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Besov, Sobolev and Potentials Type Spaces on Chébli-Trimèche Hypergroups

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Dedicated to Professor J. Rodríguez on the occasion of his 60th birthday

R. S. Pathak and P. K. Pandey (1995, 1997) analyzed pseudo-differential operators and Sobolev Type spaces associated with the Bessel operators. Later on, N. Ben Salem and A. Dachraoui (1998, 2000) obtained similar results for the Jacobi differential operators. A. Dachraoui and K. Trimèche (1999) studied pseudo-differential operators associated with a singular differential operator, and they generalized the results obtained by R.S. Pathak and P. K. Pandey (1995) and N. Ben Salem and A. Dachraoui (1998). The work begun by R.S. Pathak and P. K. Pandey(1997) is continued by D.I. Cruz-Báez and J. Rodríguez (2000, 2001) being obtained new results on potentials, Sobolev type spaces, Besov and Triebel-Lizorkin type spaces associated with the Bessel operators. In this paper, we generalize and continue the previous papers. We prove a Calderón's type theorem, define Besov spaces $B^s_{p,q,m}$, $b^s_{p,q,m}$ on Chébli-Trimèche hypergroups and give a characterization of $B^s_{p,q,m}$ in terms of $b^s_{p,q,m}$. Moreover we give an embedding theorem between $B^s_{p,q,m}$ and Triebel-Lizorkin type spaces $F^s_{p,q,m}$, establish a lifting property and finally, give some applications of these results.

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1. Introduction

It is well-known that a hypergroup is a locally compact space with a certain generalized convolution structure on a finite regular Borel measures. Many examples of hypergroup structures on the half line $(0, \infty)$ arise from Sturm-Liouville boundary value problems where the solutions coincide with the characters of the hypergroup in question. Chébli-Trimèche hypergroups are a example of those (see [3]-[4] and references for more details).

Chébli-Trimèche hypergroups are a class of one-dimensional hypergroups on $(0,\infty)$ with the convolution structure related to the differential operator $-\Delta$, where

$$\Delta = \frac{1}{A(x)}\frac{d}{dx}\left[A(x)\frac{d}{dx}\right], \tag{1.1}$$

A being a continuous function on $(0,\infty)$, twice continuously differentiable on $(0,\infty)$, and satisfying the following conditions:

1. $A(0)=0$ and $A(x)>0$ for $x>0$;
2. A is increasing and unbounded;
3. $\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x)$ on a neighbourhood of 0, being $\alpha > -\frac{1}{2}$ and B is an odd C^∞-function on $\mathbb{R}$;
4. $\frac{A'(x)}{A(x)}$ is a decreasing C^∞-function on $(0,\infty)$ and $\frac{1}{2}\lim_{x\to\infty}\frac{A'(x)}{A(x)} = \rho \geq 0$.

A function that verifies these conditions is called a Chébli-Trimèche function.

A hypergroup $((0,\infty),*)$ is called a Chébli-Trimèche hypergroup, when there exists a Chébli-Trimèche function A such that for any real-valued function f on $(0,\infty)$ that is the restriction of an even non-negative C^∞-function on $\mathbb{R}$, the generalized translation $u(x,y)=\tau_x f(y)$ is the solution of the following Cauchy problem:

$$\begin{cases} (\Delta_x - \Delta_y)\, u(x,y) = 0, \\ u(x,0) = f(x),\ u_y(x,0) = 0,\ x>0. \end{cases}$$

and it comes given by

$$\tau_x f(y) = \int_0^\infty f(z)\delta_x * \delta_y(dz),\ x,y\in\mathbb{R}^+,$$

where δ_x is the unit point mass at $x\in\mathbb{R}^+$, and the convolution $*$ of two functions f and g comes defined by

$$f*g(x) = \int_0^\infty \tau_x f(y)g(y)A(y)dy.$$

Note that in the case of that $A(x)=x^{2\alpha+1}, \alpha>-\frac{1}{2}$, $x\in(0,\infty)$, we have the Bessel-Kingman hypergroup and if

$$A(x) = 2^{2(\alpha+\beta+1)}(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1},$$

$\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$, then appears the Jacobi hypergroup.

We denote $((0,\infty), *(A))$ to the Chébli-Trimèche hypergroup associated with A. Then, the multiplicative functions on $((0,\infty), *(A))$ coincide with the solutions φ_λ, $(\lambda \in \mathbb{C})$, of the following equation

$$-\Delta\varphi_\lambda(x) = (\lambda^2 + \rho^2)\ \varphi_\lambda(x),\ \ \varphi_\lambda(0) = 1 \ \text{ and } \ \varphi_\lambda^{'}(0) = 0.$$

Now, we introduce some functions spaces [3].

We denote by $\mathcal{S}_*(\mathbb{R})$ to the space of even $\mathcal{C}^\infty$ functions on $\mathbb{R}$, rapidly decreasing together with their derivatives, equipped with the usual Schwartz topology.

For $0 < r \leq 2$, let $\mathcal{S}_*^r(\mathbb{R})$ be the generalized Schwartz space defined by

$$\mathcal{S}_*^r(\mathbb{R}) = \left\{ \begin{array}{c} g \in \mathcal{C}^\infty(\mathbb{R}),\ g \text{ is even and, } \forall k,l \in \mathbb{N}, \\ n_{k,l}(g) = \sup\limits_{x\in\mathbb{R}}(1+x)^l \left|\varphi_0^{-\frac{2}{r}}(x) g^{(k)}(x)\right| < \infty \end{array} \right\}.$$

The space $\mathcal{S}_*^r(\mathbb{R})$ is topologized by means of the seminorms $n_{k,l}(g)$. The space $\mathcal{S}_*^r(\mathbb{R})$ is invariant under Δ.

A r-distribution on $(0,\infty)$ is a continuous linear functional on S_*^r and the space of r-distributions is denoted by $\left(\mathcal{S}_*^r(\mathbb{R})\right)^{'}$.

By $H_*^r, 0 < r \leq 2$, we denote the extended Schwartz space defined by all functions h that are even and holomorphic in the interior of $D_r = \left\{z \in \mathbb{C} : |\mathrm{Im}\, z| \leq \left(\frac{2}{r} - 1\right)\rho\right\}$ and such that h together with all derivatives extend continuously to D_r and satisfy

$$p_{n,m}(h) = \sup_{\lambda \in D_r}(1+|\lambda|)^n \left|h^{(m)}(\lambda)\right| < \infty.$$

Note that if $\rho = 0$ then $H_*^r = \mathcal{S}_*^r(\mathbb{R}) = \mathcal{S}_*(\mathbb{R})$, for all $0 < r \leq 2$.

On the other hand, we denote by $dm(t) = A(t)dt$ and

$$d\nu(t) = (2\pi)^{-1}\left|c(\lambda)\right|^{-2} d\lambda,$$

where $c(\lambda)$ is given in [8, p.528].

For $p \geq 1$, the Lebesgue space $L^p(m)$ is defined as usually,

$$\|f\|_{p,m} = \left(\int_0^\infty |f(t)|^p \, dm(t) \right)^{1/p} < \infty,$$

and by $L^p(d\nu)$, $p \geq 1$, we denote the space of measurable functions h on $[0, \infty)$, such that

$$\|f\|_{p,\nu} = \left(\int_0^\infty |f(\lambda)|^p \, d\nu(\lambda) \right)^{1/p} < \infty.$$

The generalized Fourier transform associated with $((0,\infty), *(A))$ is given by

$$\mathcal{F}(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)dm(x),$$

for $\forall \lambda \in \mathbb{C}$. Moreover, the inverse of the generalized Fourier transform $\mathcal{F}$ is defined by

$$\mathcal{F}^{-1}(h)(x) = \int_{-\infty}^\infty h(\lambda)\varphi_\lambda(x)d\nu(\lambda), \forall x \geq 0.$$

Note, that for all $f \in S_*^r$, $0 < r \leq 2$, we have [8, (1.26)]

$$\mathcal{F}\left(\Delta f\right)(\lambda) = -(\lambda^2 + \rho^2)\mathcal{F}(f)(\lambda) \tag{1.2}$$

In this paper, our objective is to generalize and to continue the work of the papers [1]-[2], [6]-[10]. Among other results, we prove an analogue of Calderón's theorem for the generalized Fourier transform, we define Besov spaces $B^s_{p,q,m}$, $b^s_{p,q,m}$ and give a characterization of $B^s_{p,q,m}$ in terms of $b^s_{p,q,m}$. Moreover, we give an embedding theorem between $B^s_{p,q,m}$ and Triebel-Lizorkin type spaces $F^s_{p,q,m}$ and some applications.

Throughout this paper C is denoting a positive constant, not necessarily the same in each occurrence.

2. Sobolev and potentials spaces

Next, motivated by the paper [8], we introduce a Sobolev type spaces $L^{k,p}_m$ and potentials spaces $W^{s,p}_m$ on a Chébli-Trimèche hypergroup and to deduce an analogue of the Calderón's theorem for the generalized Fourier transform.

Definition 2.1. Let $m \in \mathbf{N}$ and $1 \leq p < \infty$ we defined $L^{m,p}_\mu$ as

$$L^{k,p}_m = \left\{ T \in \left(\mathcal{S}_*^r(\mathbb{R})\right)' : T \in L^1_{loc} \text{ and } \Delta^j T \in L^p(m), 0 \leq j \leq k \right\},$$

endowed with the norm

$$\|T\|_{L_m^{k,p}} = \sum_{j=0}^{k} \left\|\Delta^j T\right\|_{p,m}.$$

Proposition 2.1. *$L_m^{k,p}$ is complete when $1 \leq p < \infty$.*

Proof. Let $\{f_l\}_{l=1}^{\infty}$ be a Cauchy sequence in $L_m^{k,p}$. Therefore $\left\{\Delta^j f_l\right\}_{l=1}^{\infty}$ is a Cauchy sequence in $L^p(m)$, $j = 0, \ldots, k$.

If we denote by g_j to the limit in $L^p(m)$ of $\left\{\Delta^j f_l\right\}_{l=1}^{\infty}$, we have, by the uniqueness of the limit, that for every $\phi \in \mathcal{S}_*^r(\mathbb{R})$ it follows

$$< \Delta^j g_0, \phi >=< g_j, \phi > .$$

Then $f_l \to g_0$ in $L_m^{k,p}$ as $l \to \infty$. ∎

Now, we establish in a similar way to that in [6]-[10], the definition of the generalized Bessel potential and potentials spaces.

Definition 2.2. Let $u \in (\mathcal{S}_*^r(\mathbb{R}))'$ and $s \in \mathbf{R}$. We define the generalized Bessel potential of order s, as follows

$$(J^s u)(x) = \mathcal{F}^{-1}\left(\left(1+\rho^2+\lambda^2\right)^{-s/2} \mathcal{F}\,(u)(\lambda)\right)(x).$$

Definition 2.3. Let $s \in \mathbf{R}$ and $1 \leq p < \infty$, then we define the potentials spaces as

$$W_m^{s,p} = \left\{\phi \in (\mathcal{S}_*^r(\mathbb{R}))' : J^{-s}\phi \in L^p(m)\right\}.$$

The norm in $W_m^{s,p}$ is given by

$$\begin{aligned} \|\phi\|_{s,p,m} &= \|\phi\|_{W_m^{s,p}} = \left\|J^{-s}\phi\right\|_{p,m} \\ &= C \cdot \left(\int_0^{\infty} \left|J^{-s}\phi\right|^p dm\right)^{1/p}. \end{aligned}$$

Next, we prove that the generalized Bessel potential verifies the classic semigroup properties and that it is an isometry between potentials spaces. Moreover, we prove that the potentials space is Banach and we give a density property.

Lemma 2.1. *Let $f \in (S_*^r(\mathbb{R}))'$. Then J^s verifies $J^s J^t f = J^{s+t} f$ and $J^0 f = f$.*

P r o o f. By definition $\left(J^t f\right)(x) = \mathcal{F}^{-1}\left(\left(1+\rho^2+\lambda^2\right)^{-t/2}\mathcal{F}(f)(\lambda)\right)(x)$.
Then

$$\begin{aligned}\left(J^s J^t f\right)(x) &= \mathcal{F}^{-1}\left(\left(1+\rho^2+\lambda^2\right)^{-s/2}\left(1+\rho^2+\lambda^2\right)^{-t/2}\mathcal{F}(u)(\lambda)\right)(x)\\ &= \mathcal{F}^{-1}\left(\left(1+\rho^2+\lambda^2\right)^{-(s+t)/2}\mathcal{F}(u)(\lambda)\right)(x)\\ &= \left(J^{s+t} f\right)(x).\end{aligned}$$

On the other hand, $J^0 f(x) = \mathcal{F}^{-1}\left(\mathcal{F}(f)(\lambda)\right)(x) = f(x)$. ■

Lemma 2.2. *The generalized Bessel potential J^t is an isometry of $W_m^{s,p}$ onto $W_m^{s+t,p}$ satisfying*

$$\left\|J^t\phi\right\|_{s+t,p,m} = \|\phi\|_{s,p,m}.$$

P r o o f. Let $\phi \in W_m^{s,p}$. By Definition 2.2 and Lemma 2.1 we obtain

$$\left\|J^t\phi\right\|_{s+t,p,m} = \left\|J^{-s-t}J^t\phi\right\|_{p,m} = \left\|J^{-s}\phi\right\|_{p,m} = \|\phi\|_{s,p,m}.$$

Now, let $g \in W_m^{s+t,p}$. Then $J^{-t}g \in W_m^{s,p}$ and $J^tJ^{-t}g = g$. Therefore we have that J^t is onto. ■

Lemma 2.3. *$W_m^{s,p}$ is a Banach space with respect to the norm $\|\cdot\|_{s,p,m}$.*

P r o o f. Let $\{\phi_l\}$ be a Cauchy sequence in $W_m^{s,p}$.

By the definition of $W_m^{s,p}$, the sequence $\{J^{-s}\phi_l\}$ is a Cauchy sequence in $L^p(m)$. As $L^p(m)$ is complete, it follows that there exists a function ϕ in $L^p(m)$ such that $J^{-s}\phi_l \to \phi$ in $L^p(m)$, as $l \to \infty$.

We denote by $g = J^s\phi$. Then by Lemma 2.1 we see that $J^{-s}g = \phi$. Therefore $J^{-s}g$ is in $L^p(m)$, that is, $g \in W_m^{s,p}$.

Then, $\phi_l \to g$ in $W_m^{s,p}$ as $l \to \infty$. ■

Lemma 2.4. *For $0 < r \le 2$, $s \in \mathbb{R}$ and $1 \le p < \infty$, S_*^r is dense in $W_m^{s,p}(I)$.*

P r o o f. Let $f \in W_m^{s,p}(I)$. Then $J^{-s}f \in L^p(m)$.

We denote by C_0^∞ the space of even C^∞-function with compact support.

Since C_0^∞ is dense in $L^p(m)$, there exists a sequence $\{\phi_j\} \in C_0^\infty$ such that

$$\phi_j \to J^{-s}f \quad \text{in} \quad L^p(m). \tag{2.1}$$

Next, we define $g_j = J^s\phi_j = \mathcal{F}^{-1}\left(\left(1+\rho^2+\lambda^2\right)^{-s/2}\mathcal{F}(\phi_j)(\lambda)\right)(x)$.

A straightforward calculation gives us that $\left(1+\rho^2+\lambda^2\right)^{-s/2}\mathcal{F}\ (\phi_j)(\lambda)$ $\in H_*^r$, for $0<r\leq 2$.

And by [8, Theorem 1.3] we have that

$$g_j=\mathcal{F}^{\ -1}\left(\left(1+\rho^2+\lambda^2\right)^{-s/2}\mathcal{F}\ (\phi_j)(\lambda)\right)(x)\in S_*^r.$$

Hence, by (2.1) we have

$$\begin{aligned}\|f-g_j\|_{W_m^{s,p}} &= C\cdot\left(\int_0^\infty\left|J^{-s}f-J^{-s}g_j\right|^p d\gamma\right)^{1/p}\\ &= C\cdot\left(\int_0^\infty\left|J^{-s}f-\phi_j\right|^p d\gamma\right)^{1/p}\to 0,\ \text{for } j\to\infty.\end{aligned}$$

■

Now, our purpose is obtain the analogue of the Calderón's theorem for the generalized Fourier transform. This result includes as a particular case to the one obtained in [6] for the Bessel differential operator.

For this, we need the following proposition.

Proposition 2.2. *Let $0\leq\mu\leq s/2$, $\mu\in\mathbf{N}$, $1\leq p<\infty$. Then $\Delta^\mu J^s$ is a continuous linear mapping of $L^p(m)$ into itself.*

Proof. Applying the multiplier theorem given in [4, p. 649] with

$$m(\lambda)=(-1)^\mu\lambda^{2\mu}(1+\rho^2+\lambda^2)^{-s/2},$$

the desired result is established. ■

Now, we are in conditions to demonstrate Calderón's theorem, that is exposed as follows.

Theorem 2.1. *Let $k\in\mathbf{N}$ and $1<p<\infty$. Then $f\in L_m^{k,p}$ if and only if $f\in W_m^{2k,p}$.*

Proof. Let $f\in W_m^{2k,p}$, then by definition, $f=J^{2k}g$, $g\in L^p(m)$.

Moreover, if $\alpha\leq k$, $\alpha\in\mathbf{N}$, by Proposition , $\Delta^\alpha f=\Delta^\alpha J^s g\in L^p(m)$ and

$$\|\Delta^\alpha f\|_{p,m}=\|\Delta^\alpha J^s g\|_{p,m}\leq C\,\|g\|_{p,m}=C\left\|J^{-2k}f\right\|_{p,m}=C\,\|f\|_{2k,p,m}.$$

Hence

$$\sum_{0\leq\alpha\leq k}\|\Delta^\alpha f\|_{p,m}\leq C\,\|f\|_{2k,p,m},$$

and therefore $f\in L_m^{k,p}$.

Conversely, we consider $f\in L_m^{k,p}$. Then $\Delta^\alpha f\in L_m^p$ for all $\alpha\in\ mathbfN$, $0\leq\alpha\leq k$.

We can see that $f\in W_m^{2k,p}$.

By definition we have that $J^{-2k}f = (1-\Delta)^k f$ and then taking norms we obtain

$$\|f\|_{2k,p,m} = \left\|J^{-2k}f\right\|_{p,m} = \left\|(1-\Delta)^k f\right\|_{p,m} \leq C \sum_{0\leq\alpha\leq k} \|\Delta^\alpha f\|_{p,m}\,.$$

■

3. Besov and Triebel-Lizorkin type spaces

We define Besov type spaces, Nikol'skij type spaces, and Triebel-Lizorkin type spaces in this section and we prove some relations among them.

Definition 3.1. Let $s \in \mathbb{R}$, $1 \leq p < \infty$, we define the sequence spaces l_p^s as

$$l_p^s = \left\{ \xi : \xi = (\xi_j)_{j=0}^{\infty}\ ,\ \xi_j \text{ complex}\ ,\ \|\xi_j\|_{l_p^s} = \left(\sum_{j=0}^{\infty} \left(2^{jsp}\,|\xi_j|^p\right)\right)^{1/p} < \infty \right\},$$

and for $p = \infty$ we have

$$l_\infty^s = \left\{ \xi : \xi = (\xi_j)_{j=0}^{\infty}\ ,\ \xi_j \text{ complex}\ ,\ \|\xi_j\|_{l_\infty^s} = \sup_j 2^{js}\,|\xi_j| < \infty \right\}.$$

In the case of $s = 0$ we denote l_p^0 by l_p.

Definition 3.2. Let Φ be the collection of all systems $\{\varphi_j(x)\}_{j=0}^{\infty} \subset S_*^r(\mathbb{R})$ with the following properties:

1. $\varphi_j(x) \in S_*^r(\mathbb{R})$, $\mathcal{F}\varphi_j(x) \geq 0$ for $j = 0, 1, 2, 3, \ldots$;
2. $\text{supp}\mathcal{F}\varphi_j \subset \left\{\lambda : \sqrt{2^{j-1}-\rho^2-1} \leq \lambda \leq \sqrt{2^{j+1}-\rho^2-1}\right\}$,

   $\text{supp}\mathcal{F}\varphi_0 \subset \left\{\lambda : \lambda \leq \sqrt{1-\rho^2}\right\}$;
3. There exists a positive number c_1 such that

   $$\left|D^i\mathcal{F}\varphi_j(x)\right| \leq c_1(1+|x|)^{-i},$$

   for $j = 1, 2, \ldots$; $N = [\alpha+1]+1$.
4. ${}_{j=0}^{\infty}\mathcal{F}\varphi_j(x) = 1$, $\forall x \in \mathbb{R}$.

Proceeding as in [11, pp. 171-172], we have that Φ is not empty.

Definition 3.3. Let $1 < p < \infty$, $1 \le q \le \infty$, $0 < r \le 2$ and $s \in \mathbb{R}$. Then for any system of functions $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$, the Besov type spaces are defined by

$$B^s_{p,q,m} = \left\{ f \in (S^r_*(\mathbb{R}))' : \|f\|_{B^s_{p,q,m}} = \|\varphi_j * f\|_{l^s_q(L^p(m))} < \infty \right\},$$

where $\|\cdot\|_{l^s_q(L^p(m))} = \left\| \|\cdot\|_{L^p(m)} \right\|_{l^s_q} = \left(\sum_{j=0}^{\infty} \left(2^{sj} \|\cdot\|_{L^p(m)} \right)^q \right)^{1/q}$.

Definition 3.4. For $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $0 < r \le 2$, we define

$$b^s_{p,q,m} = \left\{ \begin{array}{c} f \in (S^r_*(\mathbb{R}))', \ f \underset{(S^r_*(\mathbb{R}))'}{=} \sum_{i=0}^{\infty} a_i(x) : \\ \|\{a_i\}\|_{l^s_q(L^p(m))} = \left(\sum_{j=0}^{\infty} \left(2^{sj} \|a_i(x)\|_{L^p(m)} \right)^q \right)^{1/q} < \infty \end{array} \right\},$$

where

$$\text{supp}\mathcal{F}a_i \subset \left\{ x : \sqrt{2^{i-1} - \rho^2 - 1} \le x \le \sqrt{2^{i+1} - \rho^2 - 1} \right\},$$

$$\text{supp}\mathcal{F}a_0 \subset \left\{ x : x \le \sqrt{1 - \rho^2} \right\}.$$

By $f \underset{(S^r_*(\mathbb{R}))'}{=} \sum_{i=0}^{\infty} a_i(x)$ will be understood that $\sum_{i=0}^{\infty} a_i(x)$ converges in $(S^r_*(\mathbb{R}))'$ to f.

The norm of $b^s_{p,q,m}$ comes defined by

$$\|f\|_{b^s_{p,q,m}} = \inf_{f=\sum a_i} \|\{a_i\}\|_{l^s_q(L^p(m))}.$$

Theorem 3.1. *Let* $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, *and* $0 < r \le 2$, *then* $B^s_{p,q,m} = b^s_{p,q,m}$.

P r o o f. First, we show that $B^s_{p,q,m} \subset b^s_{p,q,m}$.

Let $f \in B^s_{p,q,m}$. Given $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$ we have that $\sum_{j=0}^{\infty} \mathcal{F}\varphi_j(x) = 1$.

Then

$$f = \mathcal{F}^{-1}\mathcal{F}f = \mathcal{F}^{-1}\left(\sum_{j=0}^{\infty} \mathcal{F}\varphi_j \mathcal{F}f \right) \underset{(S^r_*(\mathbb{R}))'}{=} \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\mathcal{F}\varphi_j \mathcal{F}f) = \sum_{j=0}^{\infty} \varphi_j * f.$$

Taking $a_j = \varphi_j * f$, we get

$$\begin{aligned} \|f\|_{b^s_{p,q,m}} &= \inf_{f=\sum a_i} \|\{a_j\}\|_{l^s_q(L^p(m))} \le \|\{a_j\}\|_{l^s_q(L^p(m))} \\ &= \|\{\varphi_j * f\}\|_{l^s_q(L^p(m))} = \|f\|_{B^s_{p,q,m}}. \end{aligned}$$

Thus the first inclusion is proved.

Next we demonstrate that $b^s_{p,q,m} \subset B^s_{p,q,m}$.

Let $f \in b^s_{p,q,m}$ and $f = \sum_{i=0}^{\infty} a_i(x)$ in the sense of the convergence in $\left(S^r_*(\mathbb{R})\right)'$.

If $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$ then

$$(\varphi_j * f)(x) \underset{(S^r_*(\mathbb{R}))}{=} \sum_{i=0}^{\infty}(\varphi_j * a_i)(x) = \sum_{i=j-1}^{j+1}(\varphi_j * a_i)(x),$$

since $\varphi_j * a_i = \mathcal{F}^{-1}\left(\mathcal{F}\varphi_j \mathcal{F}a_i\right) = 0$ if $i > j+1$ or $i < j-1$.

Now, we define $\varphi_j = a_j = 0$ for $j < 0$ and therefore

$$\|f\|_{B^s_{p,q,m}} = \|\{\varphi_j * f\}\|_{l^s_q(L^p(m))} \le \sum_{r=-1}^{1} \|\{\varphi_j * a_{j+r}\}\|_{l^s_q(L^p(m))}. \tag{3.1}$$

On the other hand, if we apply the multiplier theorem given in [4, p. 649] with $1 < p < \infty$ we have

$$\|\{\varphi_j * f\}\|_{L^p(m)} \le c_1 \|\{a_{j+r}\}\|_{L^p(m)}, \tag{3.2}$$

where c_1 is a suitable positive constant.

Then, taking the norm of l^s_q in (3.2) we have

$$\|\{\varphi_j * a_{j+r}\}\|_{l^s_q(L^p(m))} \le c_1 \|\{a_{j+r}\}\|_{l^s_q(L^p(m))}.$$

Therefore by (3.1) we obtain

$$\begin{aligned} \|f\|_{B^s_{p,q,m}} &= \|\{\varphi_j * f\}\|_{l^s_q(L^p(m))} \\ &\le c_1 \sum_{r=-1}^{1} \|\{a_{j+r}\}\|_{l^s_q(L^p(m))} \\ &\le c_2 \|\{a_j\}\|_{l^s_q(L^p(m))}. \end{aligned} \tag{3.3}$$

Taking the infimum on the right-hand side of (3.3), we have

$$\|f\|_{B^s_{p.q.m}} \le c_2 \|f\|_{b^s_{p,q,m}}.$$

Thus, we have proved Theorem 3.1. ■

Note that by Theorem 3.1, the spaces $B^s_{p,q,m}$ are independent of the functions $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$.

Now we introduce a Triebel-Lizorkin type spaces.

Definition 3.5. Let $1 < p < \infty$, $1 \le q \le \infty$, $0 < r \le 2$ and $s \in \mathbb{R}$. For any system of functions $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$ the Triebel-Lizorkin type spaces are defined by

$$F^s_{p,q,m} = \left\{ f \in (S^r_*(\mathbb{R}))' : \|f\|_{F^s_{p,q,m}} = \|\varphi_j * f\|_{L^p_m(l^s_q)} < \infty \right\},$$

where $\|\cdot\|_{L^p_m(l^s_q)} = \left\| \|\cdot\|_{l^s_q} \right\|_{L^p_m} = \left\| \left(\sum_{j=0}^{\infty} \left(2^{sj}(\cdot)\right)^q \right)^{1/q} \right\|_{L^p(m)}$.

Theorem 3.2. *Let $1 < p, q, < \infty$, $0 < r \le 2$ and $s \in \mathbb{R}$, then*

$$B_{p,\min\{p,q\},m} \subset F^s_{p,q,m} \subset B_{p,\max\{p,q\},m} \ ,$$

where $\subset$ means continuous embedding.

Proof. We must prove that

$$B_{p,p,m} \subset F^s_{p,q,m} \subset B_{p,q,m} \ , \tag{3.4}$$

if $p \le q$, and

$$B_{p,q,m} \subset F^s_{p,q,m} \subset B_{p,p,m} \ , \tag{3.5}$$

for $q \le p$.

To prove the previous embeddings, we will use the monotony of the l^s_q spaces, and the trivial equality $B^s_{p,p,m} \equiv F^s_{p,p,m}$.

First, we will prove (3.4). Let $f \in F^s_{p,q,m}$ and $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$,

$$\begin{aligned}
\|f\|_{B^s_{p,q,m}} &= \|\{\varphi_j * f\}\|_{l^s_q(L^p_m)} = \left(\sum_{j=0}^{\infty} \left(2^{sj} \|\{\varphi_j * f\}\|_{L^p_m} \right)^q \right)^{1/q} \\
&= \left(\sum_{j=0}^{\infty} 2^{sjq} \left(\int_0^{\infty} |\varphi_j * f|^p \, dm(t) \right)^{q/p} \right)^{1/p} \\
&= \left\| \left(\int_0^{\infty} 2^{sjp} |\varphi_j * f|^p \, dm(t) \right) \right\|^{1/p}_{l^s_{q/p}} .
\end{aligned}$$

Now, by using Minkowski's inequality we obtain

$$\begin{aligned}\|f\|_{B^s_{p,q,m}} &\leq \left(\int_0^\infty \left\|2^{sjp}\left|\varphi_j * f\right|^p\right\|_{l^s_{q/p}} dm(t)\right)^{1/p} \\ &= \left\|\left(\sum_{j=0}^\infty 2^{sj}\left|\varphi_j * f\right|^q\right)^{1/q}\right\|_{L^p_m} \\ &= \|\{\varphi_j * f\}\|_{L^p_m(l^s_q)} = \|f\|_{F^s_{p,q,m}} \leq \|\{\varphi_j * f\}\|_{L^p_m(l^s_p)} \\ &= \|\{\varphi_j * f\}\|_{l^s_p(L^p_m)} = \|f\|_{B^s_{p,p,m}}.\end{aligned}$$

Then, to prove (3.5) we have

$$\begin{aligned}\|f\|_{B^s_{p,p,m}} &= \|\{\varphi_j * f\}\|_{l^s_p(L^p_m)} = \|\{\varphi_j * f\}\|_{l^s_p(L^p_m)} \\ &= \|\{\varphi_j * f\}\|_{L^p_m(l^s_p)} \leq \|\{\varphi_j * f\}\|_{L^p_m(l^s_q)} \\ &= \left\|\sum_{j=0}^\infty \left(2^{sj}\left|\varphi_j * f\right|\right)^q\right\|^{1/q}_{L^{p/q}_m} \leq \left(\sum_{j=0}^\infty 2^{sjq}\left\|\left|\varphi_j * f(x)\right|^q\right\|_{L^{p/q}_m}\right)^{1/p} \\ &= \|\{\varphi_j * f\}\|_{l^s_q(L^p_m)} = \|f\|_{B^s_{p,q,m}}.\end{aligned}$$

■

Now, we establish a lifting property for the Bessel potential spaces and Besov type spaces.

Theorem 3.3. *Let $\sigma, s \in \mathbb{R}$, $0 < r \leq 2$, $1 < p < \infty$ and $1 \leq q \leq \infty$. Then J^σ is a linear bounded one-to-one operator from $W^{s,p}_m$ onto $W^{s+\sigma,p}_m$ and from $B^s_{p,q,m}$ onto $B^{s+\sigma}_{p,q,m}$.*

Proof. Let $\{\varphi_j\}_{j=0}^\infty \in \Phi$. If we define $\{\psi_j\}_{j=0}^\infty$ as

$$\psi_j = \varphi_j * \mathcal{F}^{-1}\left(\left(1+\rho^2+\lambda^2\right)^{\sigma/2} 2^{j\sigma}\right),$$

by a straightforward calculation, we have that $\{\psi_j\}_{j=0}^\infty \in \Phi$.

Then,

$$\begin{aligned}J^\sigma f * \psi_j &= \mathcal{F}^{-1}\left(\mathcal{F}\psi_j \mathcal{F} J^\sigma f\right) = \mathcal{F}^{-1}\left(\mathcal{F}\psi_j\left(1+\rho^2+\lambda^2\right)^{-\sigma/2}\mathcal{F}(f)\right) \\ &= \mathcal{F}^{-1}\left(2^{j\sigma}\mathcal{F}\varphi_j\mathcal{F}(f)\right) = 2^{j\sigma} f * \varphi_j.\end{aligned}$$

And the result follows immediately as in [11, pp. 180-181]. ■

4. Applications

In this section we give some applications of the potential and Sobolev spaces.

Theorem 4.1. *Let* $P(\Delta_x) = \sum_{j=0}^{k} a_j \Delta_x^j$, $k \neq 0$ *a differential operator with constant coefficients* a_j, *given* Δ_x *by (1.1) and the symbol* $P(\lambda) = \sum_{j=0}^{k} a_j \lambda^j \neq 0$, $\lambda \in (0,\infty)$. *If* $u \in L^2(m)$, $P(-\Delta_x)u = f$, *and* $f \in L^2(m)$, *then* $u \in W_m^{k,2}$.

P r o o f. For $C_1 > 0$ and $\forall \lambda \in (0,\infty)$, we have that [10, p. 109]

$$|P(\lambda)| \geq C_1 \cdot \lambda^k. \tag{4.1}$$

Let $g \in \mathcal{S}_*^r(\mathbb{R})$

$$\|g\|^2_{W_m^{k,2}} = C \cdot \int_0^\infty \left| J^{-k} g \right|^2 dm$$

$$= C \int_0^\infty \mathcal{F}^{-1}((1+\rho^2+\lambda^2)^{k/2} \mathcal{F}g(\lambda))(x) dm(x).$$

Then by the Plancharel formula ([8, p.531], [4, p.645]) it follows that

$$\|g\|^2_{W_m^{k,2}} = C \cdot \int_0^\infty (1+\rho^2+\lambda^2)^k \left|\mathcal{F}g(\lambda)\right|^2 d\nu(\lambda).$$

If we consider $R \geq 1$, we have

$$\|g\|^2_{W_m^{k,2}} \leq C \cdot \int_0^R (1+\rho^2+\lambda^2)^k \left|\mathcal{F}g(\lambda)\right|^2 d\nu(\lambda) + C \cdot \int_R^\infty (1+\rho^2+\lambda^2)^k \left|\mathcal{F}g(\lambda)\right|^2 d\nu(\lambda).$$

Now, if $\lambda \leq R$, we use that $(1+\rho^2+\lambda^2)^k \leq (1+\rho^2+R^2)^k$ and for $\lambda \geq R$, $(1+\rho^2+\lambda^2)^k \leq (1+\rho^2)^k \cdot \lambda^{4k}$, obtaining

$$\|g\|^2_{W_m^{k,2}} \leq C \cdot (1+\rho^2+R^2)^k \int_0^R \left|\mathcal{F}g(\lambda)\right|^2 d\nu(\lambda)$$

$$+ C \cdot (1+\rho^2)^k \int_R^\infty \lambda^{4k} \left|\mathcal{F}g(\lambda)\right|^2 d\nu(\lambda).$$

Using again the Plancharel formula ([8, p.531], [4, p.645], (1.2) and (4.1) we have

$$\begin{aligned}\|g\|^2_{W^{k,2}_m} &\leq C\cdot\left(\int_0^\infty |g(x)|^2\,dm(x)+\int_0^\infty \left|(\rho^2+\lambda^2)^k\mathcal{F}g(\lambda)\right|^2 d\nu(\lambda)\right)\\ &\leq C\cdot\left(\|g\|^2_{L^2(m)}+\frac{1}{C_1}\int_0^\infty \left|P\left((\rho^2+\lambda^2)\right)\cdot\mathcal{F}g(\lambda)\right|^2 d\nu(\lambda)\right)\\ &\leq C\cdot\left(\|g\|^2_{L^2(m)}+\frac{1}{C_1}\int_0^\infty \left|\mathcal{F}\left(P\left(-\Delta_x\right)g\right)(\lambda)\right|^2 d\nu(\lambda)\right).\end{aligned}$$

Again, by using the Plancharel formula we obtain

$$\|g\|^2_{W^{k,2}_m}\leq C\cdot\left(\|g\|^2_{L^2(m)}+\|(P\left(-\Delta_x\right)g))\|^2_{L^2(m)}\right),$$

where $C>0$.

Now, we complete the proof using Lemma 2.4, that is, that S^r_* is dense in $W^{k,2}_m$. ■

Theorem 4.2. *Let $f\in B^s_{p,q,m}$, then there exists $g\in(\mathcal{S}^r_*(\mathbb{R}))'$ such that*

$$(I-\Delta)^k g=f,$$

where I is the identity operator and $k\in\mathbb{N}-\{0\}$.

Proof. Let us consider $f\in B^s_{p,q,m}$. We want to obtain $g\in(\mathcal{S}^r_*(\mathbb{R}))'$ such that

$$(I-\Delta)^k g=f.$$

Applying the generalized Fourier transform we have

$$(1+\rho^2+\lambda^2)^k\mathcal{F}g=\mathcal{F}f.$$

Now, using the inverse transform we get

$$g=\mathcal{F}^{-1}(1+\rho^2+\lambda^2)^{-k}\mathcal{F}f=J^{2k}f. \tag{4.2}$$

On the other hand, by Theorem 3.3, we obtain that $g\in B^{s+2m}_{p,q,m}$. Thus the proof is complete. ■

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