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# On the Solutions of Higher Order Rational System of Recursive Sequences

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In this paper we deal with the solutions of the systems of the difference equations

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}},$$

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \ z_{n+1} = \frac{x_{n-1}y_{n-1}}{x_{n-2}y_{n-2}z_{n-2}},$$

and

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \ z_{n+1} = \frac{x_{n-1}y_{n-1}}{x_{n-2}y_{n-2}z_{n-2}}, \ t_{n+1} = \frac{x_{n-2}y_{n-2}z_{n-2}}{x_{n-3}y_{n-3}z_{n-3}t_{n-3}},$$

with a nonzero real numbers initial conditions. Also, the periodicity of the general k variable will be considered.

*Key Words*: difference equations, periodic solutions

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## 1. Introduction

Our aim in this paper is to investigate the periodic nature of solutions of the following systems of rational difference equations

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}},$$

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \ z_{n+1} = \frac{x_{n-1}y_{n-1}}{x_{n-2}y_{n-2}z_{n-2}},$$

and

$$x_{n+1} = \frac{1}{y_n},$$
$$y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}},$$
$$z_{n+1} = \frac{x_{n-1}y_{n-1}}{x_{n-2}y_{n-2}z_{n-2}},$$
$$t_{n+1} = \frac{x_{n-2}y_{n-2}z_{n-2}}{x_{n-3}y_{n-3}z_{n-3}t_{n-3}},$$

with a nonzero real numbers initial conditions. Also, the periodicity of the general k variable will be considered.

The periodicity of the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{m}{y_n}, \quad y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}},$$

was studied by Cinar in [3].

Also, Cinar [4] has obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{1}{z_n}, \; y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \; z_{n+1} = \frac{1}{x_{n-1}}.$$

Elabbasy et al. [6] has obtained the solution of particular cases of the following general system of difference equations

$$x_{n+1} = \frac{a_1 + a_2 y_n}{a_3 z_n + a_4 x_{n-1} z_n},$$
$$y_{n+1} = \frac{b_1 z_{n-1} + b_2 z_n}{b_3 x_n y_n + b_4 x_n y_{n-1}},$$
$$z_{n+1} = \frac{c_1 z_{n-1} + c_2 z_n}{c_3 x_{n-1} y_{n-1} + c_4 x_{n-1} y_n + c_5 x_n y_n}.$$

Özban [8] has investigated the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{1}{y_{n-k}}, \; y_{n+1} = \frac{y_n}{x_{n-m} y_{n-m-k}}.$$

Özban [9] has investigated the solutions of the following system

$$x_{n+1} = \frac{a}{y_{n-3}}, \; y_{n+1} = \frac{b y_{n-3}}{x_{n-q} y_{n-q}}.$$

Yang et al.[11] has investigated the positive solutions of the systems

$$x_n = \frac{a}{y_{n-p}}, \quad y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}.$$

Similar nonlinear systems of rational difference equations were investigated see [1]-[11].

**Definition (Periodicity)** A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period $p$ if $x_{n+p} = x_n$ for all $n \geq -k$.

## 2. Main Results

### 2.1. First System

In this section, we study the periodicity of the solutions of the system of two difference equations

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \tag{1}$$

with a nonzero real numbers initial conditions.

**Theorem 1.** *Suppose that $\{x_n, y_n\}$ are solutions of system (1). Also, assume that $x_{-1}$, $x_0$, $y_{-1}$, and $y_0$ are arbitrary nonzero real numbers. Then all solutions of equation system (1) are periodic with period four.*

Proof. From Eq.(1) we have

$$\begin{aligned}
x_{n+1} &= \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \\
x_{n+2} &= \frac{1}{y_{n+1}} = \frac{x_{n-1}y_{n-1}}{y_n}, \quad y_{n+2} = \frac{y_{n+1}}{x_n y_n} = \frac{1}{x_{n-1}y_{n-1}x_n}, \\
x_{n+3} &= \frac{1}{y_{n+2}} = x_{n-1}y_{n-1}x_n, \quad y_{n+3} = \frac{y_{n+2}}{x_{n+1}y_{n+1}} = \frac{1}{x_n}, \\
x_{n+4} &= \frac{1}{y_{n+3}} = x_n, \quad y_{n+4} = \frac{y_{n+3}}{x_{n+2}y_{n+2}} = y_n.
\end{aligned}$$

The proof is complete. ∎

### 2.2. Second System

In this section we deal with the solutions of the system of the difference equations

$$x_{n+1}=\frac{1}{y_n},\quad y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}},\quad z_{n+1}=\frac{x_{n-1}y_{n-1}}{x_{n-2}y_{n-2}z_{n-2}},\tag{2}$$

with a nonzero real numbers initial conditions.

**Theorem 2.** *Suppose that $\{x_n,y_n,z_n\}$ are solutions of system (2). Also, assume that $x_{-2}$, $x_{-1}$, $x_0$, $y_{-2}$, $y_{-1}$, $y_0$, $z_{-2}$, $z_{-1}$, and $z_0$ are arbitrary nonzero real numbers. Then $\{x_n,y_n\}$ are also periodic with period four and $\{z_n\}$ is periodic with period twelve.*

Proof. It is easy to see that $\{x_n,y_n\}$ are periodic with period four. So, to prove theorem we prove that $\{z_n\}$ is periodic with period twelve. From the given equations we see that

$$\begin{aligned}
z_{n+1} &= \frac{x_{n-1}y_{n-1}}{x_{n-2}y_{n-2}z_{n-2}},\\
z_{n+2} &= \frac{x_ny_n}{x_{n-1}y_{n-1}z_{n-1}},\\
z_{n+3} &= \frac{x_{n+1}y_{n+1}}{x_ny_nz_n}=\frac{\frac{1}{y_n}\frac{y_n}{x_{n-1}y_{n-1}}}{x_ny_nz_n}=\frac{1}{x_{n-1}y_{n-1}x_ny_nz_n},
\end{aligned}$$

$$z_{n+4}=\frac{x_{n+2}y_{n+2}}{x_{n+1}y_{n+1}z_{n+1}}=$$

$$=\frac{x_{n-1}y_{n-1}}{y_nx_{n-1}y_{n-1}x_n}\frac{1}{y_n\frac{y_n}{x_{n-1}y_{n-1}}\frac{x_{n-1}y_{n-1}}{x_{n-2}y_{n-2}z_{n-2}}}=\frac{x_{n-2}y_{n-2}z_{n-2}}{y_nx_n},$$

$$z_{n+5}=\frac{x_{n+3}y_{n+3}}{x_{n+2}y_{n+2}z_{n+2}}=$$

$$=\frac{x_{n-1}y_{n-1}x_n}{x_n\frac{x_{n-1}y_{n-1}}{y_n}\frac{1}{x_{n-1}y_{n-1}x_n}\frac{x_ny_n}{x_{n-1}y_{n-1}z_{n-1}}}=x_{n-1}^2y_{n-1}^2z_{n-1},$$

$$z_{n+6}=\frac{x_{n+4}y_{n+4}}{x_{n+3}y_{n+3}z_{n+3}}=\frac{x_ny_n}{x_{n-1}y_{n-1}x_n\frac{1}{x_n}\frac{1}{x_{n-1}y_{n-1}x_ny_nz_n}}=x_n^2y_n^2z_n,$$

$$z_{n+7} = \frac{x_{n+5}y_{n+5}}{x_{n+4}y_{n+4}z_{n+4}} =$$

$$= \frac{y_n,}{y_n x_{n-1}y_{n-1}x_n y_n \dfrac{x_{n-2}y_{n-2}z_{n-2}}{y_n x_n}} = \frac{1}{x_{n-1}y_{n-1}x_{n-2}y_{n-2}z_{n-2}},$$

$$z_{n+8} = \frac{x_{n+6}y_{n+6}}{x_{n+5}y_{n+5}z_{n+5}} =$$

$$= \frac{x_{n-1}y_{n-1}}{y_n x_{n-1}y_{n-1}x_n \dfrac{1}{y_n}\dfrac{y_n}{x_{n-1}y_{n-1}} x_{n-1}^2 y_{n-1}^2 z_{n-1}} = \frac{1}{x_n y_n x_{n-1}y_{n-1}z_{n-1}},$$

$$z_{n+9} = \frac{x_{n+7}y_{n+7}}{x_{n+6}y_{n+6}z_{n+6}} = \frac{x_{n-1}y_{n-1}x_n}{x_n \dfrac{x_{n-1}y_{n-1}}{y_n}\dfrac{1}{x_{n-1}y_{n-1}x_n} x_n^2 y_n^2 z_n} = \frac{x_{n-1}y_{n-1}}{x_n y_n z_n},$$

$$z_{n+10} = \frac{x_{n+8}y_{n+8}}{x_{n+7}y_{n+7}z_{n+7}} =$$

$$= \frac{x_n y_n}{x_{n-1}y_{n-1}x_n \dfrac{1}{x_n}\dfrac{1}{x_{n-1}y_{n-1}x_{n-2}y_{n-2}z_{n-2}}} = x_n y_n x_{n-2}y_{n-2}z_{n-2},$$

$$\begin{aligned}
z_{n+11} &= \frac{x_{n+9}y_{n+9}}{x_{n+8}y_{n+8}z_{n+8}} = \frac{y_n}{y_n x_{n-1}y_{n-1}x_n y_n \dfrac{1}{x_n y_n x_{n-1}y_{n-1}z_{n-1}}} = z_{n-1}, \\
z_{n+12} &= \frac{x_{n+10}y_{n+10}}{x_{n+9}y_{n+9}z_{n+9}} = \frac{x_{n-1}y_{n-1}}{y_n x_{n-1}y_{n-1}x_n \dfrac{1}{y_n}\dfrac{y_n}{x_{n-1}y_{n-1}}\dfrac{x_{n-1}y_{n-1}}{x_n y_n z_n}} = z_n.
\end{aligned}$$

The proof is complete. ■

### 2.3. Third System

In this section, we obtain the solutions of the system of the difference equations

$$x_{n+1}=\frac{1}{y_n},\ \ y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}},$$

$$z_{n+1}=\frac{x_{n-1}y_{n-1}}{x_{n-2}y_{n-2}z_{n-2}},\ t_{n+1}=\frac{x_{n-2}y_{n-2}z_{n-2}}{x_{n-3}y_{n-3}z_{n-3}t_{n-3}}, \tag{3}$$

with a nonzero real numbers initial conditions.

**Theorem 3.** *Suppose that $\{x_n, y_n, z_n\}$ are solutions of system (3). Also, assume that $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0,\ z_{-3},\ z_{-2},\ z_{-1},\ z_0,$ $t_{-3},\ t_{-2},\ t_{-1}$ and $t_0$ are arbitrary nonzero. Then $\{x_n, y_n\}$ are also periodic with period four, $\{z_n\}$ is periodic with period twelve, and $\{t_n\}$ is periodic with period 24.*

Proof. As the proof of Theorem 2. ■

Numerical Example:-

We consider the difference equation system (4) with the initial conditions $x_{-3}=7,\ x_{-2}=-5,\ x_{-1}=1.1,\ x_0=3,\ y_{-3}=1.5,\ y_{-2}=13,\ y_{-1}=-8,\ y_0=6,\ z_{-3}=-0.3,\ z_{-2}=0.011,\ z_{-1}=.07,\ z_0=4,\ t_{-3}=9,\ t_{-2}=-14,\ t_{-1}=-17$ *and* $t_0=2.3$. (See Fig. 1-4).

### 2.4.General System

In this section, we investigate the solutions of the system of the difference equations

$$x^1_{n+1}=\frac{1}{x^2_n},\ x^2_{n+1}=\frac{x^2_n}{x^1_{n-1}x^2_{n-1}},\ x^3_{n+1}=\frac{x^1_{n-1}x^2_{n-1}}{x^1_{n-2}x^2_{n-2}x^3_{n-2}}, \tag{4}$$

$$x^4_{n+1}=\frac{x^1_{n-2}x^2_{n-2}x^3_{n-2}}{x^1_{n-3}x^2_{n-3}x^3_{n-3}x^4_{n-3}},\dots,x^j_{n+1}=\frac{\prod_{i=1}^{j-1}x^i_{n-(j-2)}}{\prod_{i=1}^{j}x^i_{n-(j-1)}},\dots,x^k_{n+1}=\frac{\prod_{i=1}^{k-1}x^i_{n-(k-2)}}{\prod_{i=1}^{k}x^i_{n-(k-1)}},$$

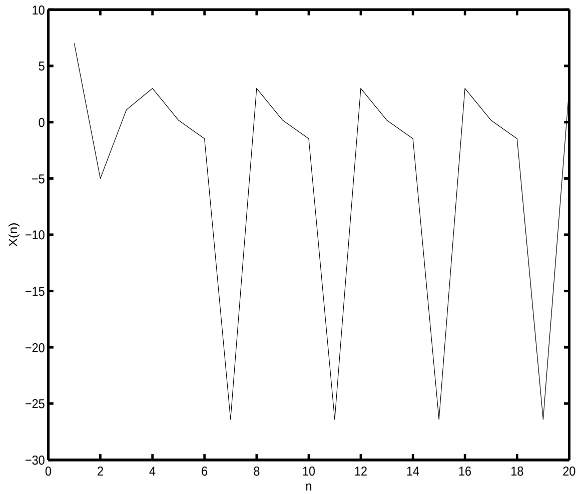


Figure 1:

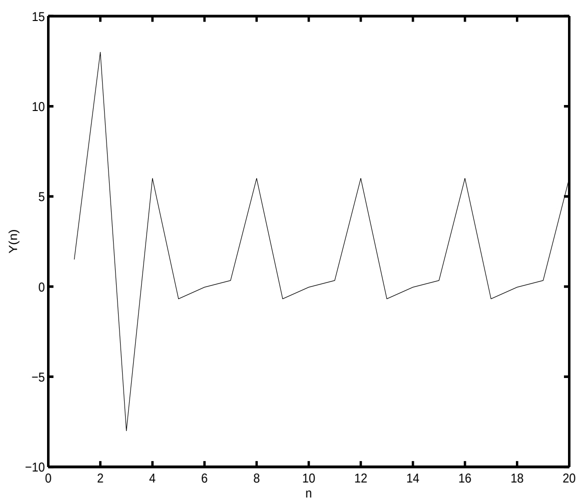


Figure 2:

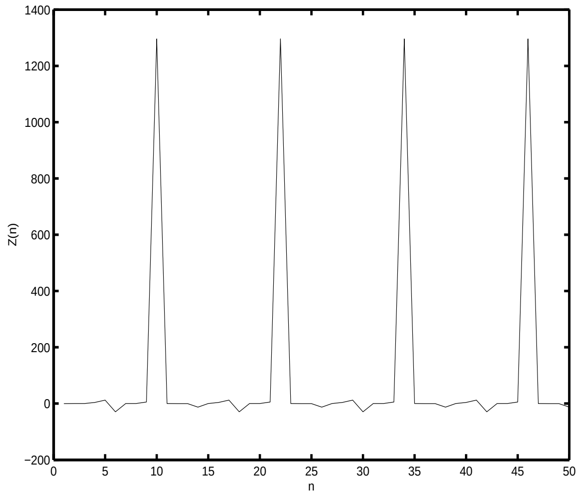


Figure 3:

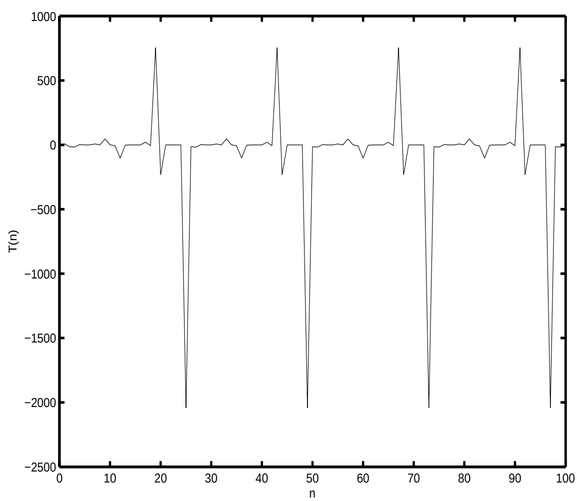


Figure 4:

with a nonzero real numbers initial conditions.

As the above calculations it is easy to prove the following theorem.

**Theorem 4.** *Suppose that* $\{x_n^1, x_n^2, x_n^3, ..., x_n^k\}$ *are solutions of the system (4). Then* $\{x_n^1, x_n^2\}$ *are periodic with period four and* $\{x_n^j\}$ *is periodic with period* $(P(x_n^{j-1}) + 4(j-1))$, *where* $P(x_n^j)$ *the period of* $(x_n^{j-1})$, $j = 3, 4, 5, ...$ .

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