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Isometric Immersion of Three-Dimensional Quasi-Sasakian Manifolds

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Presented by G. Ganchev

In this paper we study a three-dimensional quasi-Sasakian manifold which is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1.

Key Words: isometric immersion, quasi-Sasakian manifold,Riemannian manifold of constant curvature 1, minimal immersion, Sasakian manifold, para-Sasakian manifold

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1. Introduction

T. Takahashi and S.Tano[9] introduced the notion of isometric immersion on K-contact manifolds. D. E. Blair, T. Koufogiorgos[4] studied isometric immersion for three dimensional contact manifolds satisfying $\phi Q = Q\phi$. In this paper we like to study isometric immersion for a quasi-Sasakian manifold of dimension three. On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Z. Olszak[7] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat[8]. Next he has proved that if the manifold is additionally conformally flat with $\beta = constant$, then (a) the manifold is locally a product of R and a 2-dimensional Kählerian space of constant Gauss curvature (the cosymplectic case), or, (b) the manifold is of constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure).

The object of the present paper is to study a three-dimensional quasi-Sasakian manifold which is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1. The present paper is organized as follows:

Section-1 is the introductory section. In section-2 we recall some preliminary results. Section-3 deals with the notion of three-dimensional quasi-Sasakian manifolds. In section-4 we derive some results of three-dimensional quasi-Sasakian manifolds isometrically immersed in four-dimensional Riemannian manifold of constant curvature 1. In this section we also derive a necessary and sufficient condition for the immersion to be minimal. We also prove that if a three-dimensional quasi-Sasakian manifold is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1 then the manifold is either Sasakian or para-Sasakian. Section-5 is devoted for an example which illustrates some results obtained in Section-4.

2. Preliminaries

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact structure (ϕ, ξ, η, g), where ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that [1], [2]

$$\phi^2(X) = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad X, Y \in T(M). \tag{2.2}$$

Then also

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(X) = g(X, \xi). \tag{2.3}$$

Let Φ be the fundamental 2-form defined by

$$\Phi(X, Y) = g(X, \phi Y), \qquad X, Y \in T(M).$$

M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η, g) is normal and the fundamental 2-form Φ is closed ($d\Phi = 0$), which was first introduced by Blair[3]. The normality condition gives that the induced almost contact structure $M \times R$ is integrable or equivalently, the torsion tensor field $N[\phi, \phi] + 2\xi \otimes d\eta$ vanishes identically on M. The rank of a quasi Sasakian structure is always odd[3], it is equal to 1 if the structure is cosymplectic and it is equal to $(2n+1)$ if the structure is Sasakian.

3. Quasi-Sasakian structure of dimension three

An almost contact metric manifold M of dimension three is quasi-Sasakian if and only if [7]

$$\nabla_X \xi = -\beta\phi X, \qquad X \in T(M), \tag{3.1}$$

for a certain function β on M such that $\xi\beta = 0$, ∇ being the operator of the covariant differention with respect to the Levi-Civita connection on M. Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. As a consequence of (3.1), we have[7]

$$(\nabla_X\phi)(Y) = \beta(g(X,Y)\xi - \eta(Y)X), \qquad X,Y \in T(M). \tag{3.2}$$

In a three-dimensional Riemannian manifold, we always have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y - \tfrac{r}{2}(g(Y,Z)X - g(X,Z)Y), \end{aligned} \tag{3.3}$$

where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. Let M be a three-dimensional quasi-Sasakian manifold. The Ricci tensor S of M is given by [8]

$$\begin{aligned} S(Y,Z) &= (\tfrac{r}{2} - \beta^2)g(Y,Z) + (3\beta^2 - \tfrac{r}{2})\eta(Y)\eta(Z) \\ &\quad -\eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y), \end{aligned} \tag{3.4}$$

where r is the scalar curvature of M. As a consequence of (3.4), we get for the Ricci operator Q

$$QX = (\frac{r}{2} - \beta^2)X + (3\beta^2 - \frac{r}{2})\eta(X)\xi + \eta(X)(\phi grad\beta) - d\beta(\phi X)\xi, \tag{3.5}$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X) = g(gradf, X)$. From (3.4) it follows that

$$S(X,\xi) = 2\beta^2\eta(X) - d\beta(\phi X). \tag{3.6}$$

Moreover as a consequence of (3.3) – (3.5), we note that for a three-dimensional quasi-Sasakian manifold

$$\begin{aligned} R(X,Y)\xi &= \beta^2(\eta(Y)X - \eta(X)Y) + d\beta(\phi Y)\eta(X)\xi - d\beta(\phi X)\eta(Y)\xi \\ &\quad + d\beta(\phi X)Y - d\beta(\phi Y)X, \end{aligned} \tag{3.7}$$

for $X, Y \in T(M)$.

4. Isometric immersion of three-dimensional quasi-Sasakian manifolds

Definition 4.1. Let M and M' be smooth manifolds of dimension m and m' respectively. If $f : M \to M'$ is a smooth map and $f_{*x} : T_x(M) \to T_{f(x)}M'$ is the tangential map at $x \in M$ then f is said to be an immersion if f_{*x} is injective for each $x \in M$.

Let M and M' be Riemannian manifolds with Riemannian metric g and g' respectively. A mapping $f : M \to M'$ is called isometric at a point x of M if $g(X, Y) = g'(f_*X, f_*Y)$, for all $X, Y \in T_xM$.

An immersion f which is isometric at every point of M is called an isometric immersion[10].

If X and Y are two vector fields on a manifold M which is immersed in a Riemannian manifold M' then we know that [10] $B(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where B is the second fundamental form and $\tilde{\nabla}$ and ∇ denote the covariant differentiation with respect to the Levi-Civita connection in M and M' respectively.

We consider a three-dimensional quasi-Sasakian manifold which is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1. Then we can write the Gauss and Codazzi equations as [5]

$$R(X, Y) = X \wedge Y + AX \wedge AY, \tag{4.1}$$

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(AY, Z)AX - g(AX, Z)AY, \tag{4.2}$$

$$(\nabla_X A)(Y) = (\nabla_Y A)(X), \tag{4.3}$$

where A is a $(1, 1)$ tensor field associated with second fundamental form B given by $B(X, Y) = g(AX, Y)$. A is symmetric with respect to g. If the trace of A vanishes then the imersion is called minimal. The type number of the immersion is equal to the rank of A. From (4.2) it follows that

$$\begin{aligned} g(R(X, Y)Z, U) &= g(Y, Z)g(X, U) - g(X, Z)g(Y, U) \\ &+ g(AY, Z)g(AX, U) - g(AX, Z)g(AY, U). \end{aligned}$$

In the above equation putting $X = U = e_i$, where $\{e_i\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space at each point of the manifold M and taking summation over i we get

$$S(Y, Z) = 2g(Y, Z) + g(AY, Z)\theta - g(AAY, Z), \tag{4.4}$$

where θ is the trace of A. Replacing Z by ξ we have from (4.4)

$$S(Y,\xi) = 2g(Y,\xi) + g(AY,\xi)\theta - g(AAY,\xi).$$

Considering (3.6) we note from above

$$2\beta^2\eta(Y) - d\beta(\phi Y) = 2g(Y,\xi) + g(AY,\xi)\theta - g(AAY,\xi).$$

For $g(gradf, X) = df(X)$, symmetry of A and skew-symmetry of ϕ, the above equation implies

$$2\beta^2 g(Y,\xi) + g(Y,\phi grad\beta) = 2g(Y,\xi) + g(Y,A\xi)\theta - g(Y,AA\xi), \tag{4.5}$$

which yields

$$2(\beta^2-1)\xi + \phi grad\beta = \theta A\xi - AA\xi. \tag{4.6}$$

If $\theta = 0$ the above equation reduces to

$$2(\beta^2-1)\xi + \phi grad\beta + AA\xi = 0. \tag{4.7}$$

Thus we can state the following:

Theorem 4.1. *If a three-dimensional quasi-Sasakian manifold is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1 and if the immersion is minimal then* (4.7) *holds.*

We now suppose that the relation (4.7) holds. Then in view of (4.6), $\theta A\xi = 0$. Therefore either $\theta = 0$ or $A\xi = 0$. If $A\xi = 0$, then from (4.6) we get

$$2(\beta^2-1)\xi = -\phi grad\beta. \tag{4.8}$$

Applying ϕ on both sides of the above relation we obtain $\phi^2 grad\beta = 0$. Hence by (2.1)

$$-grad\beta + g(grad\beta,\xi)\xi = 0.$$

Since $g(grad\beta, X) = d\beta(X) = X\beta$, we obtain from above $-grad\beta + \xi\beta\xi = 0$. Now for a three-dimensional quasi-Sasakian manifold we know that $\xi\beta = 0$. Therefore $grad\beta = 0$. Thus from (4.8) we obtain $2(\beta^2-1) = 0$. Hence $\beta = \pm 1$. Thus we have the following:

Theorem 4.2. *If a three-dimensional quasi-Sasakian manifold is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1 and if* (4.7) *holds then the manifold is either Sasakian or para-Sasakian or the immersion is minimal.*

By virtue of (2.3) we obtain from (3.4)

$$S(\phi Y, \phi Z) = (\frac{r}{2} - \beta^2) g(\phi Y, \phi Z). \tag{4.9}$$

From (4.4) we also have

$$S(\phi Y, \phi Z) = 2g(\phi Y, \phi Z) + g(A\phi Y, \phi Z)\theta - g(AA\phi Y, \phi Z). \tag{4.10}$$

From (4.9), and (4.10) we obtain

$$(\frac{r}{2} - \beta^2 - 2) g(\phi Y, \phi Z) + \theta g(\phi A \phi Y, Z) - g(\phi AA\phi Y, Z) = 0.$$

From above it follows that

$$(\frac{r}{2} - \beta^2 - 2) g(\phi^2 Y, Z) - \theta g(\phi A \phi Y, Z) + g(\phi AA\phi Y, Z) = 0.$$

We obtain from above

$$(\frac{r}{2} - \beta^2 - 2)\phi^2 - \theta\phi A\phi + \phi AA\phi = 0. \tag{4.11}$$

If $\theta = 0$, then (4.11) reduces to

$$(\frac{r}{2} - \beta^2 - 2)\phi^2 + \phi AA\phi = 0. \tag{4.12}$$

Thus we can state the following:

Theorem 4.3. *If a three-dimensional quasi-Sasakian manifold is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1 and if the immersion is minimal then* (4.12) *holds.*

Next let (4.12) holds. Then from (4.11) we note that $\theta\phi A\phi = 0$. Hence either $\theta = 0$, i.e., the immersion is minimal or $\phi A\phi = 0$. Hence we can state the following:

Theorem 4.4. *If a three-dimensional quasi-Sasakian manifold is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1 and if* (4.12) *holds then either the immersion is minimal or* $\phi A\phi = 0$.

Combining Theorem 4.3 and Theorem 4.4 we get a necessary and sufficient condition for the immersion to be minimal as the following:

Theorem 4.5. *If a three-dimensional quasi-Sasakian manifold is isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1, then the immersion is minimal if and only if* (4.12) *holds, provided that* $\phi A\phi \neq 0$.

From (4.2) we have

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY.$$

For $Z = \xi$, using (3.7) we obtain from above

$$\begin{aligned} &\beta^2(\eta(Y)X - \eta(X)Y) + d\beta(\phi Y)\eta(X)\xi \\ &-d\beta(\phi X)\eta(Y)\xi + d\beta(\phi X)Y - d\beta(\phi Y)X \\ &= \eta(Y)X - \eta(X)Y + \eta(AY)AX - \eta(AX)AY. \end{aligned} \tag{4.13}$$

Putting $Y = \xi$ we obtain from (4.13)

$$(1-\beta^2)(X - \eta(X)\xi) + \eta(A\xi)AX - \eta(AX)A\xi = 0. \tag{4.14}$$

Now $g(AX,Y) = B(X,Y)$ and we know that $B(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$. Hence

$$g(A\xi,\xi) = B(\xi,\xi) = \tilde{\nabla}_\xi \xi - \nabla_\xi \xi, \tag{4.15}$$

and

$$g(AX,\xi) = B(X,\xi) = \tilde{\nabla}_X \xi - \nabla_X \xi. \tag{4.16}$$

Using(4.15), (4.16) in (4.14) we obtain

$$(1-\beta^2)(X - \eta(X)\xi) + (\tilde{\nabla}_\xi \xi - \nabla_\xi \xi)AX - (\tilde{\nabla}_X \xi - \nabla_X \xi)A\xi = 0. \tag{4.17}$$

From [6] we know that $2\tilde{\nabla}_X X = gradf$, where $f = g(X,X)$ is a smooth function on a Riemannian manifold endowed with a metric g. Then for $X = \xi$ and

$g(\xi,\xi) = 1$, we get $\tilde{\nabla}_\xi \xi = 0$, since $grad 1 = 0$. Also from (3.1) it follows that $\nabla_\xi \xi = 0$. Hence applying ϕ on both sides of (4.17) we obtain

$$(1-\beta^2)(\phi X) = 0.$$

Since $\phi X \neq 0$, unless $X = \xi$, we have $\beta = \pm 1$. Hence we can state the following:

Theorem 4.6. *If a three dimensional quasi-Sasakian manifold is isommetrically immersed in a four-dimensional Riemannian manifold of constant curvature 1, then the manifold is either Sasakian or para-Sasakian.*

5. Example

In this section we give an example which illustrates the results obtained in Theorem 4.1 and Theorem 4.2.

Let us consider the 3-dimensional manifold $M = \{(x, y, z \in R^3)\}$, where (x, y, z) are the standerd coordinates in R^3. The vector fields

$$e_1 = \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial z}, \quad e_3 = 2\frac{\partial}{\partial x} - y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y},$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any Z belongs to $\chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.

Let ∇ be the Levi-civita connection with respect to the Riemannian metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1.$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X)$$
$$-Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y),$$

which is known as Koszul's formula. Taking $e_3 = \xi$ and using the above formula for Riemannian metric g, it can be easily calculated that

$$\begin{array}{lll} \nabla_{e_1} e_3 = -e_2, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = 0, \\ \nabla_{e_2} e_3 = e_1, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = -e_1, & \nabla e_3 e_1 = e_2. \end{array}$$

We see that the (ϕ, ξ, η, g) structure satisfies the formula $\nabla_X \xi = -\beta\phi X$, where $\beta = -1$.

Hence $M(\phi, \xi, \eta, g)$ is a 3-dimensional non-cosymplectic quasi-Sasakian manifold with the structure function β as constant. If the manifold be isometrically immersed in a four-dimensional Riemannian manifold of constant curvature 1 then by (4.6) we have

$$2(\beta^2 - 1)\xi + \phi grad\beta = \theta A\xi - AA\xi.$$

Now for $\beta = -1$, we get from above $AA\xi = \theta A\xi$. Thus in the manifold under consideration $\beta = -1$ and $AA\xi = \theta A\xi$. Hence for $\theta = 0$, (4.7) is satisfied. Again, since $\beta = -1$ the manifold is para-Sasakian. In this way the manifold agrees with Theorem 4.1 and Theorem 4.2.

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