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On Extremal Properties for the Derivative of Polynomials [1]

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If $P(z)$ be a polynomial of degree n, having no zero in $|z| < k$, $k \leq 1$, then Govil [Proc. Natl. Acad. Sci., 50 (1980), 50–52] proved that

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P(1/\overline{z})}$. In this paper, we consider a class of polynomials of the type $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$ and obtain generalization as well as improvement of above inequality. Also we generalize a result of Govil [J. Math. Phy. Sci., 14, no. 2 (1980), 183–187] in this direction.

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1. Introduction

If $P(z)$ be a polynomial of degree n, then according to a well known result of S. Bernstein (for reference see [2]), we have

$$\max_{|z|=1} \left|P'(z)\right| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

The result is best possible and equality holds for $P(z) = \alpha z^n$, where $|\alpha| = 1$.

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If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then inequality (1.1) can be sharpened. In fact, P. Erdös conjectured and later Lax [7] proved that if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{2} \max_{|z|=1} |P(z)| . \tag{2}$$

The result is best possible and equality holds for $P(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

As a generalization of inequality (1.2), Malik [8] considered that if $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)| . \tag{3}$$

The result is best possible and equality holds for $P(z) = (z + k)^n$.

While seeking for the inequality analogous to (1.3) for the polynomials not vanishing in $|z| < k$, $k \leq 1$, Govil [4] proved the following result.

Theorem A . *Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n, having no zero in $|z| < k$, $k \leq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)| . \tag{4}$$

For the polynomials having all its zeros on $|z| = k$, $k \leq 1$, Govil [5] was able to prove the following result.

Theorem B . *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n, having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |P(z)| . \tag{5}$$

In this paper, we generalize as well as improve upon Theorems A and also generalize Theorem B for the polynomials of the type $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$.

Theorem 1 . *Let* $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu < n$ *is a polynomial of degree* n, *having no zero in* $|z| < k$, $k \le 1$ *and* $Q(z) = z^n \overline{P(1/\overline{z})}$. *If* $|P'(z)|$ *and* $|Q'(z)|$ *becomes maximum at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{1 + k^{n-\mu+1}} \max_{|z|=1} |P(z)| . \tag{6}$$

Remark 1 . If we take $\mu = 1$ in Theorem 1, then inequality (1.6) reduces to inequality (1.4) due to Govil [4].

Theorem 2 . *Let* $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu < n$ *is a polynomial of degree* n, *having no zero in* $|z| < k$, $k \le 1$ *and* $Q(z) = z^n \overline{P(1/\overline{z})}$. *If* $|P'(z)|$ *and* $|Q'(z)|$ *becomes maximum at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{1 + k^{n-\mu+1}} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\} . \tag{7}$$

If we take $\mu = 1$ in Theorem 2, then we get the following result, which was also proved by Aziz and Rather [1].

Corollary 1 . *Let* $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ *is a polynomial of degree* n, *having no zero in* $|z| < k$, $k \le 1$ *and* $Q(z) = z^n \overline{P(1/\overline{z})}$. *If* $|P'(z)|$ *and* $|Q'(z)|$ *becomes maximum at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{1 + k^{n}} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\} .$$

Theorem 3 . *If* $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu < n$ *is a polynomial of degree* n, *having all its zeros on* $|z| = k$, $k \le 1$, *then*

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |P(z)| . \tag{8}$$

Remark 2 . Again if we take $\mu = 1$ in Theorem 3, then inequality (1.8) reduces to inequality (1.5) due to Govil [5].

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 1 . *If $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ is a polynomial of degree n, having all its zeros in the disk $|z| \le k$, $k \ge 1$, then for $|z| = 1$*

$$k^{n+\mu-3} \left|Q'(z)\right| \le \left|P'(k^2 z)\right|, \tag{2.1}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof. Since the polynomial $P(z)$ has all its zeros in $|z| \le k$, $k \ge 1$. Therefore the polynomial $F(z) = P(kz)$ has all its zeros in the unit disk $|z| \le 1$. Now if $G(z) \equiv z^n \overline{F(1/\overline{z})} \equiv z^n \overline{P(k/\overline{z})} \equiv k^n Q(z/k)$, then all the zeros of $G(z)$ lie in $|z| \ge 1$. Since $|F(z)| = |G(z)|$ on $|z| = 1$, it follows by maximum modulus principle that $|G(z)| \le |F(z)|$ on $|z| \ge 1$. Hence for every complex number λ with $|\lambda| > 1$, it follows by Rouche's theorem that the polynomial $G(z) - \lambda F(z)$ has all its zeros in $|z| < 1$. By Gauss Lucas theorem the polynomial $G'(z) - \lambda F'(z)$ has all its zeros in $|z| < 1$, which implies

$$\left|G'(z)\right| \le \left|F'(z)\right| \quad \text{for} \quad |z| \ge 1. \tag{2.2}$$

Substituting for $F(z)$ and $G(z)$ in (2.2), we get

$$k^{n-1} \left|Q'(z/k)\right| \le k \left|P'(kz)\right| \quad \text{for} \quad |z| \ge 1. \tag{2.3}$$

Since $c_1 = c_2 = \cdots = c_{\mu-1} = 0$, from (2.3), we get

$$k^{n-1} \left|Q'(z/k)\right| \le k^\mu \left|\sum_{\nu=\mu}^{n} \nu c_\nu (kz)^{\nu-\mu}\right| \quad \text{for} \quad |z| \ge 1. \tag{2.4}$$

In fact (2.4) holds for $|z| = 1$. But $\sum_{\nu=\mu}^{n} \nu c_\nu (kz)^{\nu-\mu} \ne 0$ in $|z| > 1$, by maximum modulus principle it also holds for $|z| > 1$. Taking kz instead of z in (2.4), we have

$$k^{n-1} \left|Q'(z)\right| \le k^\mu \left|\sum_{\nu=\mu}^{n} \nu c_\nu (k^2 z)^{\nu-\mu}\right| \quad \text{for} \quad |z| \ge 1/k.$$

In particular,

$$k^{n-1} \left|Q'(z)\right| \le k^\mu \left|\sum_{\nu=\mu}^{n} \nu c_\nu (k^2 z)^{\nu-\mu}\right| \quad \text{for} \quad |z| = 1,$$

this implies

$$k^{n-1}\left|Q'(z)\right| \leq k^{2-\mu}\left|\sum_{\nu=\mu}^{n} \nu c_\nu (k^2 z)^{\nu-1}\right| \quad \text{for} \quad |z|=1.$$

Consequently

$$k^{n+\mu-3}\left|Q'(z)\right| \leq \left|P'(k^2 z)\right| \quad \text{for} \quad |z|=1.$$

This completes the proof of Lemma 1. ■

Lemma 2 . *If $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \leq \mu < n$ is a polynomial of degree n, having all its zeros in the disk $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=1}\left|Q'(z)\right| \leq k^{n-\mu+1}\max_{|z|=1}\left|P'(z)\right|,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

P r o o f. By Lemma 1, we have

$$\max_{|z|=1}\left|Q'(z)\right| \leq \frac{1}{k^{n+\mu-3}}\max_{|z|=k^2}\left|P'(z)\right|. \tag{2.5}$$

If $H(z)$ is a polynomial of degree n, then it is a simple deduction from maximum modulus principle (see [9, P. 158, Prob. 269]) that

$$\max_{|z|=R\geq 1}|H(z)| \leq R^n \max_{|z|=1}|H(z)|. \tag{2.6}$$

Applying inequality (2.6) to the polynomial $P'(z)$, which is of degree $n-1$, with $R = k^2 \geq 1$, we have

$$\max_{|z|=k^2}\left|P'(z)\right| \leq k^{2n-2}\max_{|z|=1}\left|P'(z)\right|.$$

Combining this with (2.5), the lemma follows. ■

Lemma 3 . *If $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ is a polynomial of degree n, having no zero in $|z| < k$, $k \leq 1$, then*

$$k^{n-\mu+1}\max_{|z|=1}\left|P'(z)\right| \leq \max_{|z|=1}\left|Q'(z)\right|,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof. If $P(z)$ has no zero in $|z| < k$, $k \le 1$, then $Q(z) = z^n\overline{P(1/\overline{z})}$ has all its zeros in $|z| \le 1/k$, $1/k \ge 1$. Thus applying Lemma 2 to the polynomial $Q(z)$, we get

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{1}{k^{n-\mu+1}} \max_{|z|=1} \left|Q'(z)\right|,$$

and the lemma follows. ∎

Lemma 4 . *If $P(z)$ is a polynomial of degree n, then for $|z| = 1$*

$$\left|P'(z)\right| + \left|Q'(z)\right| \le n \max_{|z|=1} |P(z)|,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

The above lemma is a special case of a result due to Govil and Rahman [6].

Lemma 5 . *If $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu \le n$ is a polynomial of degree n, having no zero in the disk $|z| < k$, $k \ge 1$, then for $|z| = 1$*

$$k^\mu \left|P'(z)\right| \le \left|Q'(z)\right|,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

The above lemma is due to Chan and Malik [3].

Lemma 6 . *If $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$ is a polynomial of degree n, having all its zeros on $|z| = k$, $k \le 1$, then for $|z| = 1$*

$$\left|Q'(z)\right| \le k^\mu \left|P'(z)\right|,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Proof. Since $P(z)$ has all its zeros on $|z| = k$, $k \le 1$, therefore $Q(z) = z^n\overline{P(1/\overline{z})}$ has all its zeros on $|z| = 1/k$, $1/k \ge 1$. Now applying Lemma 5 to polynomial $Q(z)$ and result follows. ∎

3. Proofs of the Theorems

P r o o f. o f T h e o r e m 1. Since by hypothesis $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, if we choose z_0 be a point on $|z| = 1$ such that $|P'(z_0)| = \max_{|z|=1} |P'(z)|$ and $|Q'(z_0)| = \max_{|z|=1} |Q'(z)|$, then by Lemma 4, we have

$$\left|P'(z_0)\right| + \left|Q'(z_0)\right| \le n \max_{|z|=1} |P(z)| . \tag{3.1}$$

Combining inequality (3.1) with Lemma 3, we get

$$\left|p'(z_0)\right| + k^{n-\mu+1} \max_{|z|=1} \left|P'(z)\right| \le n \max_{|z|=1} |P(z)| ,$$

which implies

$$\left(1 + k^{n-\mu+1}\right) \max_{|z|=1} \left|P'(z)\right| \le n \max_{|z|=1} |P(z)| ,$$

and the result follows. ■

P r o o f. o f T h e o r e m 2. If $P(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \le 1$ and if $m = \min_{|z|=k} |P(z)|$, then for every α with $|\alpha| < 1$, the polynomial $P(z) - \alpha m$ has no zero in $|z| < k$, $k \le 1$. This result is clear if $P(z)$ has a zero on $|z| = k$, for then $m = 0$ and therefore $P(z) - \alpha m = P(z)$. In case $P(z)$ has no zero on $|z| = k$, then for every α with $|\alpha| < 1$, we have $|P(z)| > m\,|\alpha|$ on $|z| = k$ and on applying Rouche's theorem the result will follows. Thus $P(z) - \alpha m$ has no zero in $|z| < k$, $k \le 1$ and hence, applying Theorem 1 to $P(z) - \alpha m$, we get

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{1 + k^{n-\mu+1}} \max_{|z|=1} |P(z) - \alpha m| . \tag{3.2}$$

If we choose a point z_0 on $|z| = 1$ such that $|P(z_0)| = \max_{|z|=1} |P(z)|$, then (3.2) in particular gives

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{1 + k^{n-\mu+1}} |P(z_0) - \alpha m| . \tag{3.3}$$

Now choosing argument of α on right hand side (3.3), we get

$$\max_{|z|=1} \left|P'(z)\right| \ge \frac{n}{1 + k^{n-\mu+1}} \left\{|P(z_0)| - |\alpha|\, m\right\} ,$$

and letting $|\alpha| \to 1$, we get

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{1 + k^{n-\mu+1}} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\} .$$

This completes the proof of Theorem 2. ■

Proof. of Theorem 3. If z_0 be a point on $|z| = 1$ such that $|Q'(z_0)| = \max_{|z|=1} |Q'(z)|$, then by Lemma 4, we have

$$\left|P'(z_0)\right| + \max_{|z|=1} \left|Q'(z)\right| \leq n \max_{|z|=1} |P(z)| . \tag{3.4}$$

Combining inequality (3.4) with Lemma 6, we get

$$\left(\frac{1}{k^\mu}\right) \left|Q'(z_0)\right| + \max_{|z|=1} \left|Q'(z)\right| \leq n \max_{|z|=1} |P(z)| ,$$

which is equivalent to

$$\left(\frac{1}{k^\mu} + 1\right) \max_{|z|=1} \left|Q'(z)\right| \leq n \max_{|z|=1} |P(z)| . \tag{3.5}$$

Inequality (3.5), when combined with Lemma 3, gives

$$\left(\frac{1}{k^\mu} + 1\right) k^{n-\mu+1} \max_{|z|=1} \left|P'(z)\right| \leq n \max_{|z|=1} |P(z)| ,$$

which implies

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |P(z)| .$$

This completes the proof of the Theorem 3. ■

Remark 3 . For $\mu = n$ Theorem 1, 2 and 3 holds if polynomial satisfy the condition $|c_0| \leq k\, |c_n|$.

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