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On Almost γ-Continuous Multifunctions

Idris Zorlutuna

Presented by P. Boyvalenkov

The purpose of this paper is to study upper (lower) almost γ-continuous multifunctions. Some new covering properties have been introduced and their behaviour under upper almost γ-continuous multifunctions are investigated.

Key Words: multifunctions, $\gamma-$continuity, covering properties

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1. Introduction

Several weak and strong variants of continuity of multifunctions occur in the literature ([1],[8], [11], [12]). A weak form of continuous multifunctions called upper (lower) γ-continuous multifunctions was introduced and studied by Abd El-Monsef and Nasef [1]. Recently, Ekici and Park [6] have introduced and investigated the notions of upper and lower almost γ-continuous multifunctions. The notion of b-open sets has been introduced by Andrijevic [2]. This notion was also called γ-open sets in the sense of El-Atik [7] and called sp-open sets in the sense of Dontchev and Przemski [5]. In this paper, we shall continue the investigations carried out in [6]. We give some new characterizations and introduce new covering properties. Further, we study behaviour of these covering properties under the upper almost γ-continuous multifunctions.

Let U be a subset of a topological space (X,τ), $Int(U)$ (resp. $Cl(U)$) will denote the interior (resp. closure) of U in (X,τ). And U is called regular open (resp. regular closed) if $U = Int(Cl(U))$ (resp. $U = Cl(Int(U))$). The family of all regular open subsets of (X,τ) forms a base for a smaller topology τ_s on X, called the semi-regularization of τ [9]. Sometimes we write X_s for (X,τ_s).

A subset A is called γ-open [7] if $A \subset Cl(Int(A)) \cup Int(Cl(A))$. The complement of a γ-open set is called γ-closed. The intersection of all γ-closed sets of X containing A is called the γ-closure [2] of A and is denoted by $bCl(A)$. The union of all γ-open sets of X contained in A is called the γ-interior [2] of A and is denoted by $bInt(A)$. Define $BO(X,x) = \{U \subseteq X : x \in U$ and U is γ-open$\}$

A point $x \in X$ is called a $\delta-$cluster point of a subset A of X if $A \cap U \neq \emptyset$ for every regular open neighbourhood U of x. The $\delta-$closure of A, denoted by $Cl_\delta(A)$, is the set all $\delta-$cluster points of A [15], and A is $\delta-$closed if $A = Cl_\delta(A)$. A subset A is called $\delta-$open if the complement of it is $\delta-$closed. The union of all regular open sets of X contained in A is called the δ-interior of A and is denoted by $Int_\delta(A)$.

The net $(x_\alpha)_{\alpha \in I}$ is $\delta-$convergent to x [10] (resp. γ-convergent [7]) if for each regular open (resp. γ-open) set U containing x, there exists a $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in U$.

A topological space is called a $P-$space if every $G_\delta-$set is open.

A space X is called almost paralindelöf [13] (resp. almost paracompact) if each open cover Ψ of X has a locally countable (resp. locally finite) open refinement Ω such that $X = \cup\{Cl(V) : V \in \Omega\}$.

If $A \subset X$, A is called an $APL-$set (resp. $PL-$set) [4] in X if every open cover Ψ of A in X has a locally countable open refinement Ω in X such that $A \subset \cup\{Cl(V) : V \in \Omega\}$ (resp. $A \subset \cup\{V : V \in \Omega\}$). A is called α-paracompact [16] if every open cover of A in X has a locally finite open refinement in X which covers A.

A space X is called nearly compact [14] (resp. γ-compact [7]) if every regular open (resp. γ-open) cover of X has a finite subcover. The notion of nearly compactness was localized by Carnahan [3], by introducing N-closed subsets of a space. Recall that a subset A of X is called N-closed in X if every regular open cover of A in X has a finite subcover. Similarly, a subset A of X is called γ-compact in X if every γ-open cover of A in X has a finite subcover.

A multifunction $F : X \rightsquigarrow Y$ is a point to set correspondence, and we always assume that $F(x) \neq \emptyset$ for every point $x \in X$. For each subset A of X and each subset B of Y, let $F(A) = \cup\{F(x) : x \in A\}$, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. For $y \in Y$, we use $F^-(y)$ instead of $F^-(\{y\})$.

The concept of almost continuity for multifunctions was firstly introduced by Popa [12]. A multifunction $F : X \rightsquigarrow Y$ is called upper almost continuous, abbreviated as u.a.c., (resp. lower almost continuous, or l.a.c.) at $x \in X$ if for each open $V \subset Y$ with $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$), there is an open neighbourhood U of x such that $F(U) \subset Int(Cl(V))$ (resp. $F(z) \cap Int(Cl(V)) \neq \emptyset$ for $z \in U$). F is u.a.c. (resp. l.a.c.) iff it is u.a.c. (resp. l.a.c.) at each point of X. Then F is called almost continuous iff it is both u.a.c.and l.a.c.

A multifunction $F : X \rightsquigarrow Y$ is called upper $\delta-$continuous [8] abbreviated as u.δ.c.(resp. lower $\delta-$continuous or l.δ.c. [8]) at a point $x \in X$ if for each open set $V \subseteq Y$ with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there is an open set U containing x such that $F(Int(Cl(U))) \subseteq Int(Cl(V))$ (resp. $F(z) \cap Int(Cl(V)) \neq \emptyset$ for each $z \in Int(Cl(U))$).

A multifunction $F : X \rightsquigarrow Y$ is called upper γ-continuous [8] abbreviated as u.γ.c.(resp. lower γ-continuous or l.γ.c. [1]) at a point $x \in X$ if for each open set $V \subseteq Y$ with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there is a γ-open set U containing x such that $F(U) \subseteq V$ (resp. $F(z) \cap V \neq \emptyset$ for each $z \in U$).

2. Characterizations

Definition 1. [6] A multifunction $F : X \rightsquigarrow Y$ is called

(a) upper almost γ-continuous (briefly, u.a.γ-c.) at a point $x \in X$ if for each open subset V of Y with $F(x) \subseteq V$, there is a γ-open set U containing x such that $F(U) \subseteq Int(Cl(V))$.

(b) lower almost γ-continuous (briefly, l.a.γ-c.) at a point $x \in X$ if for each open subset V of Y with $F(x) \cap V \neq \emptyset$, there is a γ-open set U containing x such that $F(z) \cap Int(Cl(V)) \neq \emptyset$ for every point $z \in U$.

(c) almost γ-continuous at $x \in X$ if it is both u.a.γ-c. and l.a.γ-c. at $x \in X$.

(d) almost γ-continuous if it is almost γ-continuous at each point $x \in X$.

Theorem 1. *For a multifunction $F : X \rightsquigarrow Y$, the following statements are equivalent;*

(1) F is l.a.γ-c.;

(2) For each regular open subset V of Y, $F^-(V)$ is γ-open;

(3) For each regular closed subset K of Y, $F^+(K)$ is γ-closed;

(4) For each $\delta-$open subset V of Y, $F^-(V)$ is γ-open;

(5) For each $\delta-$closed subset K of Y, $F^+(K)$ is γ-closed;

(6) $F : X \rightsquigarrow Y_s$ is l.γ-c.;

(7) For any subset B of Y, $bCl(F^+(B)) \subseteq F^+(Cl_\delta(B))$;

(8) For any subset B of Y, $F^-(Int_\delta(B)) \subseteq bInt(F^-(B))$;

(9) For any subset A of X, $F(bCl(A)) \subseteq Cl_\delta(F(A))$.

Proof. (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3)$\Leftrightarrow$(6) is in [6]

(2)$\Rightarrow$(4): Let V be any $\delta-$open set in Y and $x \in F^-(V)$. Then $F(x) \cap V \neq \emptyset$ and there exists a $y \in F(x) \cap V$. Since V is $\delta-$open set, there exists a regular open set W such that $y \in W \subseteq V$. Therefore, we have $F(x) \cap W \neq \emptyset$.

By (2), $F^{-}(W)$ is γ-open in X and $x \in F^{-}(W) \subseteq F^{-}(V)$. This shows that $F^{-}(V)$ is γ-open set in X.

(4)⇔(5): They follow from equality $F^{-}(Y\backslash K) = X\backslash F^{+}(K)$.

(5)⇒(7): Let B be any subset of Y. Then by (4) $F^{+}(Cl_{\delta}(B))$ is γ-closed subset of X. Since $F^{+}(B) \subseteq F^{+}(Cl_{\delta}(B))$, then

$$bCl(F^{+}(B)) \subseteq bCl(F^{+}(Cl_{\delta}(B))) = F^{+}(Cl_{\delta}(B)).$$

(7)⇔(8): These follow from the facts that $F^{-}(Y\backslash K) = X\backslash F^{+}(K)$, $Y\backslash(Cl_{\delta}(B)) = Int_{\delta}(Y\backslash B)$ for $B \subseteq Y$ and $X\backslash(bCl(A)) = bInt(X\backslash A)$ for each subset A of X.

(8)⇒(9):Under the assumption (8), suppose (9) is not true i.e. for some $A \subseteq X$. $F(bCl(A)) \nsubseteq Cl_{\delta}(F(A))$. Then there exists a $y_0 \in Y$ such that $y_0 \in F(bCl(A))$ but $y_0 \notin Cl_{\delta}(F(A))$. So $Y\backslash Cl_{\delta}(F(A))$ is an $\delta-$open set containing y_0. By (8), we have $F^{-}(Y\backslash Cl_{\delta}(F(A))) = F^{-}(Int_{\delta}(Y\backslash Cl_{\delta}(F(A)))) \subseteq bInt(F^{-}(Y\backslash Cl_{\delta}(F(A))))$ and $F^{-}(y_0) \subseteq F^{-}(Y\backslash Cl_{\delta}(F(A)))$. Since $F^{-}(Y\backslash Cl_{\delta}(F(A))) \cap F^{+}(F(A)) = \emptyset$ and $A \subset F^{+}(F(A))$ we have $F^{-}(Y\backslash Cl_{\delta}(F(A)))$ is γ-open set, clearly we have that $F^{-}(Y\backslash Cl_{\delta}(F(A))) \cap bCl(A) = \emptyset$. On the other hand, because of $y_0 \in F(bCl(A))$, we have $F^{-}(y_0) \cap bCl(A) \neq \emptyset$. But this is a contradiction with $F^{-}(Y\backslash Cl_{\delta}(F(A))) \cap bCl(A) = \emptyset$. Thus $y \in F(bCl(A))$ implies $y \in Cl_{\delta}(F(A))$. Consequently $bCl(F(A)) \subseteq Cl_{\delta}(F(A))$.

(9)⇒(3): Let $K \subseteq Y$ be a regular closed set. Since we always have $F(F^{+}(K)) \subset K$, $Cl_{\delta}(F(F^{+}(K))) \subseteq Cl_{\delta}(K)$ and by (9), $F(bCl(F^{+}(K))) \subseteq Cl_{\delta}(F(F^{+}(K))) \subseteq Cl_{\delta}(K) = K$. Hence, $bCl(F^{+}(K)) \subseteq F^{+}(F(bCl(F^{+}(K)))) \subset F^{+}(K)$ and so $F^{+}(K)$ is γ-closed in X. ■

Theorem 2. *If the multifunction $F : X \rightsquigarrow Y$ is l.a.γ-c., then for each $y \in F(x)$ and for every net $(x_{\alpha})_{\alpha\in I}$ γ-converging to x, there exists a subnet $(z_{\beta})_{\beta\in\xi}$ of the net $(x_{\alpha})_{\alpha\in I}$ and a net $(y_{\beta})_{(\beta,V)\in\xi}$ in Y with $y_{\beta} \in F(z_{\beta})$ is $\delta-$convergent to y. Moreover, if (intersection of two γ-open sets in X is γ-open in X) X is a γ-space, then the converse of the above implication is also true.*

Proof. (⇒): Suppose F is l.a.γ-c. at x_0. Let $(x_{\alpha})_{\alpha\in I}$ be a net γ-converging to x_0. Let $y \in F(x_0)$ and V be any regular open set containing y. So we have $F(x_0) \cap V \neq \emptyset$. Since F is l.a.γ-c. at x_0, there exists a γ-open set U such that $x_0 \in U \subseteq F^{-}(V)$. Since the net $(x_{\alpha})_{\alpha\in I}$ is γ-convergent to x_0, for this U, there exists $\alpha_0 \in I$ such that $\alpha \geq \alpha_0 \Rightarrow x_{\alpha} \in U$. Therefore, we have the implication $\alpha \geq \alpha_0 \Rightarrow x_{\alpha} \in F^{-}(V)$. For each regular open set $V \subseteq Y$ containing y, define the sets $I_V = \{\alpha_0 \in I : \alpha \geq \alpha_0 \Rightarrow x_{\alpha} \in F^{-}(V)\}$ and $\xi = \{(\alpha, V) : \alpha \in I_V, y \in V \text{ and } V \text{ is regular open }\}$ and order " $\geq$ " on ξ

as follows: "$(\acute{\alpha}, \acute{V}) \geq (\alpha, V) \Leftrightarrow \acute{V} \subseteq V$ and $\acute{\alpha} \geq \alpha$". Define $\varphi : \xi \longrightarrow I$, by $\varphi((\beta, V)) = \beta$. Then φ is increasing and cofinal in I, so φ defines a subnet of $(x_\alpha)_{\alpha \in I}$. We denote the subnet $(z_\beta)_{(\beta,V)\in\xi}$. On the other hand, for any $(\beta, V) \in \xi$, if $\beta \geq \beta_0 \Rightarrow x_\beta \in F^-(V)$ and we have $F(z_\beta) \cap V = F(x_\beta) \cap V \neq \phi$. Pick $y_\beta \in F(z_\beta) \cap V \neq \phi$. Then the net $(y_\beta)_{(\beta,V)\in\xi}$ is $\delta-$convergent to y. To see this, let V_0 be a regular open set containing y. Then there exists $\beta_0 \in I$ such that $\varphi((\beta_0, V_0)) = \beta_0$ and $y_{\beta_0} \in V$. If $(\beta, V) \geq (\beta_0, V_0)$ this means that $\beta \geq \beta_0$ and $V \subseteq V_0$. Therefore, $y_\beta \in F(z_\beta) \cap V = F(x_\beta) \cap V \subseteq F(x_\beta) \cap V_0$, so $y_\beta \in V_0$. Thus $(y_\beta)_{(\beta,V)\in\xi}$ is $\delta-$convergent to y.

($\Leftarrow$): Suppose F is not l.a.γ-c. at x_0. Then there exists an open set $V \subseteq Y$ so that $x_0 \in F^-(V)$ and for each γ-open set $U \subseteq X$ containing x_0, there is a point $x_U \in U$ for which $x_U \notin F^-(Int(Cl(V)))$. Let us consider the net $(x_U)_{U\in BO(X,x_0)}$. Obviously $(x_U)_{U\in BO(X,x_0)}$ is γ-convergent to x_0. Let $y_0 \in F(x_0) \cap V$. By hypothesis, there is a subnet $(z_w)_{w\in W}$ of $(x_U)_{U\in BO(X,x_0)}$ and $y_w \in F(z_w)$ such that $(y_w)_{w\in W}$ is $\delta-$convergent to y_0. As $y_0 \in V$ and $Int(Cl(V)) \subseteq Y$ is a regular open set, there is $w_0' \in W$ so that $w \geq w_0'$ implies $y_w \in Int(Cl(V))$. On the other hand, $(z_w)_{w\in W}$ is a subnet of the net $(x_U)_{U\in BO(X,x_0)}$ and so there is a function $h : W \longrightarrow BO(X, x_0)$ such that $z_w = x_{h(w)}$. By the definition of the net $(x_U)_{U\in BO(X,x_0)}$, we have $F(z_w) \cap Int(Cl(V)) = F(x_{h(w)}) \cap Int(Cl(V)) = \emptyset$ and this means that $y_w \notin Int(Cl(V))$. This is a contradiction and so F is l.a.γ-c. at x_0. ■

Similarly, we can obtain a characterization of lower (upper) almost γ-continuity for multifunctions (see [6]).

Theorem 3. *If the multifunction $F : X \rightsquigarrow Y$ is l.a.γ-c. (resp. u.a.γ-c.), then for each net $(x_\alpha)_{\alpha\in I}$ γ-convergent to x and for each regular open subset V of Y with $F(x) \cap V \neq \emptyset$ (resp. $F(x) \subseteq V$), there is an $\alpha_0 \in I$ such that $F(x_\alpha) \cap V \neq \emptyset$ (resp. $F(x_\alpha) \subseteq V$) for all $\alpha \geq \alpha_0$. Moreover, if (intersection of two γ-open sets in X is γ-open in X) X is a γ-space, then the converse of the above implication is also true.*

Proof. ($\Rightarrow$): [6]

($\Leftarrow$): We prove only for lower almost γ-continuity. The other is entirely analogous. Suppose that F is not l.a.γ-c. Then there is a regular open set V in Y with $x \in F^-(V)$ such that for each γ-open set U of X containing x, $x \in U \nsubseteq F^-(V)$ i.e. there is a $x_U \in U$ such that $x_U \notin F^-(V)$. Define $D = \{(x_U, U) : U \in BO(X), x_U \in U, x_U \notin F^-(V)\}$. Now the order " $\leq$ " defined by $(x_{U_1}, U_1) \leq (x_U, U) \Leftrightarrow U \subseteq U_1$ is a direction on D and g defined by $g : D \longrightarrow X$, $g((x_U, U)) = x_U$ is a net on X. The net $(x_U)_{(x_U,U)\in D}$ is γ-convergent to x. But $F(x_U) \cap V = \emptyset$ for all $(x_U, U) \in D$. This is a contradiction. ■

From the above definitions, it is obvious that upper (lower) γ-continuity implies upper (lower) almost γ-continuity. But the converse is not true in general.

Example 1. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Define a multifunction $F : X \rightsquigarrow X$ by $F(x) = \begin{cases} \{a, c\} & ; x = a \\ \{b\} & ; x = b \\ X & ; x = c \end{cases}$. Then F is u.a.γ-c. But $F^+(\{a, b\}) = \{b\}$ is not γ-open subset in X which implies F is not u.γ-c. at $x = b$.

Example 2. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Define a multifunction $F : X \rightsquigarrow X$ by $F(x) = \begin{cases} \{b\} & ; x = a \\ \{a, c\} & ; x = b \\ \{c\} & ; x = c \end{cases}$. Then F is l.a.γ-c. But $F^-(\{a\}) = \{b\}$ is not γ-open subset in X which implies F is not l.γ-c. at $x = b$.

3. Some applications

Definition 2. The graph $G(F)$ of the multifunction $F : X \rightsquigarrow Y$ is almost γ-closed with respect to X if for each $(x, y) \notin G(F)$, there exist a γ-open set U containing x and an open set V containing y such that $(U \times Int(Cl(V))) \cap G(F) = \emptyset$.

Theorem 4. *If $F : X \rightsquigarrow Y$ is a u.a.γ-c. multifunction into a Hausdorff space Y and $F(x)$ is α-paracompact for each $x \in X$, then the graph $G(F)$ is almost γ-closed with respect to X.*

Proof. Let $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Therefore, every $y \in F(x_0)$, there exists a regular open set $V(y)$ and an open set $W(y)$ in Y containing y and y_0 respectively, such that $V(y) \cap W(y) = \emptyset$. Then $\{V(y) | y \in F(x_0)\}$ is a regular open cover of $F(x_0)$, thus there is a locally finite cover $\Psi = \{U_\beta | \beta \in \Delta\}$ of $F(x_0)$ which refines $\{V(y) | y \in F(x_0)\}$. So there exists an open neighborhood W_0 of y_0 such that W_0 intersect only finitely many members $U_{\beta_1}, U_{\beta_2}, ..., U_{\beta_n}$ of Ψ. Chose finitely many points $y_1, y_2, ..., y_n$ of $F(x_0)$ such that $U_{\beta_k} \subset V(y_k)$ of each $1 \leq k \leq n$. And set $W = W_0 \cap [\bigcap_{k=1}^{n} W(y_k)]$. Then W is an open neighborhood of y_0 such that $W \cap (\cup\Psi) = \emptyset$, hence $Int(Cl(W)) \cap (\cup\Psi) = \emptyset$. Since F is u.a.γ-c., then there exists a γ-open set U containing x_0 such that $F(U) \subset \cup\Psi$. Therefore, we have that $(U \times Int(Cl(W))) \cap G(F) = \emptyset$. Thus, $G(F)$ is almost γ-closed set with respect to X. ∎

In the upper Theorem, for upper almost γ-continuous multifunction F, if F is taken as a point closed multifunction and Y is taken as a regular space, then we get also same result.

Theorem 5. *Let $F : X \rightsquigarrow Y$ be a point N-closed and u.a.γ-c. multifunction. If A is a γ-compact set relative to X, then $F(A)$ is N-closed in Y.*

Proof. Let A be a γ-compact set relative to X and Φ be a regular open cover of $F(A)$. If $a \in A$, then we have $F(a) \subseteq \cup\Phi$. Thus Φ is a regular open cover of $F(a)$. Since $F(a)$ is N-closed, there exists a finite subfamily $\Phi_{n(a)}$ of Φ such that $F(a) \subseteq \cup\Phi_{n(a)} = V_a$. V_a is an δ-open in Y. Since F is u.a.γ-c., $F^+(V_a)$ is a γ-open set in X. Therefore, $\Omega = \{F^+(V_a) : a \in A\}$ is a γ-open cover of A. Since A is γ-compact set relative to X, there exist points $a_1, a_2, ..., a_n \in A$ such that $A \subset \cup\{F^+(V_{a_i}) : a_i \in A,\ i = 1, 2, ..., n\}$. So we obtain $F(A) \subseteq F(\cup\{F^+(V_{a_i}) : i = 1, 2, ..., n\}) \subset \cup\{V_{a_i} : i = 1, 2, ..., n\} \subset \cup\{\Phi_{n(a_i)} : i = 1, 2, ..., n\}$. Thus $F(A)$ is N-closed in Y. ■

Corollary 1. *Let $F : X \rightsquigarrow Y$ be a point N-closed and u.a.γ-c. multifunction. If X is γ-compact and F is surjective, then Y is nearly compact.*

Definition 3. A multifunction $F : X \rightsquigarrow Y$ is called

(a) strongly γ-open if for each γ-open subset U of X, $F(U)$ is open in Y.

(b) strongly γ-closed if for each γ-closed subset K of X, $F(K)$ is closed in Y.

Proposition 1. *A multifunction $F : X \rightsquigarrow Y$ is strongly γ-closed if and only if for each point $y \in Y$ and each γ-open subset U of X with $F^-(y) \subseteq U$, there exists an open neighbourhood V_y of y such that $F^-(V_y) \subseteq U$.*

Proof. ($\Rightarrow$): Suppose that F is strongly γ-closed, $y \in Y$ is any point and U is any γ-open subset of X with $F^-(y) \subseteq U$. Then $Y - F(X - U)$ is an open neigborhood of y. Set $V_y = Y - F(X - U)$. Then $F^-(V_y) = F^-(Y - F(X - U)) = X - F^+(F(X - U)) \subseteq U$.

($\Leftarrow$): Let K be any γ-closed subset of X and $G = Y - F(K)$. Then for each $y \in G$, $F^-(y) \subseteq X - K$. By hypothesis, there exists an open neighborhood V_y of y such that $F^-(V_y) \subseteq X - K$. Let $V = \cup\{V_y : y \in G\}$. Then V is open and $F^-(V) \subseteq X - K$. Therefore, $F(K) \subseteq Y - V$. On the other hand, $G = Y - F(K) \subseteq V$ which implies that $F(K) = Y - V$. Therefore, $F(K)$ is a closed subset of Y. ■

Lemma 1. *Let a multifunction $F : X \rightsquigarrow Y$ be a strongly γ-closed surjection such that $F^-(y)$ is γ-compact set in X for each $y \in Y$. If $\Psi = \{U_\alpha : \alpha \in \Delta\}$ is a γ-open locally finite family, then $F(\Psi) = \{F(U_\alpha) : \alpha \in \Delta\}$ is a locally finite family.*

Proof. Let $\Psi = \{U_\alpha : \alpha \in \Delta\}$ be any γ-open locally finite family in X. Let $y \in Y$ be any point in Y. Then for each $x \in F^-(y)$, we can choose an open neighbourhood $G(x)$ of x such that $G(x)$ intersects only finitely many members of Ψ. Let $\Delta(x)$ be a finite subset of Δ such that $G(x) \cap U_\alpha \neq \emptyset$ for $\alpha \in \Delta(x)$, and $G(x) \cap U_\alpha = \emptyset$ for $\alpha \in \Delta \backslash \Delta(x)$. The family $\{G(x) : x \in F^-(y)\}$ is an open and so γ-open cover of $F^-(y)$. Since $F^-(y)$ is γ-compact in X, there exists a finite number of points $x_1, x_2, ..., x_n$ of $F^-(y)$ such that $F^-(y) \subseteq \cup\{G(x_i) : 1 \leq i \leq n\}$. Set $G = \cup\{G(x_i) : 1 \leq i \leq n\}$. Then G is open and so γ-open set in X containing $F^-(y)$ such that $G \cap U_\alpha = \emptyset$ for all $\alpha \in \Delta \backslash \cup \{\Delta(x_i) : 1 \leq i \leq n\}$. If $G = X$, then the family Ψ is finite, hence $F(\Psi)$ is finite and so locally finite. Let $G \neq X$. Since F is strongly γ-closed, there is an open neighbourhood V of y such that $F^-(V) \subseteq G$. Thus we have $V \cap F(U_\alpha) = \emptyset$ for every $\alpha \in \Delta \backslash \cup \{\Delta(x_i) : 1 \leq i \leq n\}$. This implies $\{F(U_\alpha) : \alpha \in \Delta\}$ is locally finite. ∎

Definition 4. A space X is called γ-paracompact if every γ-open cover of X has a locally finite γ-open refinement which covers X.

Theorem 6. *Let $F : X \rightsquigarrow Y$ be a strongly γ-open, strongly γ-closed, u.a.γ-c. multifunction of a γ-paracompact space X onto a space Y such that $F(x)$ is α-paracompact for each $x \in X$ and $F^-(y)$ is γ-compact set in X for each $y \in Y$. Then Y is almost paracompact space.*

Proof. Let Ψ be any open cover of Y. For each $x \in X$, since $F(x)$ is α-paracompact, then it has an open locally finite in Y cover $\Psi(x)$ such that $\Psi(x)$ refines Ψ. Set $G(x) = Int(Cl(\cup\Psi(x))$. Then $F(x) \subseteq G(x)$ and since F is u.a.γ-c., $\{F^+(G(x)) : x \in X\}$ is a γ-open cover of X. Since X is γ-paracompact, there exists a locally finite γ-open refinement $\Omega = \{W_\beta : \beta \in \Lambda\}$ of $\{F^+(G(x)) : x \in X\}$ such that $X = \cup\Omega$. By the above lemma, $F(\Omega) = \{F(W_\beta) : \beta \in \Lambda\}$ is locally finite. Since F is strongly γ-open, $F(\Omega)$ is a locally finite open covering of Y. For each $\beta \in \Lambda$, there exists a point $x_\beta \in X$ such that $W_\beta \subseteq F^+(G(x_\beta))$. Therefore, we have $F(W_\beta) \subset F(F^+(G(x_\beta))) \subseteq G(x_\beta) = Int(Cl(\cup\Psi(x_\beta))$.

Let $\Re_\beta = \{F(W_\beta) \cap V : V \in \Psi(x_\beta)\}$ for each $\beta \in \Lambda$ and $\Re = \{R : R \in \Re_\beta, \beta \in \Lambda\}$ for some $\beta \in \Lambda$. Then $\Re$ is an open refinemet of Ψ. Moreover,

$$\begin{aligned} Y &= F(X) \subseteq F(\cup W_\beta) = \cup F(W_\beta) \\ &= \cup[F(W_\beta) \cap Cl(\cup\Psi(x_\beta)] \subseteq \cup[Cl(F(W_\beta) \cap (\cup\Psi(x_\beta))] \\ &= \cup Cl[\cup\{R : R \in \Re_\beta\}] = \cup[\cup\{Cl(R) : R \in \Re_\beta\}] = \cup\{Cl(R) : R \in \Re\} \end{aligned}$$

Hence Y is almost paracompact. ■

Definition 5. A space X is called γ-paralindelöf if every γ-open cover of X has a locally countable γ-open refinement which covers X.

Definition 6. A subset A of a space X is called

(a) γ-paralindelöf in X if every γ-open cover Ψ of A in X has a locally countable γ-open refinement Ω in X such that $A \subseteq \cup\{V : V \in \Omega\}$.

(b) γ-Lindelöf in X if every γ-open cover of A in X has a countable subcover.

Theorem 7. *Let $F : X \rightsquigarrow Y$ be a strongly γ-open, strongly γ-closed, u.a.γ-c. multifunction of a X into a $P-$space Y such that $F(x)$ is a $PL-$set in Y for each $x \in X$ and $F^{-}(y)$ is γ-Lindelöf in X for each $y \in Y$. If A is a γ-paralindelöf set in X, then $F(A)$ is an $APL-$set in Y.*

Proof. Let Ψ be any open cover of $F(A)$ in Y. For each $x \in X$, since $F(x)$ is a $PL-$set in Y, then it has a locally countable open cover $\Psi(x)$ in Y such that $\Psi(x)$ refines Ψ. Since F is u.a.γ-c., there exists a γ-open set $U(x)$ containing x such that $F(U(x)) \subset Int(Cl(\cup\Psi(x))$. Then $\{U(x) : x \in A\}$ is a γ-open cover of A, so it has a locally countable γ-open refinement $\Omega = \{W_\alpha : \alpha \in \Delta\}$ in X such that $A \subset \cup_{\alpha\in\Delta} W_\alpha$. Then for each $\alpha \in \Delta$, there exists a point $x_\alpha \in A$ such that $W_\alpha \subset U(x_\alpha)$. Therefore we have $F(W_\alpha) \subset F(U(x_\alpha)) \subset Int(Cl(\Psi(x_\alpha))$. Let $\Re_\alpha = \{F(W_\alpha) \cap V : V \in \Psi(x_\alpha)\}$ for each $\alpha \in \Delta$ and $\Re = \{R : R \in \Re_\alpha\}$ for some $\alpha \in \Delta$. Since F is strongly γ-open, then $\Re$ is an open refinemet of Ψ. To see that $\Re$ is locally countable in Y, let $y \in Y$ and for each $x \in F^{-}(y)$, since Ω is locally countable in X, we can choose an open neighbourhood $G(x)$ of x such that $G(x)$ intersects only countably many members of Ω. Since $F^{-}(y)$ is γ-Lindelöf in X, there are countably many points $x_1, x_2, ..., x_n...$ of $F^{-}(y)$ such that $F^{-}(y) \subset \cup_{k=1}^{\infty} G(x_k)$. Let $G = \cup_{k=1}^{\infty} G(x_k)$, then G is open and so γ-open set and intersects only countably many members $W_{\alpha_1}, W_{\alpha_2}, W_{\alpha_3}, ..., W_{\alpha_n}, ...$ of Ω. By the strong γ-closedness of F, there is an open neighbourhood H_0 of y such that $F^{-}(H_0) \subset G$. It follows that H_0 intersects at most countably many members $F(W_{\alpha_1}), F(W_{\alpha_2}), F(W_{\alpha_3}), ..., F(W_{\alpha_n}), ...$ of the family $\{F(W_\alpha) : \alpha \in \Delta\}$. Furthermore, each $\Re_{\alpha_k}$ $(k = 1, 2, 3, ...)$ is locally countable, hence there exists an open neighbourhood H_k $(k = 1, 2, 3, ...)$ of y such that H_k intersects only countably many members of $\Re_{\alpha_k}$ $(k = 1, 2, 3, ...)$. Finally $H = \cap_{k=1}^{\infty} H_k$ is an open neighbourhood of y and intersects at most countably many members of $\Re$. Therefore $\Re$ is locally countable. Thus

$$\begin{aligned} F(A) &\subseteq F(\cup W_\alpha) = \cup F(W_\alpha) \subseteq [\cup F(W_\alpha) \cap Cl(\cup \Psi(x_\alpha)] \\ &= \cup [F(W_\alpha) \cap Cl(\cup \Psi(x_\alpha)] \subseteq \cup [Cl(F(W_\alpha) \cap (\cup \Psi(x_\alpha))] \\ &= \cup Cl[\cup \{R : R \in \Re_\alpha\}] = \cup [\cup \{Cl(R) : R \in \Re_\alpha\}] \\ &= \cup \{Cl(R) : R \in \Re\}. \end{aligned}$$

Hence $F(A)$ is an $APL-$set in Y. ■

Corollary 2. *Let $F : X \rightsquigarrow Y$ be a strongly γ-open, strongly γ-closed, u.a.γ-c. multifunction of a X into a $P-$space Y such that $F(x)$ is a $PL-$set in Y for each $x \in X$ and $F^{-}(y)$ is γ-Lindelof in X for each $y \in Y$. If X is γ-paralindelöf and F is surjective, then Y is almost paralindelöf.*

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Cumhuriyet University, Faculty of Science
Department of Mathematics
58140 Sivas, TURKEY
Received 13.07.2007
E-mail: izorlu@cumhuriyet.edu.tr