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A Fixed Point Theorem in Fuzzy Metric Spaces via an Implicit Relation

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We prove a fixed point Theorem for a self-map in complete fuzzy metric spaces satisfying an implicit relation, next we give a metric version of this Theorem.

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1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [14] in 1965. To use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. George and Veeramani [7] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [9] and defined the Hausdoff topology of fuzzy metric spaces which have very important applications in quantum particle physics particularly in connections with both string and $E-$infinity theory which were given and studied by El Naschie [3, 4, 5, 12]. They showed also that every metric induces a fuzzy metric. Grabiec [2] extended the well known fixed point theorem of Banach [1] and Edelstein [2] to fuzzy metric spaces in the sense of [9].

In this paper, we prove a fixed point theorem for a self-map in complete fuzzy metric space X satisfying an implicit relation, next we give a metric version of this Theorem.

Definition 1.1. ([11]) A binary operation $*: [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t-$norm if it satisfies the following conditions:

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous $t-$norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 1.2. ([7]) A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous $t-$norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, .) : (0, \infty) \longrightarrow [0,1]$ is continuous

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

If $(X, M, *)$ is a fuzzy metric space, let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then, τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable.

A sequence $\{x_n\}$ in X converges to x [7] if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$.

It is called a Cauchy sequence if for all $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Since $*$ is continuous, it follows from (FM-4) that the limit of a sequence in a fuzzy metric space is unique

E x a m p l e 1.3. ([7]) Let $X = \mathbb{R}$ and $\forall a, b \in [0, 1]$, $a * b = ab$. Define for all $x, y \in X$ and $t > 0$.

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

Lemma 1.4. *([8]) Let $(X, M, *)$ be a fuzzy metric space. Then, $M(x, y, t)$ is increasing with respect to t for all x, y in X.*

Definition 1.5. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t).$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t)$$

Lemma 1.6. *([8]) Let $(X, M, *)$ be a fuzzy metric space. Then, M is continuous function on $X^2 \times (0, \infty)$.*

Let $(X, M, *)$ be a fuzzy metric space and $B(X)$ be the set of all nonempty bounded subsets of X. As in Fisher [6], for $A, B \in B(X)$ and $\forall t > 0$

$$\delta_M(A, B, t) = \inf\{M(a, b, t) : a \in A, b \in B\}.$$

If A is consisted of a single point a, we write $\delta_M(A, B, t) = \delta_M(a, B, t)$. If B is consisted also of a single point b, we write $\delta_M(A, B, t) = M(a, b, t)$. It follows immediately from the definition that

$$\begin{aligned} \delta_M(A, B, t) &= \delta_M(B, A, t) \geq 0, \\ \delta_M(A, B, t) &= 1 \Longleftrightarrow A = B = \{a\}, \end{aligned}$$

for all $A, B \subseteq X$. In particular if $\emptyset \neq S = A = B \subset X$, we obtain

$$\delta_M(S, t) = \inf\{M(x, y, t) : x, y \in S, t > 0\}.$$

If S is consisted of a single point a, then $\delta_M(S,t) = 1$ for all $t > 0$. If S is consisted of a two points a, b, then $\delta_M(S,t) = M(a,b,t)$. It follows immediately from the definition that:

(i) :If $A \subseteq B$, then $\delta_M(A,t) \geq \delta_M(B,t)$.

(ii) : $0 \leq \delta_M(S,t) \leq 1$, for all nonempty subset S of X.

(iii) : $\delta_M(S,t)$ is increasing with respect to t. That is, if $0 < t_1 \leq t_2$, therefore $\delta_M(S,t_1) \leq \delta_M(S,t_2)$.

For a sequence $A_n = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$ in a fuzzy metric space $(X, M, *)$, let $r_n(t) = \delta_M(A_n, t)$ for $n \in \mathbb{N}$ and $t > 0$. Then

(a) : By (i), since $A_n \supseteq A_{n+1}$, we have $r_n(t) \leq r_{n+1}(t)$, $\forall t > 0$,

(b) : $\forall n, m \geq k$, $M(x_n, x_m, t) \geq \delta_M(A_k, t) = r_k(t)$,

(c) : $\forall n \geq 1$, $0 \leq r_n(t) \leq 1$.

Therefore, $\{r_n(t)\}$ is increasing and bounded for all $n \in \mathbb{N}$ and $t > 0$ and so there exists $0 \leq r(t) \leq 1$ such that $\lim_{n\to\infty} r_n(t) = r(t)$.

Lemma 1.7. *Let $(X, M, *)$ be a fuzzy metric space. If $\lim_{n\to\infty} r_n(t) = 1$ for all $t > 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof. As $\lim_{n\to\infty} r_n(t) = 1$, given $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we get $|r_n(t) - 1| < \epsilon$. That is $1 - \epsilon < r_n(t) < 1 + \epsilon$. Then, for $l, k \geq n > n_0$ by (b) we obtain

$$M(x_l, x_k, t) \geq \inf\{M(x_i, x_j, t) : x_i, x_j \in A_n\} = r_n(t) > 1 - \epsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. ■

Motivated by a work due to Popa [13], we have observed that proving fixed point theorems using an implicit relation is a good idea since it covers several contractive conditions rather than one contractive condition.

Let Φ be the set of all continuous functions $\varphi : [0,1]^6 \longrightarrow (0,1]$ satisfying the following conditions:

(ϕ_1) : $\varphi(t_1, \cdots, t_5, t_6)$ is increasing in variables $t_1, t_2, .., t_5$.

(ϕ_2) : $\forall\ u, v \in (0,1]$, $\varphi(u,u,u,u,u,v) \leq 0$ implies that $v \geq \phi(u)$, where $\phi : (0,1] \longrightarrow (0,1]$ is an increasing continuous function with $\phi(t) > t$ for all $0 < t < 1$.

It easy to see that $\phi(1) = 1$ and if $\varphi(1,1,1,1,1,v) \leq 0$, then $v = 1$. In fact, since $\varphi(1,1,1,1,1,v) \leq 0$ by ϕ_2 we obtain $v \geq 1$ which is a contradiction if $v < 1$. Thus, $v = 1$

Example 1.8. $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = \phi_1(\min\{t_1, t_2, t_3, t_4, t_5\}) - t_6$, where $\phi_1 : (0, 1] \longrightarrow (0, 1]$ is an increasing and continuous function with $\phi(t) > t$ for $0 < t < 1$.

For example $\phi_1(t) = \sqrt{t}$ or $\phi_1(t) = t^h$ for $0 < h < 1$.

It is easy to see that φ in Example 1.8 verifies conditions (ϕ_1) and (ϕ_2).

2 Main results

Theorem 2.1. *Let $(X, M, *)$ be a complete fuzzy metric space and T be a self map of X satisfying*

(2.1)
$$\varphi(M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(y, Tx, t), M(Tx, Ty, t)) \leq 0$$

for all x, y in X, $t > 0$ and $\varphi \in \Phi$. Then, T has a unique fixed point p in X.

Proof. Let x_0 be an arbitrary point in X, $Tx_n = x_{n+1}$, $A_n = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$ and $r_n(t) = \delta_M(A_n, t)$, $n \in \mathbb{N}$ and $t > 0$. Then, we know that $\lim_{n\to\infty} r_n(t) = r(t)$ for some $0 \leq r(t) \leq 1$.

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then T has a fixed point, say $p \in X$.

Assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Taking $x = x_{n+k}$ and $y = x_{m+k}$ in (2.1) for $k \in \mathbb{N}$, we have

$$\begin{aligned}
& \varphi(M(x_{n+k}, x_{m+k}, t), M(x_{n+k}, Tx_{n+k}, t), M(x_{m+k}, Tx_{m+k}, t), \\
& M(x_{n+k}, Tx_{m+k}, t), M(x_{m+k}, Tx_{n+k}, t), M(Tx_{n+k}, Tx_{m+k}, t)) \\
= \; & \varphi(M(x_{n+k}, x_{m+k}, t), M(x_{n+k}, x_{n+k+1}, t), M(x_{m+k}, x_{m+k+1}, t), \\
& M(x_{n+k}, x_{m+k+1}, t), M(x_{m+k}, x_{n+k+1}, t), M(x_{n+k+1}, x_{m+k+1}, t)) \\
\leq \; & 0
\end{aligned}$$

Since $M(x_{n+k}, x_{m+k}, t) \geq r_k(t)$, for every $n, m \geq 0$, using (ϕ_1) we get

$$\varphi\left(r_k(t), r_k(t), r_k(t), r_k(t), r_k(t), M(x_{n+k+1}, x_{m+k+1}, t)\right) \leq 0.$$

which implies by (ϕ_2)

$$M(x_{n+k+1}, x_{m+k+1}, t) \geq \phi(r_k(t)) \quad \forall m, n \geq 0.$$

Hence

$$\inf_{m,n\geq 0} M(x_{n+k+1}, x_{m+k+1}, t) \geq \phi(r_k(t)).$$

Therefore, $r_{k+1}(t) \geq \phi(r_k(t))$. Letting $k \to \infty$, we get $r(t) \geq \phi(r(t))$. If $0 < r(t) < 1$, then $r(t) \geq \phi(r(t)) > r(t)$ which is a contradiction. Thus, $r(t) = 1$ and so $\lim_{n\to\infty} r_n(t) = 1$. Thus by Lemma 1.7, $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists a $p \in X$ such that $\lim_{n\to\infty} x_n = p$, then $\lim_{n\to\infty} Tx_n = p$. Applying inequality (2.1) we get

$$\begin{aligned} & \varphi(M(x_n,p,t), M(x_n,Tx_n,t), M(p,Tp,t), M(x_n,Tp,t), \\ & M(p,Tx_n,t), M(Tx_n,Tp,t)) \\ = \; & \varphi(M(x_n,p,t), M(x_n,x_{n+1},t), M(p,Tp,t), M(x_n,Tp,t), \\ & M(p,x_{n+1},t)), M(x_{n+1},Tp,t)) \leq 0. \end{aligned}$$

Taking $n \longrightarrow \infty$, we obtain

$$\varphi(1,1,M(p,Tp,t),M(p,Tp,t),1,M(p,Tp,t)) \leq 0.$$

By (ϕ_1) we have

$$\varphi(M(p,Tp,t),M(p,Tp,t),M(p,Tp,t),M(p,Tp,t),M(p,Tp,t),M(p,Tp,t)) \leq 0$$

and (ϕ_2) implies that $M(p,Tp,t) \geq \phi(M(p,Tp,t)) > M(p,Tp,t)$ which is a contradiction. Hence $Tp = p$. The uniqueness of p follows from inequality (2.1) and (ϕ_2). ■

Corollary 2.2. *Let $(X,M,*)$ be a complete fuzzy metric space and T be a self map of X satisfying*

$$M(Tx,Ty,t) \geq (\min\{M(x,y,t),M(x,Tx,t),M(y,Ty,t),M(x,Ty,t),M(y,Tx,t)\})^h$$

for all x,y in X, $0 < h < 1$ and $t > 0$. Then, T has a unique fixed point p in X.

Proof. We take $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = (\min\{t_1,t_2,t_3,t_4,t_5\})^h - t_6$. ■

Corrollary 2.3. *Let $(X,M,*)$ be a complete fuzzy metric space and T be a self map of X satisfying*

$$M(Tx,Ty,t) \geq \sqrt{M(x,y,t)}$$

for all x,y in X and $t > 0$. Then, T has a unique fixed point p in X.

Proof. We take $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = \sqrt{t_1} - t_6$. ■

Example 2.4. Let $(X, M, *)$ be a fuzzy metric space, where $X = \mathbb{R}$ and $M(x, y, t) = e^{-\frac{|x-y|}{t}}$. Define a self-map T on X by: $Tx = \dfrac{x+5}{6}$ for all $x \in X$. Then, we have

$$M(Tx, Ty, t) = e^{-\frac{|x-y|}{6t}} \geq e^{-\frac{|x-y|}{2t}}.$$

Therefore, all conditions of Corollary 2.3 are satisfied and 1 is the unique fixed point of T.

3. A metric version

Now, we give a metric version of Theorem 2.1.

Let (X, d) be a metric space and $B(X)$ be the set of all nonempty bounded subsets of X. As in Fisher [6], for $A, B \in B(X)$

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

If A is consisted of a single point a, we write $\delta(A, B) = \delta(a, B)$. If B is consisted also of a single point b, we write $\delta(A, B) = d(a, b)$. It follows immediately from the definition that

$$\begin{aligned} \delta(A, B) &= \delta(B, A) \geq 0, \\ \delta(A, B) &= 1 \Longleftrightarrow A = B = \{a\}, \end{aligned}$$

for all $A, B \subseteq X$. In particular if $\emptyset \neq S = A = B \subset X$, we obtain

$$\delta(S) = \sup\{d(x, y) : x, y \in S\}.$$

If S is consisted of a single point a, then $\delta(S) = 1$. If S is consisted of a two points a, b, then $\delta(S) = d(a, b)$. It follows immediately from the definition that:

(i) If $A \subseteq B$, then $\delta(A) \leq \delta(B)$.

(ii) $\delta(S) \geq 0$, for all nonempty subset S of X.

For a sequence $A_n = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$ in a metric space (X, d), let $r_n = \delta(A_n)$ for $n \in \mathbb{N}$. Then

(a) : By (i), since $A_n \supseteq A_{n+1}$, we have $r_{n+1} \leq r_n$,

(b) : $\forall n, m \geq k$, $d(x_n, x_m) \leq \delta(A_k) = r_k$,

(c) : $\forall n \in \mathbb{N}$, $r_n \geq 0$.

From (a) and (c), there exists $r \geq 0$ such that $\lim_{n\to\infty} r_n = r$.

Lemma 3.1. *Let (X, d) be a metric space. If $\lim_{n\to\infty} r_n = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

P r o o f. As $\lim_{n\to\infty} r_n = 0$, given $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we get $0 \le r_n < \epsilon$. That is Hence, for $l, k \ge n > n_0$ by (b) we obtain

$$d(x_l, x_k) \le \sup\{d(x_i, x_j) : x_i, x_j \in A_n\} = r_n < \epsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. ■

Let Ψ be the set of all continuous functions $\psi : [0,\infty)^6 \longrightarrow \mathbb{R}$ satisfying the following conditions:

$(\psi_1) : \psi(t_1, \cdots, t_5, t_6)$ is increasing in variables $t_1, t_2, .., t_5$.

$(\psi_2) : \forall\ u, v \ge 0,\ \psi(u, u, u, u, u, v) \ge 0$ implies that $v \le f(u)$, where $f : [0,\infty) \longrightarrow [0,\infty)$ is an increasing and upper semi-continuous function with $f(t) < t$ for all $t > 0$.

It easy to see that $f(0) = 0$ and if $\psi(0, 0, 0, 0, 0, v) \le 0$, then $v = 0$.

E x a m p l e 3.2. $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = \psi_1(\max\{t_1, t_2, t_3, t_4, t_5\}) - t_6$,

where $\psi_1 : [0,\infty) \longrightarrow [0,\infty)$ is an increasing and upper semi-continuous function with $\psi_1(t) < t$ for all $t > 0$. For example $\psi_1(t) = kt$, $0 < k < 1$ for all $t > 0$.

It is easy to see that ψ in Example 3.2 verifies conditions (ψ_1) and (ψ_2).

Theorem 3.3. *Let (X, d) be a complete metric space and T be a self map of X satisfying*

$$\psi(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(Tx, Ty)) \ge 0 \tag{3.1}$$

for all x, y in X and $\psi \in \Psi$. Then, T has a unique fixed point in X.

P r o o f. Let x_0 be an arbitrary point in X, $Tx_n = x_{n+1}$, $A_n = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$ and $r_n = \delta(A_n)$, $n \in \mathbb{N}$. Then, we know that $\lim_{n\to\infty} r_n = r$ for some $r \ge 0$.

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then T has a fixed point, say $p \in X$.

Assume that $x_{n+1} \ne x_n$ for each $n \in \mathbb{N}$. Taking $x = x_{n+k}$ and $y = x_{m+k}$ in (3.1) for $k \in \mathbb{N}$, we have

$$\begin{aligned} & \psi(d(x_{n+k}, x_{m+k}), d(x_{n+k}, Tx_{n+k}), d(x_{m+k}, Tx_{m+k}), \\ & d(x_{n+k}, Tx_{m+k}), d(x_{m+k}, Tx_{n+k}), d(Tx_{n+k}, Tx_{m+k})) \\ = \ & \psi(d(x_{n+k}, x_{m+k}), d(x_{n+k}, x_{m+k+1}), d(x_{m+k}, x_{m+k+1}), \\ & d(x_{n+k}, x_{m+k+1}), d(x_{m+k}, x_{n+k+1}), d(x_{n+k+1}, x_{m+k+1})) \ge 0. \end{aligned}$$

Since $d(x_{n+k}, x_{m+k}) \le r_k$, for every $n, m \ge 0$, using (ψ_1) we get

$$\psi\left(r_k, r_k, r_k, r_k, r_k, d(x_{n+k+1}, x_{m+k+1}\right) \ge 0$$

which implies by (ψ_2)

$$d(x_{n+k+1}, x_{m+k+1}) \le f(r_k), \ \ \forall m, n \ge 0.$$

Hence

$$\sup_{n,m\ge 0} d(x_{n+k+1}, x_{m+k+1}) \le \psi(r_k).$$

Therefore, $r_{k+1} \le f(r_k)$. Letting $k \to \infty$, we get $r \le f(r)$. If $0 < r$, then $r < r$ which is a contradiction. Thus, $r(t) = 1$ and so $\lim_{n\to\infty} r_n = 0$. Thus by Lemma 1.7, $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$, then $\lim_{n\to\infty} Tx_n = p$. Applying inequality (3.1) we get

$$\begin{aligned} & \psi(d(x_n, p, d(x_n, Tx_n), d(p, Tp), d(x_n, Tp), \\ & d(p, Tx_n), d(Tx_n, Tp)) \\ = \ & \psi(d(x_n, p), d(x_n, x_{n+1}), d(p, Tp), d(x_n, Tp), \\ & d(p, x_{n+1}), d(x_{n+1}, Tp)) \ge 0. \end{aligned}$$

Taking $n \longrightarrow \infty$, we obtain

$$\psi(0, 0, d(p, Tp), d(p, Tp), 0, d(p, Tp)) \ge 0.$$

By (ψ_1) we have

$$\psi(d(p, Tp), d(p, Tp), d(p, Tp), d(p, Tp), d(p, Tp,), d(p, Tp)) \ge 0$$

and (ψ_2) implies that $d(p, Tp) \le \psi(d(p, Tp)) < d(p, Tp)$ which is contradiction. Hence $Tp = p$.

The uniqueness of p follows from inequality (3.1) and (ψ_2). ■

Corollary 3.4. *Let (X, d) be a complete metric space and T be a self map of X satisfying*

$$d(Tx, Ty) \le h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all x, y in X and $0 < h < 1$. Then, T has a unique fixed point p in X.

Proof. We take $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = k \max\{t_1, t_2, t_3, t_4, t_5\} - t_6$, $0 < k < 1$. ■

Corollary 3.5. *Let (X, d) be a complete metric space and T^n be a self map of X satisfying for some $n \ge 1$*

$$d(T^n x, T^n y) \le h \max\{d(x, y), d(x, T^n x), d(y, T^n y), d(x, T^n y), d(y, T^n x)\}$$

for all x, y in X and $0 < h < 1$. Then, T has a unique fixed point p in X.

Proof. By Corollary 3.4, T^n has a unique fixed point p in X. That is, $T^n p = p$. Thus, $T^n(Tp) = T(T^n p) = Tp$. Since p is unique, we get $Tp = p$.

Example 3.6. Let $(X, d) = (\mathbb{R}, |.|)$. Define a self-map T on X by:

$$Tx = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

For all $n \geq 2$ and $x \in \mathbb{R}$ we have $T^n x = 1$ and so

$$d(T^n x, T^n y) \leq h \max\{d(x, y), d(x, T^n x), d(y, T^n y), d(x, T^n y), d(y, T^n x)\}.$$

Hence, by Corollary 3.5, T has a unique fixed point $x = 1$.

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