 0), then only one of the numbers a_i, an already excluded one, equals 0 (resp. ∞).) For the points of the interior of $\overline{\Delta_j}$ or for the finite points without zero coordinate of its boundary the same fact follows from part 2) of Lemma 21.. By Brower's fixed point theorem the mapping $\Phi : \overline{\Delta_j} \rightarrow \overline{\Delta_j}$ has a fixed point, and this point belongs to the interior of $\overline{\Delta_j}$. Hence this point defines a polynomial for which the numbers x_i are positive, distinct and up to permutation equal to the numbers $(-a_i)$. Hence, x_i are the roots of a polynomial defining an eigenvector of the mapping Φ.

4.4. Proof of part 5)

Part 4) of the theorem and part 3) of Remark 7. imply that when j is even, then 1 is a nonroot of Q_j. Indeed, all roots of Q_j are positive, distinct and form couples of the form $(x_0, 1/x_0)$. In such a couple one cannot have $x_0 = 1$ because then $x_0 = 1/x_0 = 1$, i.e. Q_j would have a double root. In the case when j is odd in the same way one shows that 1 is a simple root of Q_j. This proves the second statement of part 5).

The first and the third statements of part 5) of the theorem follow now from part 2) of Remark 7.. For the third statement notice that $(x+1)^{n-j-2}Q_j$ is a degree $n-2$ polynomial.

4.5. Proof of part 6)

1^0. Set $Q_j = x^j + h_1x^{j-1} + \cdots + h_j$. If the polynomial $x(x+1)^{n-j-2}Q_j$ defines an eigenvector of the mapping Φ corresponding to the eigenvalue λ_{j+2}, then its factorization (2) contains exactly one composition factor K_0, exactly $n-j-3$ composition factors K_1, exactly one composition factor K_∞ and j composition factors K_{a_i} such that for the symmetric functions σ_ν of the numbers a_i one has $\sigma_\nu = \lambda_{j+2}h_\nu$.

2^0. Equality (1) allows not to write the composition factors K_1. Hence equality (2) written for the polynomial $x(x+1)^{n-j-2}Q_j$ reads

$$(\lambda_{j+2})^{-1}x(x+1)^{n-j-2}Q_j = K_0 \mathbin{\overset{*}{_n}} K_\infty \mathbin{\overset{*}{_n}} K_{a_1} \mathbin{\overset{*}{_n}} \cdots \mathbin{\overset{*}{_n}} K_{a_j} \tag{11}$$

For each power $\mu = 1, \ldots, n-1$ the coefficient of x^μ in the left-hand (resp. in the right-hand) side of (11) is an affine function $G_\mu(h_1, \ldots, h_j)$ (an affine function $H_\mu(\sigma_1, \ldots, \sigma_j)$) whose coefficients are rational functions in n with rational coefficients (see Subsection 5.1.).

3^0. Applying the equality $\sigma_\nu = \lambda_{j+2}h_\nu$, one can consider H_μ to be rational functions with rational coefficients not of $(\sigma_1,\ldots,\sigma_j)$, but of $(h_1,\ldots,h_j)$ (remember that λ_{j+2} is a rational function of n with rational coefficients).

Set $F_\mu := G_\mu - H_\mu$. Denote by (Σ) the system of linear equations $F_\mu(h_1,\ldots,h_j) = 0$ (it contains $n-1$ equations). Replace the rational functions of n (i.e. the coefficients of the variables h_ν) by their series in $1/n$ (convergent about the origin). Multiply each equation $F_\mu = 0$ by a suitably chosen integer power of n so that it takes the form

$$\sum_{l=0}^{\infty} F_\mu^l(h_1,\ldots,h_j)(1/n)^l = 0 \tag{12}$$

where the affine functions F_μ^l (with rational coefficients) are independent of n. Find the maximal possible number of equations whose affine functions F_μ^0 have linearly independent linear parts. Denote by $\mathcal{F}_1$ the set of these affine functions, by $\mu_1,\ldots,\mu_{k_1}$ their indices μ and by $\mathcal{S}(\mathcal{F}_1) \subset \mathbf{R}^j$ the linear space spanned by their linear parts.

4^0. Suppose that $\mathrm{card}(\mathcal{F}_1) = j$. Then the system of equations $F_\mu = 0$, $\mu = \mu_1,\ldots,\mu_j$ has as limit when $n \to \infty$ the nondegenerate linear system $F_\mu^0 = 0$, $\mu = \mu_1,\ldots,\mu_j$ from which one finds the values of the coefficients h_ν for $Q_j^* = \lim_{n\to\infty} Q_j$. This last system is with rational coefficients, hence h_ν are rational. The polynomials Q_j^* are limits of self-reciprocal monic hyperbolic polynomials, hence they are such as well. More exactly, the equality $(Q_j)^R = (-1)^j Q_j$ valid for every n implies $(Q_j^*)^R = (-1)^j Q_j^*$. All roots of Q_j are positive, hence all roots of Q_j^* are nonnegative. In fact they are positive because $Q_j(0) = (-1)^j$ for all values of n, hence $Q_j^*(0) = (-1)^j \neq 0$. For j odd one has $Q_j(1) = 0$, hence $Q_j^*(1) = 0$ for j odd.

5^0. Suppose that $\mathrm{card}(\mathcal{F}_1) < j$. Then the left-hand sides of the remaining equations (12) (i.e. with $\mu \neq \mu_i, i = 1,\ldots,k_1$) cannot all belong to the $(1/n)$-module generated by the functions F_μ, $\mu = \mu_1,\ldots,\mu_j$. Indeed, this would imply that the system of equations (Σ) is not of maximal rank; hence, for every n it either has more than one linearly independent solution $(h_1,\ldots,h_j)$ or it has no solution. This is impossible because the unique solution of this system is the polynomial Q_j defining the unique up to multiplication eigenvector of Φ corresponding to its eigenvalue λ_{j+2}. (The polynomial Q_j is a unique solution because it is presumed monic.)

Hence one can choose a function $F_{\mu_{k_1+1}}$, $\mu_{k_1+1} \neq \mu_i, i = 1,\ldots,k_1$ and polynomials $\varphi_i(1/n)$ such that

$$\tilde{F}_{\mu_{k_1+1}} := F_{\mu_{k_1+1}} - \sum_{i=1}^{k_1} \varphi_i F_{\mu_i} = \sum_{l=l_0}^{\infty} \tilde{F}^l_{\mu_{k_1+1}}(h_1, \ldots, h_j)(1/n)^l \ ,$$

where $l_0 \in \mathbf{N}$ and the linear part of the affine function $\tilde{F}^{l_0}_{\mu_{k_1+1}}$ does not belong to the space $\mathcal{S}(\mathcal{F}_1)$.

6^0. Replace in system (Σ) the equation $F_{\mu_{k_1+1}} = 0$ by the equation $n^{l_0}\tilde{F}_{\mu_{k_1+1}} = 0$. This defines a new system equivalent to (Σ). Add the function $\tilde{F}^{l_0}_{\mu_{k_1+1}}$ to the set $\mathcal{F}_1$. This defines the set $\mathcal{F}_2$ with card$(\mathcal{F}_2)$ >card$(\mathcal{F}_1)$ and dim$(\mathcal{S}(\mathcal{F}_2))$ >dim$(\mathcal{S}(\mathcal{F}_1))$. After a finite number of changing $\mathcal{F}_k$ by $\mathcal{F}_{k+1}$ one obtains a system equivalent to (Σ) with card$(\mathcal{F}_p) = j$ and one applies the reasoning from 4^0.

5. Comments and open questions

5.1. How to compute explicitly the matrix M^{-1}

To write down the matrix M^{-1} one can use equality (2). Its left-hand side equals $(x+1)P_1$ and the coefficient of x^k there is $c_{n-k}+c_{n-k-1}$, see Notation 3.. The coefficient of x^k in the right-hand side equals

$$(13)(\prod_{i=1}^{n-1}(C^k_{n-1}a_i + C^{k-1}_{n-1}))/(C^k_n)^{n-2} = \sum_{j=0}^{n-1}((C^k_{n-1})^j(C^{k-1}_{n-1})^{n-1-j}/(C^k_n)^{n-2})\sigma_j$$

Use the notation of Remark 5.. The vector-column of the coefficients of x, x^2, …, x^{n-1} in the left-hand side of (2) equals $(I+N)\tilde{c}+Z$ where N is the lower-triangular matrix with units on the first subdiagonal and zeros elsewhere and Z does not depend on the coefficients c_i for $i > 0$. Denote by A the matrix $\{A_{k,j}\}_{k=1}^{n-1}{}_{j=1}^{n-1}$ where $A_{k,j} = (C^k_{n-1})^j(C^{k-1}_{n-1})^{n-1-j}/(C^k_n)^{n-2}$, see the right-hand side of (13). The matrix M^{-1} equals $(I+N)^{-1}A = (I - N + N^2 - \cdots + (-N)^{n-2})A$.

5.2. The roots of the polynomials Q_j

For $n = 5, \ldots, 10$ we list below the roots of the polynomials Q_j (computed with precision 10^{-10}). Notice that

1) for n fixed the roots of Q_j interlace with the ones of Q_{j+1};

2) the roots $x_{j-j_1+1} < x_{j-j_1+2} < \cdots < x_j$, $j_1 = [j/2]$, $1 < x_{j-j_1+1}$, of the polynomials Q_j computed for two consecutive values of n, interlace; the same is true about their roots $x_1 < \cdots < x_{j_1} < 1$.

It would be interesting to prove or disprove that these properties hold for any n and j. It would be also worth knowing whether

3) all coefficients of all polynomials Q_j^* are not only rational but integer;

4) all roots of all polynomials Q_j^* are simple (if this is true, then one would have in particular $Q_j^*(1) \neq 0$ for j even);

5) whether the roots of Q_j^* interlace with the ones of Q_{j+1}^*.

	$n = 5$						
Q_1	1.						
Q_2	0.4312706956	2.318729304					
	$n = 6$						
Q_1	1.						
Q_2	0.4202041029	2.379795897					
Q_3	0.2679491924	1.	3.732050808				
	$n = 7$						
Q_1	1.						
Q_2	0.4132004518	2.420132882					
Q_3	0.2556906500	1.	3.910976017				
Q_4	0.1952834097	0.6108227089	1.637136252	5.120762700			
	$n = 8$						
Q_1	1.						
Q_2	0.4083673674	2.448775490					
Q_3	0.2476429769	1.	4.038071309				
Q_4	0.1833814682	0.6006099942	1.664973959	5.453113720			
Q_5	0.1554820965	0.4344139008	1.	2.301952120	6.431608673		
	$n = 9$						
Q_1	1.						
Q_2	0.4048305224	2.470169478					
Q_3	0.2419521980	1.	4.133047802				
Q_4	0.1752850843	0.5937539720	1.684199260	5.704991978			
Q_5	0.1443957348	0.4214487836	1.	2.372767555	6.925412316		
Q_6	0.1306471244	0.3370249030	0.7064754847	1.415477284	2.967139790	7.654205973	
	$n = 10$						
Q_1	1.						
Q_2	0.4021298312	2.486759058					
Q_3	0.2377143223	1.	4.206730122				
Q_4	0.1694263691	0.5887904231	1.698397190	5.902268965			
Q_5	0.1365772354	0.4126132985	1.	2.423576757	7.321864419		
Q_6	0.1205562417	0.3233130653	0.6979347398	1.432798717	3.092977388	8.294883663	
Q_7	0.1136295820	0.2766203885	0.5424930753	1.	1.843341502	3.615062525	8.800525203

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Université de Nice
Laboratoire Jean-Alexandre Dieudonné
UMR 6621
Parc Valrose
06108 Nice Cedex 2
France
Received 01.02.2008
E-mail: kostov@math.unice.fr