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Uniqueness of Meromorphic Functions Sharing Three Sets

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Using the notion of weighted sharing and truncated sharing of sets, we investigate the problem of uniqueness of meromorphic functions sharing three sets and obtain three results which are improving and supplement a result of Lin-Yi [12].

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1. Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. The notation $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure.

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities, then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$,

$E_{\infty)}(a;f)=E_{\infty)}(a;g)$, we say that f and g share the value a CM. For a set S of distinct elements of $\mathbb{C}$ we define $E_{m)}(S,f)=\bigcup_{a\in S}E_{m)}(a,f)$.

In 1976, F. Gross [4] posed the following question:

Question A. *Can one find two finite sets* S_j $(j=1,2)$ *such that any two nonconstant entire functions* f *and* g *satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1,2$ *must be identical ?*

For meromorphic functions it is natural to ask the following question.

Question B. ([14]) *Can one find three finite sets* S_j $(j=1,2,3)$ *such that any two nonconstant meromorphic functions* f *and* g *satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1,2,3$ *must be identical ?*

In 1994, Yi [14] gave an affirmative answer to *Question B* and proved that there exist three finite sets S_1 (with 7 elements), S_2 (with 2 elements) and S_3 (with 1 element) such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1$, 2, 3 must be identical.

During the last couple of years or so, a considerable amount of work has been done to investigate the possible answer of *Question B* and continuous effort is being carried out to relax the hypothesis of the result, cf. [1], [2], [3], [9], [12], [13], [14], [16]. In 2001 an idea of gradation of sharing known as weighted sharing has been introduced in [7], [8] which measure how close a shared value is to being share CM or to being shared IM. In the following definition we explain the notion.

Definition 1.1. ([7],[8]) Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$, and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f,g share the value a with weight k.

We write f, g share (a,k) to mean that f,g share the value a with weight k. Clearly if f, g share (a,k), then f,g share (a,p) for any integer p, $0\leq p<k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively.

Definition 1.2. ([7]) Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S,k)$ the set $\bigcup_{a\in S}E_k(a;f)$.

Clearly $E_f(S)=E_f(S,\infty)$ and $\overline{E}_f(S)=E_f(S,0)$.

In 2007 the present author [1] has provided an affirmative answer to *Question B* by the notion of weighted sharing in which he has proved that if two non constant meromorphic functions share one set S_1 (containing 1 element) CM, and two other sets S_2 (containing 1 element) and S_3 (containing 4 elements) with finite weight, then $f\equiv g$. But to serve the purpose an extra condition on the ramification index of f and g has been taken into consideration. So it will be interesting to investigate whether *Question B* can be answered affirmatively without the help of any extra condition imposing on the ramification index of f and g and if such a situation arises then it will also be a natural query to investigate the cardinalities of the range sets.

Suppose that the polynomial $P(w)$ is defined by

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2, \tag{1.1}$$

where $n \geq 3$ is an integer and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. In fact, we consider the following rational function

$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{1.2}$$

where α_1 and α_2 are two distinct roots of

$$n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0.$$

It can be easily shown that the polynomial $P(w)$ has only simple zeros (see [12]). Clearly, from (1.1) and (1.2) we have

$$R(w) - 1 = \frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}. \tag{1.3}$$

In 2003, Lin and Yi [12] answered *Question B* and proved the following result which is an improvement of that of Yi [14].

Theorem A. ([12]) *Let $S_1 = \{0\}$, $S_2 = \{\infty\}$ and $S_3 = \{w \mid P(w) = 0\}$, where $P(w)$ is given by (1.1) and $n \geq 5$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ $(j = 1, 3)$ and $E_f(S_2, 0) = E_g(S_2, 0)$, then $f \equiv g$.*

It is worth mentioning that recently in [2], we have improved *Theorem A* by relaxing the nature of sharing the sets. In this paper we improve *Theorem A* in a different way. The following three theorems present our here.bn

Theorem 1.1. *Let S_1, S_2 and S_3 be defined as in* Theorem A *and $n \geq 5$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S_1, 5) = E_g(S_1, 5)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_{5)}(S_3, f) = E_{5)}(S_3, g)$, then $f \equiv g$.*

Theorem 1.2. *Let S_1, S_2 and S_3 be defined as in* Theorem A *and $n \geq 5$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S_1,\infty) = E_g(S_1,\infty)$, $E_f(S_2,1) = E_g(S_2,1)$ and $E_{4)}(S_3,f) = E_{4)}(S_3,g)$, then $f \equiv g$.*

Theorem 1.3. *Let S_1, S_2 and S_3 be defined as in* Theorem A *and $n \geq 5$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S_1,3) = E_g(S_1,3)$, $E_f(S_2,0) = E_g(S_2,0)$ and $E_{6)}(S_3,f) = E_{6)}(S_3,g)$, then $f \equiv g$.*

Although for the standard definitions and notations of the value distribution theory we refer to [5], we now explain some notations which are used in the paper.

Definition 1.3. ([6]) For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid = 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r,a;f \mid \leq m)(N(r,a;f \mid \geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r,a;f \mid \leq m)$ $(\overline{N}(r,a;f \mid \geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f \mid < m)$, $N(r,a;f \mid > m)$, $\overline{N}(r,a;f \mid < m)$ *and* $\overline{N}(r,a;f \mid > m)$ are defined analogously.

Definition 1.4. ([1]) We denote by $\overline{N}(r,a;f \mid = k)$ the reduced counting function of those a-points of f whose multiplicity is exactly k, where $k \geq 2$ is an integer.

Definition 1.5. Let f and g be two nonconstant meromorphic functions such that f and g share a value a IM, where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ $(\overline{N}_L(r,a;g))$ the counting function of those a-points of f and g where $p > q$ $(q > p)$, each a-point is counted only once.

Definition 1.6. Let f and g be two nonconstant meromorphic functions and m be a positive integer such that $E_{m)}(a;f) = E_{m)}(a;g)$, where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity $p > 0$, an a-point of g with multiplicity $q > 0$. We denote by $\overline{N}_L^{m)}(r,a;f)$ $(\overline{N}_L^{m)}(r,a;g))$ the counting function of those a-points of f and g where $p > q$ $(q > p)$, each a-point is counted only once.

Definition 1.7. ([2]) Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_E^{(m+1}(r,1;f)$ the counting function of those 1-points of f and g, where $p = q \geq m+1$, each point in this counting function is counted only once.

Definition 1.8. ([8]) We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)$.

Definition 1.9. Let m be a positive integer. Also let z_0 be a zero of $f(z)-a$ of multiplicity p and a zero of $g(z)-a$ of multiplicity q. We denote by $\overline{N}_{f\geq m+1}(r,a;f\mid g\neq a)$ ($\overline{N}_{g\geq m+1}(r,a;g\mid f\neq a)$) the reduced counting functions of those a-points of f and g for which $p\geq m+1$ and $q=0$ ($q\geq m+1$ and $p=0$).

Definition 1.10. ([7] [8]) Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly, $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

For $E_{m)}(a;f)=E_{m)}(a;g)$, we can define $\overline{N}_*(r,a;f,g)$ in a similar manner and we note that here $\overline{N}_*(r,a;f,g)=\overline{N}_L^{m)}(r,a;f)+\overline{N}_L^{m)}(r,a;g)+\overline{N}_{f\geq m+1}(r,a;f\mid g\neq a)+\overline{N}_{g\geq m+1}(r,a;g\mid f\neq a)$.

Definition 1.11. ([10]) Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.12. ([10]) Let $a,b_1,b_2,\ldots,b_q\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid g\neq b_1,b_2,\ldots,b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i=1,2,\ldots,q$.

2. Lemmas

In this section we present some lemmas which are needed in the sequel. Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$. Henceforth, we shall denote by H, Φ and V the following three functions

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right),$$

$$\Phi=\frac{F'}{F-1}-\frac{G'}{G-1}$$

and

$$V=\left(\frac{F'}{F-1}-\frac{F'}{F}\right)-\left(\frac{G'}{G-1}-\frac{G'}{G}\right)=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}.$$

Lemma 2.1. ([11]) *For* $E_{m)}(1;F)=E_{m)}(1;G)$ *and* $H\not\equiv 0$*, then*

$$N(r,1;F\mid=1)=N(r,1;G\mid=1)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.2. *Let* $E_{m)}(1;f)=E_{m)}(1;g)$ *and* $3\leq m<\infty$*. Then*

$$\begin{aligned} & \overline{N}(r,1;f\mid=2)+2\overline{N}(r,1;f\mid=3)+\ldots+ \\ & (m-1)\overline{N}(r,1;f\mid=m)+m\overline{N}_L^{m)}(r,1;f)+(m+1)\overline{N}_L^{m)}(r,1;g) \\ & +m\overline{N}_E^{(m+1}(r,1;f)+m\overline{N}_{g\geq m+1}(r,1;g\mid\ f\neq 1) \\ \leq\ & N(r,1;g)-\overline{N}(r,1;g). \end{aligned}$$

Proof. Since $E_{m)}(1;f)=E_{m)}(1;g)$, we note that the common zeros of $f-1$ and $g-1$ up to multiplicity m are same. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $q=m+1$ the possible values of , p are as follows: (i) $p=m+1$, (ii) $p\geq m+2$, (iii) $p=0$. Similarly, when $q=m+2$, the possible values of p are: (i) $p=m+1$, (ii) $p=m+2$, (iii) $p\geq m+3$, (iv) $p=0$. If $q\geq m+3$ we can similarly find the possible values of p. Now the lemma follows from the above explanation. ∎

Let f and g be two nonconstant meromorphic functions and

$$F=R(f),\qquad G=R(g), \tag{2.1}$$

where $R(w)$ is given by (1.2). From (1.2) and (2.1) it is clear that

$$T(r,f)=\frac{1}{n}T(r,F)+S(r,f),\quad T(r,g)=\frac{1}{n}T(r,G)+S(r,g). \tag{2.2}$$

Lemma 2.3. *Let* F*,* G *be given by (2.1) and* $\omega_1,\omega_2\ldots\omega_n$ *be the roots of* $P(w)=0$*. If* $E_{m)}(1;F)=E_{m)}(1;G)$*, where* $1\leq m<\infty$*, then*

(i) $\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)\leq\dfrac{1}{m}\ [\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_{\otimes}(r,0;f^{'})]+S(r,f)$

(ii) $\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\leq\dfrac{1}{m}\ [\overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_{\otimes}(r,0;g^{'})]+S(r,g)$,

where $N_{\otimes}(r,0;f^{'})=N(r,0;f^{'}\mid f\neq 0,\omega_1,\omega_2\ldots\omega_n)$*.* $N_{\otimes}(r,0;g^{'})$ *is defined similarly.*

P r o o f. We omit the proof since it can be carried out in the line of proof of Lemma 2.4, [2]. ■

Lemma 2.4. ([2]) *Let F and G be given by (2.1) f, g share* $(0,0)$ *and* 0 *be not a Picard exceptional value of f and g. Then $\Phi \equiv 0$ implies $F \equiv G$.*

Lemma 2.5. *Let F, G be given by (2.1) and $H \not\equiv 0$. If $E_{m)}(1;F) = E_{m)}(1;G)$, f, g share (∞,k) and $(0,p)$, where $1 \leq m < \infty$ and $0 \leq p < \infty$, then*

$$\begin{aligned} & [np+n-1]\,\overline{N}(r,0;f\mid\geq p+1) \\ = & [np+n-1]\overline{N}(r,0;g\mid\geq p+1) \\ \leq & \overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)+\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1) \\ & +\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)+\overline{N}_*(r,\infty;f,g)+\overline{N}(r,\alpha_1;f) \\ & +\overline{N}(r,\alpha_2;f)+\overline{N}(r,\alpha_1;g)+\overline{N}(r,\alpha_2;g) \\ & +S(r,f)+S(r,g). \end{aligned}$$

P r o o f. We omit the proof since it is similar to the proof of Lemma 2.6, [2]. ■

Lemma 2.6. ([2]) *Let F and G be given by (2.1) f, g share* $(\infty,0)$ *and* ∞ *be not a Picard exceptional value of f and g. Then $V \equiv 0$ implies $F \equiv G$.*

Lemma 2.7. *Let F, G be given by (2.1) and $H \not\equiv 0$. If $E_{m)}(1;F) = E_{m)}(1;G)$, f, g share (∞,k) and $(0,p)$, where $1 \leq m < \infty$, $0 \leq k < \infty$, then*

$$\begin{aligned} & [(n-2)k+n-3)]\,\overline{N}(r,\infty;f\mid\geq k+1) \\ = & [(n-2)k+n-3)]\overline{N}(r,\infty;g\mid\geq k+1) \\ \leq & \overline{N}_*(r,0;f,g)+\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G) \\ & +\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1) \\ & +\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)+S(r,f)+S(r,g). \end{aligned}$$

P r o o f. We omit the proof which is similar to that of Lemma 2.8, [2]. ■

Lemma 2.8. *Let F, G be given by (2.1) and $H \not\equiv 0$. If $E_{m)}(1;F) = E_{m)}(1;G)$, and f, g share (∞,k), $(0,p)$ where $1 \leq m < \infty$, then*

$$\begin{aligned} N(r,1;F\mid=1) \leq & \overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)+\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G) \\ & +\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1) \\ & +\overline{N}(r,b;f)+\overline{N}(r.b;g)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'}), \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ *denotes the reduced counting function corresponding to the zeros of* $f^{'}$ *which are not the zeros of* $f(f-b)$ *and* $F-1$, $\overline{N}_0(r,0;g^{'})$ *is defined similarly.*

Proof. It is in lines similar to these for the proof of Lemma 2.9, [2]. ■

Lemma 2.9. *Let* F, G *be given by (2.1) and* $H\not\equiv 0$. *If* $E_{m)}(1;F)=E_{m)}(1;G)$, f, g *share* (∞,k), $(0,p)$, *where* $3\le m<\infty$. *Then*

$$\begin{aligned}
& (n+1)T(r,f)+T(r,g)\\
\le\ & \overline{N}(r,0;f)+2\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+2\overline{N}(r,b;g)\\
& +\overline{N}(r,\infty;g)+\overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)-m(r,1;G)\\
& -\overline{N}(r,1;F\mid=3)-\ldots-(m-2)\overline{N}(r,1;F\mid=m)\\
& -(m-2)\overline{N}_L^{m)}(r,1;F)-(m-1)\overline{N}_L^{m)}(r,1;G)\\
& -(m-1)\overline{N}_E^{(m+1}(r,1;F)+2\overline{N}_{F\ge m+1}(r,1;F\mid G\ne 1)\\
& -(m-1)\overline{N}_{G\ge m+1}(r,1;G\mid F\ne 1)+S(r,f)+S(r,g).
\end{aligned}$$

Proof. By the second fundamental theorem, we get

$$\begin{aligned}
(2.3)\qquad & (n+1)T(r,f)+(n+1)T(r,g)\\
\le\ & \overline{N}(r,1;F)+\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G)\\
& +\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;g)-N_0(r,0;f^{'})-N_0(r,0;g^{'})\\
& +S(r,f)+S(r,g).
\end{aligned}$$

Using Lemma 2.2 and Lemma 2.8, we see that

$$\begin{aligned}
(2.4)\qquad & \overline{N}(r,1;F)+\overline{N}(r,1;G)\\
\le\ & N(r,1;F\mid=1)+\overline{N}(r,1;F\mid=2)+\ldots+\overline{N}(r,1;F\mid=m)+\overline{N}_E^{(m+1}(r,1;F)\\
& +\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)+\overline{N}_{F\ge m+1}(r,1;F\mid G\ne 1)+\overline{N}(r,1;G)\\
\le\ & \overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)+\overline{N}(r,b;f)+\overline{N}(r,b;g)+\overline{N}_L^{m)}(r,1;F)\\
& +\overline{N}_L^{m)}(r,1;G)+\overline{N}_{F\ge m+1}(r,1;F\mid G\ne 1)+\overline{N}_{G\ge m+1}(r,1;G\mid F\ne 1)\\
& +\overline{N}(r,1;F\mid=2)+\ldots+\overline{N}(r,1;F\mid=m)+\overline{N}_E^{(m+1}(r,1;F)+\overline{N}_L^{m)}(r,1;F)\\
& +\overline{N}_L^{m)}(r,1;G)+\overline{N}_{F\ge m+1}(r,1;F\mid G\ne 1)+T(r,G)-m(r,1;G)+O(1)\\
& -\overline{N}(r,1;F\mid=2)-2\overline{N}(r,1;F\mid=3)-\ldots-(m-1)\overline{N}(r,1;F\mid=m)\\
& -m\overline{N}_E^{(m+1}(r,1;F)-m\overline{N}_L^{m)}(r,1;F)-(m+1)\overline{N}_L^{m)}(r,1;G)\\
& -m\overline{N}_{G\ge m+1}(r,1;G\mid F\ne 1)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g)
\end{aligned}$$

$$\begin{aligned}
\leq\ & \overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)+\overline{N}(r,b;f)+\overline{N}(r,b;g)\\
&+nT(r,g)-m(r,1;G)-\overline{N}(r,1;F\mid=3)-2\overline{N}(r,1;F\mid=4)-\ldots\\
&-(m-2)\overline{N}(r,1;F\mid=m)-(m-2)\overline{N}_L^{m)}(r,1;F)-(m-1)\overline{N}_L^{m)}(r,1;G)\\
&-(m-1)\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)+2\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)\\
&-(m-1)\overline{N}_E^{(m+1}(r,1;F)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Using (2.4) in (2.3), the lemma follows. ■

Lemma 2.10. ([15]) *If* $H\equiv 0$*, then* F*,* G *share* $(1,\infty)$*.*

Lemma 2.11. ([2]) *Let* F*,* G *be given by (2.1) and* $H\equiv 0$*. If* f*,* g *share* $(0,0)$*, then* f *and* g *share* $(0,\infty)$*.*

3. Proofs of the theorems

Proof of Theorem 1.1. Let F and G be given by (2.1). Since $E_{5)}(S_3,f)=E_{5)}(S_3,f)$ from (1.3) and (2.1) it follows that $E_{5)}(1;F)=E_{5)}(1;G)$. Suppose $H\not\equiv 0$. Then by Lemma 2.9 for $m=5$, $k=0$, $p=5$ we get

$$\begin{aligned}
(3.1)\qquad & (n+1)T(r,f)+T(r,g)\\
\leq\ & \overline{N}(r,0;f)+2\overline{N}(r,b;f)+\overline{N}(r,0;g)+2\overline{N}(r,b;g)+\overline{N}(r,0;f\mid\geq 6)\\
&+3\overline{N}(r,\infty;f)-3\overline{N}_L(r,1;F)-4\overline{N}_L(r,1;G)+2\overline{N}_{F\geq 6}(r,1;F\mid G\neq 1)\\
&-4\overline{N}_{G\geq 6}(r,1;G\mid F\neq 1)+S(r,f)+S(r,g).
\end{aligned}$$

Using Lemma 2.5 for $p=5$ and Lemma 2.7 for $k=0$ and noting that $\overline{N}_*(r,\infty;f,g)\leq\frac{1}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}(r,\infty;g)$ we get

$$\begin{aligned}
(3.2)\qquad & (n+1)T(r,f)+T(r,g)\\
\leq\ & 3T(r,f)+3T(r,g)+\overline{N}(r,0;f\mid\geq 6)+\frac{3}{n-3}\Big[\overline{N}_L^{5)}(r,1;F)\\
&+\overline{N}_L^{5)}(r,1;G)+\overline{N}_{F\geq 6}(r,1;F\mid G\neq 1)N_{G\geq 6}(r,1;G\mid F\neq 1)\\
&++\overline{N}(r,0;f\mid\geq 6)\Big]-3\overline{N}_L^{5)}(r,1;F)-4\overline{N}_L^{5)}(r,1;G)\\
&+2\overline{N}_{F\geq 6}(r,1;F\mid G\neq 1)-4\overline{N}_{G\geq 6}(r,1;G\mid F\neq 1)\\
&+S(r,f)+S(r,g)
\end{aligned}$$

$$
\begin{aligned}
\leq\ & 3T(r,f)+3T(r,g)+\left(\frac{n}{n-3}\right)\ \overline{N}(r,0;f\mid\geq 6)\\
&+\frac{3}{n-3}\Big[\overline{N}_L^{5)}(r,1;F)+\overline{N}_L^{5)}(r,1;G)\\
&+\overline{N}_{F\geq 6}(r,1;F\mid G\neq 1)+\overline{N}_{G\geq 6}(r,1;G\mid F\neq 1)\Big]\\
&-3\overline{N}_L^{5)}(r,1;F)-4\overline{N}_L^{5)}(r,1;G)+2\overline{N}_{F\geq 6}(r,1;F\mid G\neq 1)\\
&-4\overline{N}_{G\geq 6}(r,1;G\mid F\neq 1)+S(r,f)+S(r,g)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & 3T(r,f)+3T(r,g)+\frac{n}{(n-3)(6n-1)}\{2T(r,f)\\
&+2T(r,g)+\overline{N}_*(r,\infty;f,g)\}+\frac{19n-3}{(n-3)(6n-1)}\\
&\Big[\overline{N}_L^{5)}(r,1;F)+\overline{N}_L^{5)}(r,1;G)+\overline{N}_{F\geq 6}(r,1;F\mid G\neq 1)\\
&+\overline{N}_{G\geq 6}(r,1;G\mid F\neq 1)\Big]-3\overline{N}_L^{5)}(r,1;F)\\
&-4\overline{N}_L^{5)}(r,1;G)+2\overline{N}_{F\geq 6}(r,1;F\mid G\neq 1)\\
&-4\overline{N}_{G\geq 6}(r,1;G\mid F\neq 1)+S(r,f)+S(r,g)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \left(3+\frac{\frac{5}{2}n}{(n-3)(6n-1)}\right)T(r,f)+\left(3+\frac{\frac{5}{2}n}{(n-3)(6n-1)}\right)T(r,g)\\
&+\left(2+\frac{19n-3}{(n-3)(6n-1)}\right)\ \{\overline{N}_{F\geq 6}(r,1;F\mid G\neq 1)\}\\
&-\left(4-\frac{19n-3}{(n-3)(6n-1)}\right)\ \{\overline{N}_{G\geq 6}(r,1;G\mid F\neq 1)\}\\
&+S(r,f)+S(r,g).
\end{aligned}
$$

In a similar manner, we can obtain

$$
\begin{aligned}
(3.3)\qquad & T(r,f)+(n+1)T(r,g)\\
\leq\ & \left(3+\frac{\frac{5}{2}n}{(n-3)(6n-1)}\right)T(r,f)+\left(3+\frac{\frac{5}{2}n}{(n-3)(6n-1)}\right)T(r,g)\\
&+\left(2+\frac{19n-3}{(n-3)(6n-1)}\right)\ \overline{N}_{G\geq 6}(r,1;G\mid F\neq 1)\\
&-\left(4-\frac{19n-3}{(n-3)(6n-1)}\right)\ \overline{N}_{F\geq 6}(r,1;F\mid G\neq 1)+S(r,f)+S(r,g).
\end{aligned}
$$

Adding (3.2) and (3.3) and using Lemma 2.3 for $m = 5$, we get

$$\begin{aligned}
&\left(n-4-\frac{5n}{(n-3)(6n-1)}\right)\{T(r,f)+T(r,g)\}\\
\le\ &\left(\frac{38n-6-2(n-3)(6n-1)}{5(n-3)(6n-1)}\right)\big[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\\
&+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\big]+S(r,f)+S(r,g).
\end{aligned}$$

That is,

$$\begin{aligned}
&\left(n-4-\frac{5n}{(6n-1)(n-3)}-\frac{76n-12-4(n-3)(6n-1)}{5(n-3)(6n-1)}\right)\\
&\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction for $n \ge 5$. So $H \equiv 0$. Hence by Lemma 2.10 and Lemma 2.11 we respectively get F and G share $(1,\infty)$ and f, g share $(0,\infty)$. So $E_f(S_3,\infty) = E_g(S_3,\infty)$ and the theorem follows from Theorem A. ■

Proof of Theorem 1.2. Let F and G be given by (2.1). Since $E_{4)}(S_3,f) = E_{4)}(S_3,f)$ from (1.3) and (2.1) it follows that $E_{4)}(1;F) = E_{4)}(1;G)$. Suppose $H \not\equiv 0$. Then by Lemma 2.9 for $m = 4$, $k = 1$, $p = \infty$ we get

$$\begin{aligned}
(3.4)\quad &(n+1)T(r,f)+T(r,g)\\
\le\ &\overline{N}(r,0;f)+2\overline{N}(r,b;f)+\overline{N}(r,0;g)+2\overline{N}(r,b;g)+2\overline{N}(r,\infty;f)\\
&+\overline{N}(r,\infty;f\mid\ge 2)-2\overline{N}_L^{4)}(r,1;F)-3\overline{N}_L^{4)}(r,1;G)\\
&+2\overline{N}_{F\ge5}(r,1;F\mid G\ne1)-3\overline{N}_{G\ge5}(r,1;G\mid F\ne1)+S(r,f)+S(r,g).
\end{aligned}$$

Now using Lemma 2.7 for $k = 1$ and again for $k = 0$ we get

$$\begin{aligned}
(3.5)\qquad &(n+1)T(r,f)+T(r,g)\\
\le\ &3T(r,f)+3T(r,g)+\frac{2}{n-3}\Big[\overline{N}_L^{4)}(r,1;F)+\overline{N}_L^{4)}(r,1;G)\\
&+\overline{N}_{F\ge5}(r,1;F\mid G\ne1)+\overline{N}_{G\ge5}(r,1;G\mid F\ne1)\Big]\\
&+\frac{1}{2n-5}\Big[\overline{N}_L^{4)}(r,1;F)+\overline{N}_L^{4)}(r,1;G)\\
&+\overline{N}_{F\ge5}(r,1;F\mid G\ne1)+\overline{N}_{G\ge5}(r,1;G\mid F\ne1)\Big]\\
&-2\overline{N}_L^{4)}(r,1;F)-3\overline{N}_L^{4)}(r,1;G)+2\overline{N}_{F\ge5}(r,1;F\mid G\ne1)\\
&-3\overline{N}_{G\ge5}(r,1;G\mid F\ne1)+S(r,f)+S(r,g)
\end{aligned}$$

$$\leq 3T(r,f)+3T(r,g)+\left(2+\frac{5n-13}{(n-3)(2n-5)}\right)\overline{N}_{F\geq 5}(r,1;F\mid G\neq 1)$$
$$-\left(3-\frac{5n-13}{(n-3)(2n-5)}\right)\overline{N}_{G\geq 5}(r,1;G\mid F\neq 1)+S(r,f)+S(r,g).$$

In a similar manner, we can obtain

$$(3.6)\qquad T(r,f)+(n+1)T(r,g)$$
$$\leq 3T(r,f)+3T(r,g)+\left(2+\frac{5n-13}{(n-3)(2n-5)}\right)\overline{N}_{G\geq 5}(r,1;G\mid F\neq 1)$$
$$-\left(3-\frac{5n-13}{(n-3)(2n-5)}\right)\overline{N}_{F\geq 5}(r,1;F\mid G\neq 1)+S(r,f)+S(r,g).$$

Adding (3.5) and (3.6) and using Lemma 2.3 for $m=4$, we get

$$\left(n-4-\frac{10n-26-(n-3)(2n-5)}{2(n-3)(2n-5)}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which is a contradiction for $n\geq 5$. So $H\equiv 0$. Hence by Lemma we get F and G share $(1,\infty)$. Hence $E_f(S_3,\infty)=E_g(S_3,\infty)$ and the theorem follows from Theorem A. ∎

Proof of Theorem 1.3. Let F and G be given by (2.1). Since $E_{6)}(S_3,f)=E_{6)}(S_3,g)$ from (1.3) and (2.1) it follows that $E_{6)}(1;F)=E_{6)}(1;G)$. Suppose $H\not\equiv 0$. Then using Lemma 2.9 for $m=6$, $k=0$, $p=3$; Lemma 2.5 for $p=3$ and Lemma 2.7 for $k=0$, we obtain

$$(3.7)\qquad (n+1)T(r,f)+T(r,g)$$
$$\leq 3T(r,f)+3T(r,g)+\overline{N}(r,0;f\mid\geq 4)+\frac{3}{n-3}\Big[\overline{N}_L^{6)}(r,1;F)+\overline{N}_L^{6)}(r,1;G)$$
$$+\overline{N}_{F\geq 7}(r,1;F\mid G\neq 1)+\overline{N}_{G\geq 7}(r,1;G\mid F\neq 1)+\overline{N}(r,0;f\mid\geq 4)\Big]$$
$$-4\overline{N}_L^{6)}(r,1;F)-5\overline{N}_L^{6)}(r,1;G)+2\overline{N}_{F\geq 7}(r,1;F\mid G\neq 1)$$
$$-5\overline{N}_{G\geq 7}(r,1;G\mid F\neq 1)+S(r,f)+S(r,g)$$
$$\leq \left(3+\frac{\frac{5}{2}n}{(n-3)(4n-1)}\right)T(r,f)+\left(3+\frac{\frac{5}{2}n}{(n-3)(4n-1)}\right)T(r,g)$$
$$+\left(2+\frac{13n-3}{(n-3)(4n-1)}\right)\{\overline{N}_{F\geq 7}(r,1;F\mid G\neq 1)\}$$
$$-\left(5-\frac{13n-3}{(n-3)(4n-1)}\right)\{\overline{N}_{G\geq 7}(r,1;G\mid F\neq 1)\}+S(r,f)+S(r,g).$$

In a similar manner, we can obtain

$$
\begin{aligned}
(3.8)\qquad & T(r,f)+(n+1)T(r,g)\\
\le\ & \left(3+\frac{\frac{5}{2}n}{(n-3)(4n-1)}\right)T(r,f)+\left(3+\frac{\frac{5}{2}n}{(n-3)(4n-1)}\right)T(r,g)\\
&+\left(2+\frac{13n-3}{(n-3)(4n-1)}\right)\overline{N}_{G\ge 7}(r,1;G\mid F\ne 1)\\
&-\left(5-\frac{13n-3}{(n-3)(4n-1)}\right)\overline{N}_{F\ge 7}(r,1;F\mid G\ne 1)\\
&+S(r,f)+S(r,g).
\end{aligned}
$$

Adding (3.7) and (3.8), we get using Lemma 2.3 for $m=6$

$$
\begin{aligned}
&\left(n-4-\frac{5n}{(n-3)(4n-1)}-\frac{26n-6-3(n-3)(4n-1)}{3(n-3)(4n-1)}\right)\\
&\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),
\end{aligned}
$$

which is a contradiction for $n\ge 5$. So $H\equiv 0$. Now we follow the proof of Theorem 1.1. ∎

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