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A Unified Theory of Upper and Lower Almost Nearly Continuous Multifunctions

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Recently Ekici [10] has introduced the notion of upper/lower almost nearly continuous multifunctions as a generalization of upper/lower nearly continuous multifunctions [9] and upper/lower almost continuous multifunctions [28]. In this paper, we define and investigate the unified form of several generalizations of upper/lower almost nearly continuous multifunctions.

Key words: m-open, m-structure, m-space, N-continuous, upper/lower almost nearly continuous, upper/lower almost nearly m-continuous, multifunction.

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1. Introduction

The notion of N-closed sets in a topological space was introduced in [6]. The notion of N-continuous functions was introduced in [17] and studied in [21], [23] and other papers. Recently, Ekici [9] introduced and studied upper/lower nearly continuous multifunctions as a generalization of upper/lower semi-continuous multifunctions and N-continuous functions. Furthermore, Ekici [10] introduced the notions of upper/lower almost nearly continuous multifunctions as a generalization of upper/lower nearly continuous multifunctions and upper/lower almost continuous multifunctions [28]. Quite recently, Rychlewicz [35] has introduced the notions of upper/lower almost nearly quasi-continuous multifunctions as a generalization of upper/lower almost nearly continuous multifunctions and upper/lower almost quasi continuous multifunctions [33]. In [31], the present authors introduced and studied the notions of upper/lower m-continuous multifunctions.

In this paper, we introduce and study the notion of upper/lower almost nearly m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space. The multifunction is a generalization of upper/lower m-continuous multifunctions and upper/lower almost nearly continuous multifunctions. We obtain several characterizations and properties of such multifunctions. They turn out generalizations of the results established in [10] and [35]. In the last section, we recall some types of modifications of open sets and define new forms of upper/lower almost nearly continuous multifunctions. Futhermore, we point out that characterizations and properties of upper/lower almost nearly m-continuous multifunctions can be applied to these new forms.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $\mathrm{Int}(\mathrm{Cl}(A)) = A$ (resp. $\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{A}$).

Definition 2.1. A subset A of a topological space (X, τ) is said to be *N-closed relative to X* (briefly *N-closed*) [6] if every cover of A by regular open sets of X has a finite subcover.

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be *α-open* [20] (resp. *semi-open* [15], *preopen*[18], *β-open* [1] or *semi-preopen* [3], *b-open* [4]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)), A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$, $\mathrm{BO}(X)$).

Definition 2.3. The complement of a semi-open (resp. preopen, α-open, β-open , semi-preopen, b-open) set is said to be *semi-closed* [8] (resp. *preclosed* [11], *α-closed* [19], *β-closed* [1], *semi-preclosed* [3], *b-closed* [4]).

Definition 2.4. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, semi-preclosed, b-closed) sets of X containing A is called the *semi-closure* [8] (resp. *preclosure* [11], *α-closure* [19], *β-closure* [2], *semi-preclosure* [3], *b-closure* [4]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $\mathrm{spCl}(A)$, $\mathrm{bCl}(A)$).

Definition 2.5. The union of all semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *semi-preinterior*, *b-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_{\beta}\mathrm{Int}(A)$, $\mathrm{spInt}(A)$, $\mathrm{bInt}(A)$).

The following lemma is a generalization of Lemma 1 of [35].

Lemma 2.1. *Let (X,τ) be a topological space. If V is a preopen set of X having N-closed complement, then* $\mathrm{Int}(\mathrm{Cl}(V))$ *is a regular open set having N-closed complement.*

Proof. Since V is preopen, $V \subset \mathrm{Int}(\mathrm{Cl}(V))$ and hence $X - \mathrm{Int}(\mathrm{Cl}(V)) \subset X - V$. By the hypothesis, $X - V$ is N-closed in X and $X - \mathrm{Int}(\mathrm{Cl}(V))$ is regular closed. Therefore, it follows from Theorem 2.8 of [6] that $X - \mathrm{Int}(\mathrm{Cl}(V))$ is N-closed. It is obvious that $\mathrm{Int}(\mathrm{Cl}(V))$ is regular open. ■

Definition 2.6. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be *N-continuous* at $x \in X$ [17] if for each open set V of Y containing $f(x)$ and having N-closed complement, there exists an open set U containing x such that $f(U) \subset V$. The function is said to be *N-continuous* if it has this property at each point of X.

Throughout the present paper, (X,τ) and (Y,σ) (briefly X and Y) always denote topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.7. A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be

(1) *upper nearly continuous* (briefly *u.n.c.*) at a point $x \in X$ [9] if for each open set V containing $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(U) \subset V$,

(2) *lower nearly continuous* (briefly *l.n.c.*) at a point $x \in X$ [9] if for each open set V meeting $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly continuous* on X if it has this property at each point of X.

Definition 2.8. A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be

(1) *upper almost nearly continuous* (briefly *u.a.n.c.*) [10] (resp. *upper almost nearly quasi-continuous* (briefly *u.a.n.q.c.* [35]) at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there

exists an open (resp. a semi-open) set U of X containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$,

(2) *lower almost nearly continuous* (briefly *l.a.n.c.*) [10] (resp. *lower almost nearly quasi-continuous* (briefly *l.a.n.q.c.* [35]) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists an open (resp. a semi-open) set U of X containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$,

(3) *upper/lower almost nearly continuous* (resp. *upper/lower almost nearly quasi-continuous*) on X if it has this property at each point of X.

3. Almost nearly m-continuous multifunctions

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [29], [3] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be m_X*-open* (briefly *m-open*) and the complement of an m_X-open set is said to be m_X*-closed* (briefly *m-closed*).

Remark 3.1 Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\mathrm{SPO}(X)$ are all m-structures on X.

Definition 3.2. Let (X, m_X) be an m-space. For a subset A of X, the m_X*-closure* of A and the m_X*-interior* of A are defined in [16] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A be a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$, $\mathrm{SPO}(X)$), then we have

(a) $\mathrm{mCl}(A) = \mathrm{Cl(A)}$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\mathrm{bCl}(A)$, $\mathrm{spCl}(A)$),

(b) $\mathrm{mInt}(A) = \mathrm{Int(A)}$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\mathrm{bInt}(A)$, $\mathrm{spInt}(A)$).

Lemma 3.1. *(Maki et al. [16]). Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,

(2) If $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) If $A \subset B$, then $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 3.2. *(Popa and Noiri [30]). Let (X, m_X) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [16] if the union of any family of subsets belonging to m_X belongs to m_X.

Remark 3.3. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\mathrm{SPO}(X)$ have property $\mathcal{B}$.

Lemma 3.3 *(Popa and Noiri [32]). For an m-structure m_X on a nonempty set X, the following properties are equivalent:*
(1) m_X has property $\mathcal{B}$;
(2) If m_X-Int$(A) = A$, then $A \in m_X$;
(3) If m_X-Cl$(A) = A$, then A is m_X-closed.

Definition 3.4. Let (X, m_X) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper nearly m-continuous* (briefly *u.n.m.c.*) at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an m_X-open set U containing x such that $F(U) \subset V$,

(2) *lower nearly m-continuous* (briefly *l.n.m.c.*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists an m_X-open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower nearly m-continuous* on X if it has this property at every point of X.

Definition 3.5. Let (X, m_X) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper almost nearly m-continuous* (briefly *u.a.n.m.c.*) at a point $x \in X$ if for each open set V containing $F(x)$, there exists an m_X-open set U containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$,

(2) *lower almost nearly m-continuous* (briefly *l.a.n.m.c.*) at a point $x \in X$ if for each open set V meeting $F(x)$, there exists an m_X-open set U containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$,

(3) *upper/lower almost nearly m-continuous* on X if it has this property at every point of X.

Theorem 3.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.a.n.m.c. at $x \in X$;

(2) $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y containing $F(x)$ and having N-closed complement;

(3) $x \in \mathrm{mInt}(F^+(\mathrm{sCl}(V)))$ for each open set V of Y containing $F(x)$ and having N-closed complement;

(4) $x \in \mathrm{mInt}(F^+(V))$ for each regular open set V of Y containing $F(x)$ and having N-closed complement;

(5) for each regular open set V of Y containing $F(x)$ and having N-closed complement, there exists $U \in m_X$ containing x such that $F(U) \subset V$.

Proof. (1) $\Rightarrow$ (2): Let V be any open set of Y containing $F(x)$ and having N-closed complement. There exists $U \in m_X$ containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Thus we have $x \in U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ and hence $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$.

(2) $\Rightarrow$ (3): This follows from Lemma 3.2 of [22].

(3) $\Rightarrow$ (4): Let V be any regular open set of Y containing $F(x)$ and having N-closed complement. Then it follows from Lemma 3.2 of [22] that $V = \mathrm{Int}(\mathrm{Cl}(V)) = \mathrm{sCl}(V)$.

(4) $\Rightarrow$ (5): Let V be any regular open set of Y containing $F(x)$ and having N-closed complement. By (4), $x \in \mathrm{mInt}(F^+(V))$ and hence there exists $U \in m_X$ such that $x \in U \subset F^+(V)$; hence $F(U) \subset V$.

(5) $\Rightarrow$ (1): Let V be any open set of Y containing $F(x)$ and having N-closed complement. By Lemma 2.1, $\mathrm{Int}(\mathrm{Cl}(V))$ is regular open set of Y containing $F(x)$ and having N-closed complement and hence there exists $U \in m_X$ containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. This shows that F is *u.a.n.m.c.* ■

Theorem 3.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is l.a.n.m.c. at $x \in X$;

(2) $x \in \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y meeting $F(x)$ and having N-closed complement;

(3) $x \in \mathrm{mInt}(F^-(\mathrm{sCl}(V)))$ for each open set V of Y meeting $F(x)$ and having N-closed complement;

(4) $x \in \mathrm{mInt}(F^-(V))$ for each regular open set V of Y meeting $F(x)$ and having N-closed complement;

(5) for each regular open set V of Y meeting $F(x)$ and having N-closed complement, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

P r o o f. The proof is similar to that of Theorem 3.1. ■

Theorem 3.3. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.a.n.m.c.;

(2) $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y having N-closed complement;

(3) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^-(K)$ for every closed N-closed set K of Y;

(4) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y having N-closed closure;

(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$ for every subset B of Y such that $Y - \mathrm{Int}(B)$ is N-closed;

(6) $F^+(V) = \mathrm{mInt}(F^+(V))$ for every regular open set V of Y having N-closed complement;

(7) $F^-(K) = \mathrm{mCl}(F^-(K))$ for every N-closed and regular closed set K of Y.

P r o o f. (1) $\Rightarrow$ (2): Let V be any open set of Y having N-closed complement and $x \in F^+(V)$. Then $F(x) \subset V$. By Theorem 3.1, we have $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$. This shows that $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$.

(2) $\Rightarrow$ (3): Let K be any N-closed and closed set of Y. Then, $Y - K$ is an open set of Y having N-closed complement. By (2) and Lemma 3.1, we have $X - F^-(K) = F^+(Y - K) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(Y - K)))) = \mathrm{mInt}(X - F^-(\mathrm{Cl}(\mathrm{Int}(K)))) = X - \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K))))$. Therefore, we obtain $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y having the N-closed closure. Then $\mathrm{Cl}(B)$ is a closed N-closed subset of Y and by (3) we obtain

$$\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}(B)).$$

(4) $\Rightarrow$ (5): Let B be a subset of Y such that $Y - \mathrm{Int}(B)$ is N-closed. Then since $Y - \mathrm{Int}(B)$ is closed and N-closed, we have

$$\begin{aligned} F^+(\mathrm{Int}(B)) &= X - F^-(Y - \mathrm{Int}(B)) \\ &= X - F^-(\mathrm{Cl}(Y - B)) \subset X - \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(Y - B))))) \\ &= X - \mathrm{mCl}(F^-(Y - \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B))))) = \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B))))). \end{aligned}$$

Therefore, we obtain $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$.

(5) $\Rightarrow$ (6): Let V be any regular open set of Y having N-closed complement. Then $Y - \mathrm{Int}(V)$ is N-closed and by (5) we have $F^+(V) \subset \mathrm{mInt}(F^+(V))$. By Lemma 3.1, we have $F^+(V) = \mathrm{mInt}(F^+(V))$.

(6) $\Rightarrow$ (7): Let K be any regular closed N-closed set of Y. Then $Y - K$ is a regular open set having N-closed complement. By (6) and Lemma 3.1, we

obtain $X - F^-(K) = F^+(Y - K) = \mathrm{mInt}(F^+(Y - K)) = \mathrm{mInt}(X - F^-(K)) = X - \mathrm{mCl}(F^-(K))$. Therefore, we obtain $F^-(K) = \mathrm{mCl}(F^-(K))$.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be any regular open set of Y containing $F(x)$ and having N-closed complement. Then $Y - V$ is regular closed and N-closed. By (7) and Lemma 3.1, we have $X - F^+(V) = F^-(Y - V) = \mathrm{mCl}(F^-(Y - V)) = X - \mathrm{mInt}(F^+(V))$. Therefore, we have $x \in F^+(V) = \mathrm{mInt}(F^+(V))$. Then, there exists $U \in m_X$ containing $F(x)$ such that $F(U) \subset V$. It follows from Theorem 3.1 that F is *u.a.n.m.c.* at $x \in X$. ■

Theorem 3.4. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is l.a.n.m.c.;

(2) $F^-(V) \subset \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y having N-closed complement;

(3) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^+(K)$ for every closed N-closed set K of Y;

(4) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y having N-closed closure;

(5) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$ for every subset B of Y such that $Y - \mathrm{Int}(B)$ is N-closed;

(6) $F^-(V) = \mathrm{mInt}(F^-(V))$ for every regular open set V of Y having N-closed complement;

(7) $F^+(K) = \mathrm{mCl}(F^+(K))$ for every N-closed and regular closed set K of Y.

Proof. The proof is similar to that of Theorem 3.3. ■

Corollary 3.1. *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.a.n.m.c.;

(2) $F^+(V)$ is m_X-open for each regular open set V of Y having N-closed complement;

(3) $F^-(K)$ is m_X-closed for every N-closed and regular closed set K of Y.

Proof. This is an immediate consequence of Theorem 3.3 and Lemma 3.3. ■

Corollary 3.2. *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is l.a.n.m.c.;

(2) $F^-(V)$ is m_X-open for each regular open set V of Y having N-closed complement;

(3) $F^+(K)$ is m_X-closed for every N-closed and regular closed set K of Y.

Proof. This is an immediate consequence of Theorem 3.4 and Lemma 3.3. ■

Remark 3.4. Let (X, τ) and (Y, σ) be topological spaces and $m_X = \tau$ (resp. $\mathrm{SO}(X)$).

(1) $F : (X, m_X) \to (Y, \sigma)$ is *u.a.n.m.c.*, then by Theorem 3.3 and Corollary 3.1 we obtain the results established in Theorem 3 of [10] (resp. Theorem 1 of [35]).

(2) $F : (X, m_X) \to (Y, \sigma)$ is *l.a.n.m.c.*, then by Theorem 3.4 and Corollary 3.2 we obtain the results established in Theorem 6 of [10] (resp. Theorem 2 of [35]).

Corollary 3.3. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction. If $F^-(K) = \mathrm{mCl}(F^-(K))$ (resp. $F^+(K) = \mathrm{mCl}(F^+(K))$) for every N-closed set K of Y, then F is u.a.n.m.c. (resp. l.a.n.m.c.).*

Proof. Let G be any regular open set of Y having N-closed complement. Then $Y - G$ is N-closed and regular closed. By the hypothesis, $X - F^+(G) = F^-(Y - G) = \mathrm{mCl}(F^-(Y - G)) = \mathrm{mCl}(X - F^+(G)) = X - \mathrm{mInt}(F^+(G))$ and hence, $F^+(G) = \mathrm{mInt}(F^+(G))$. It follows from Theorem 3.3 that F is *u.a.n.m.c.* The proof of lower almost near m-continuity is entirely similar. ■

Theorem 3.5. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.a.n.m.c.;

(2) $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ for every $V \in \beta(Y)$ having N-closed closure;

(3) $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ for every $V \in \mathrm{SO}(Y)$ having N-closed closure;

(4) $F^+(V)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for every $V \in \mathrm{PO}(Y)$ having N-closed complement.

Proof. (1) $\Rightarrow$ (2): Let be a β-open set of Y having N-closed closure. It follows from Theorem 2.4 of [3] that $\mathrm{Cl}(V)$ is regular closed. Since F is *u.a.n.m.c.*, by Theorem 3.3, $F^-(\mathrm{Cl}(V)) = \mathrm{mCl}(F^-(\mathrm{Cl}(V)))$. By Lemma 3.1, $\mathrm{mCl}(F^-(V)) \subset \mathrm{mCl}(F^-(\mathrm{Cl}(V))) = F^-(\mathrm{Cl}(V))$.
(2) $\Rightarrow$ (3): The proof is obvious since every semi-open set is β-open.
(3) $\Rightarrow$ (4): Let V be any preopen set having N-closed complement. Then by Lemma 2.1, $\mathrm{Int}(\mathrm{Cl}(V))$ is a regular open set having N-closed complement. Then $X - \mathrm{Int}(\mathrm{Cl}(V))$ is a regular closed and N-closed set. Therefore, $X - \mathrm{Int}(\mathrm{Cl}(V))$ is a semi-open set having N-closed closure. By (3), we have

$$\begin{aligned} X - \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) &= \mathrm{mCl}(F^-(Y - \mathrm{Int}(\mathrm{Cl}(V)))) \\ \subset F^-(\mathrm{Cl}(Y - \mathrm{Int}(\mathrm{Cl}(V)))) &= X - F^+(\mathrm{Int}(\mathrm{Cl}(V))) \subset X - F^+(V). \end{aligned}$$

Therefore, we obtain $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$.

(4) $\Rightarrow$ (1): Let V be any regular open set having N-closed complement. Then V is preopen set having N-closed complement and hence $F^+(V)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) = \mathrm{mInt}(F^+(V))$. By Lemma 3.1, $F^+(V) = \mathrm{mInt}(F^+(V))$. By Theorem 3.3, F is *u.a.n.m.c.* ■

Theorem 3.6. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is l.a.n.m.c.;

(2) $\mathrm{mCl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ for every $V \in \beta(Y)$ having N-closed closure;

(3) $\mathrm{mCl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ for every $V \in \mathrm{SO}(Y)$ having N-closed closure;

(4) $F^-(V)) \subset \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ for every $V \in \mathrm{PO}(Y)$ having N-closed complement.

Proof. The proof is similar to that of Theorem 3.5. ■

Lemma 3.4. *(Noiri [22]) For a subset V of a topological space (Y, σ), the following properties hold:*

(1) $\alpha\mathrm{Cl}(V) = \mathrm{Cl}(V)$ for every $V \in \beta(Y)$,

(2) $\mathrm{pCl}(V) = \mathrm{Cl}(V)$ for every $V \in \mathrm{SO}(Y)$.

Corollary 3.4. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.a.n.m.c.;

(2) $\mathrm{mCl}(F^-(V)) \subset F^-(\alpha\mathrm{Cl}(V))$ for every $V \in \beta(Y)$ having N-closed

closure;

(3) $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{pCl}(V))$ *for every* $V \in \mathrm{SO}(Y)$ *having N-closed closure.*

Corollary 3.5. *For a multifunction* $F : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F is l.a.n.m.c.;

(2) $\mathrm{mCl}(F^+(V)) \subset F^+(\alpha\mathrm{Cl}(V))$ *for every* $V \in \beta(Y)$ *having N-closed closure;*

(3) $\mathrm{mCl}(F^+(V)) \subset F^+(\mathrm{pCl}(V))$ *for every* $V \in \mathrm{SO}(Y)$ *having N-closed closure.*

Definition 3.6. A subset A of a topological space (X, τ) is said to be

(1) *α-paracompact*[37] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X,

(2) *α-regular* [14] if for each $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

Lemma 3.5. *(Kovačević [14])* If B is an α-regular and α-paracompact set of a topological space (Y, σ) and V is an open neighborhood of B, then there exists an open set G of Y such that $B \subset G \subset \mathrm{Cl}(G) \subset V$.

For a multifunction $F : X \to (Y, \sigma)$, a multifunction $\mathrm{Cl}F : X \to (Y, \sigma)$ is defined in [5] as follows: $(\mathrm{Cl}F)(x) = \mathrm{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{spCl}F$, and $\mathrm{bCl}F$.

Lemma 3.6. *If* $F : (X, m_X) \to (Y, \sigma)$ *is a multifunction such that* $F(x)$ *is α-paracompact and α-regular for each* $x \in X$*, then for each regular open set V of Y* $F^+(V) = G^+(V)$*, where G denotes* $\mathrm{Cl}F$*,* $\alpha\mathrm{Cl}F$*,* $\mathrm{sCl}F$*,* $\mathrm{pCl}F$*,* $\mathrm{bCl}F$ *or* $\mathrm{spCl}F$*.*

Proof. Let V be any regular open set of Y. Suppose that $x \in G^+(V)$. Then $G(x) \subset V$ and $F(x) \subset G(x) \subset V$. We have $x \in F^+(V)$ and hence $G^+(V) \subset F^+(V)$. Conversely, let $x \in F^+(V)$. Then we have $F(x) \subset V$ and by Lemma 3.5 there exists an open set H of Y such that $F(x) \subset H \subset \mathrm{Cl}(H) \subset V$. Since $G(x) \subset \mathrm{Cl}(F(x))$, $G(x) \subset V$ and hence $x \in G^+(V)$. Thus we obtain $F^+(V) \subset G^+(V)$. Therefore, $F^+(V) = G^+(V)$. ■

Theorem 3.7. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is u.a.n.m.c. if and only if $G : (X, m_X) \to (Y, \sigma)$ is u.a.n.m.c., where G denotes* ClF, αClF, sClF, pClF, bClF *or* spClF.

Proof. *Necessity.* Suppose that F is *u.a.n.m.c.* Let V be any regular open set of Y having N-closed complement. It follows from Lemma 3.6 and Theorem 3.3 that $G^+(V) = F^+(V) = \mathrm{mInt}(F^+(V)) = \mathrm{mInt}(G^+(V))$. Therefore, by Theorem 3.3, G is *u.a.n.m.c.*

Sufficiency. Suppose that G is *u.a.n.m.c.* Let V be any regular open set of Y having N-closed complement. By Lemma 3.6 and Theorem 3.3, $F^+(V) = G^+(V) = \mathrm{mInt}(G^+(V)) = \mathrm{mInt}(F^+(V))$. By Theorem 3.3, F is *u.a.n.m.c.* ■

Lemma 3.7. *If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction, then for each regular open set V of Y $F^-(V) = G^-(V)$, where G denotes* ClF, αClF, sClF, pClF, bClF *or* spClF.

Proof. Let V be any regular open set of Y. Suppose that $x \in G^-(V)$. Then $G(x) \cap V \neq \emptyset$ and hence $F(x) \cap V \neq \emptyset$ since V is open. We have $x \in F^-(V)$ and hence $G^-(V) \subset F^-(V)$. Conversely, let $x \in F^-(V)$. Then we have $\emptyset \neq F(x) \cap V \subset G(x) \cap V$ and hence $x \in G^-(V)$. Thus we obtain $F^-(V) \subset G^-(V)$. Therefore, $F^-(V) = G^-(V)$.

Theorem 3.8. *A multifunction $F : (X, m_X) \to (Y, \sigma)$ is l.a.n.m.c. if and only if $G : (X, m_X) \to (Y, \sigma)$ is l.a.n.m.c., where G denotes* ClF, αClF, sClF, pClF, bClF *or* spClF.

Proof. By using Lemma 3.7, this is shown similarly as in Theorem 3.7.

We recall the definition of the δ-closure due to Veličko [36]. Let (Y, σ) be a topological space and B a subset of Y. A point $y \in Y$ is called a δ-cluster point of B if $\mathrm{Int}(\mathrm{Cl}(V)) \cap B \neq \emptyset$ for every open set V of Y containing y. The set of all δ-cluster points of B is called the *δ-closure* of B and is denoted by $\mathrm{Cl}_\delta(B)$ [36]. A subset B is said to be *δ-closed* if $\mathrm{Cl}_\delta(B) = B$. The complement of a δ-closed set is said to be *δ-open*. The union of all δ-open sets contained in the subset B is called the δ-interior of B and is denoted by $\mathrm{Int}_\delta(B)$. ■

Theorem 3.9. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.a.n.m.c.;

(2) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B))))) \subset F^-(\mathrm{Cl}_\delta(B))$ *for every subset B of Y having N-closed δ-closure;*

(3) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}_\delta(B))$ *for every subset B of Y having N-closed δ-closure.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y such that $\mathrm{Cl}_\delta(B)$ is N-closed. By Lemma 2 of [36], $\mathrm{Cl}_\delta(B)$ is closed. Then $\mathrm{Cl}_\delta(B)$ is closed N-closed and by Theorem 3.3 we have $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B))))) \subset F^-(\mathrm{Cl}_\delta(B))$.

(2) $\Rightarrow$ (3): This is obvious since $\mathrm{Cl}(B) \subset \mathrm{Cl}_\delta(B)$.

(3) $\Rightarrow$ (1): Let K be any N-closed regular closed set of Y. Then by (3) and Theorem 2.1 of [13], we have $\mathrm{mCl}(F^-(K)) = \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) = \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(K))))) \subset F^-(\mathrm{Cl}_\delta(K)) = F^-(K)$. Hence $F^-(K) = \mathrm{mCl}(F^-(K))$. By Theorem 3.3, F is *u.a.n.m.c.* ■

Theorem 3.10. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is l.a.n.m.c.;

(2) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B))))) \subset F^+(\mathrm{Cl}_\delta(B))$ *for every subset B of Y having N-closed δ-closure;*

(3) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^+(\mathrm{Cl}_\delta(B))$ *for every subset B of Y having N-closed δ-closure.*

Proof. The proof is similar to that of Theorem 3.9. ■

4. Almost near m-continuity and near m-continuity

Definition 4.1. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper m-continuous* (briefly *u.m.c.*) [31] at a point $x \in X$ if for each open set V containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$,

(2) *lower m-continuous* (briefly *l.m.c.*) [31] at a point $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower m-continuous* if it has this property at each point of X.

Lemma 4.1. *(Popa and Noiri [25]) A multifunction $F : (X, m_X) \to (Y, \sigma)$ is u.m.c. if and only if $F^+(V) = \mathrm{mInt}(F^+(V))$ for every open set V of Y.*

Lemma 4.2. *(Popa and Noiri [25]) A multifunction $F : (X, m_X) \to (Y, \sigma)$ is l.m.c. if and only if $F^-(V) = \mathrm{mInt}(F^-(V))$ for every open set V of Y.*

Theorem 4.1. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that (Y, σ) has a base of regular open sets having N-closed complements and m_X has property $\mathcal{B}$. If F is l.a.n.m.c., then F is l.m.c.*

Proof. Let V be any open set of Y. By the hypothesis, $V = \cup_{i\in I} V_i$, where V_i is a regular open set having N-closed complement for each $i \in I$. Since m_X has property $\mathcal{B}$, by Corollary 3.2 $F^-(V_i) \in m_X$ for each $i \in I$. Moreover, $F^-(V) = F^-(\cup\{V_i : i \in I\}) = \cup\{F^-(V_i) : i \in I\}$. Therefore, we have $F^-(V) \in m_X$. Then by Lemma 4.2 and Lemma 3.3 F is *l.m.c.* ■

Remark 4.1. Let (X, τ) and (Y, σ) be topological spaces and $m_X = \tau$. If $F : (X, m_X) \to (Y, \sigma)$ be *l.a.n.m.c.*, then by Theorem 4.1 we obtain the results established in Theorem 5 of [9] and Theorem 15 of [10].

Definition 4.2. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *a*-m-continuous* if $X - F^-(\mathrm{Fr}(V)) \in m_X$ for each open set V of Y, where $\mathrm{Fr}(V)$ is the frontier of V.

Theorem 4.2. *Let X be a nonempty set with two m-structures m_1 and m_2 on X such that $U \cap V \in m_1$ whenever $U \in m_1$ and $V \in m_2$. If $F_1 : (X, m_1) \to (Y, \sigma)$ is u.a.n.m.c. and $F_2 : (X, m_2) \to (Y, \sigma)$ is a*-m-continuous, where $F_1(x) = F_2(x)$ for each $x \in X$, then $F_1 : (X, m_1) \to (Y, \sigma)$ is u.n.m.c.*

Proof. Let $x \in X$ and V be any open set of Y containing $F(x)$ and having N-closed complement. It follows that $F_2(x) \cap \mathrm{Fr}(V) = \emptyset$. Since F_1 is *u.a.n.m.c.*, there exists $G \in m_1$ containing x such that $F_1(G) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Put $U = G \cap (X - F_2^-(\mathrm{Fr}(V)))$. Since F_2 is *a*-m*-continuous, $X - F_2^-(\mathrm{Fr}(V)) \in m_2$. Then $x \in U, U \in m_1$, and $F_1(U) \subset F_1(G) \cap F_1(F_2^+(Y - \mathrm{Fr}(V))) \subset \mathrm{Int}(\mathrm{Cl}(V)) \cap (Y - \mathrm{Fr}(V)) = V$. This shows that F_1 is *u.n.m.c.* ■

Theorem 4.3. *Let X be a nonempty set with two m-structures m_1 and m_2 on X such that $U \cap V \in m_1$ whenever $U \in m_1$ and $V \in m_2$. If $F_2 : (X, m_2) \to (Y, \sigma)$ is u.a.n.m.c. and $F_1 : (X, m_1) \to (Y, \sigma)$ is a*-m-continuous, where $F_1(x) = F_2(x)$ for each $x \in X$, then $F_1 : (X, m_1) \to (Y, \sigma)$ is u.n.m.c.*

Proof. The proof is similar to that of Theorem 4.2. ■

Theorem 4.4. *Let $F : (X, m_X) \to (Y, \sigma)$ and $G : (Y, \sigma) \to (Z, \theta)$ be multifunctions. If F is u.m.c. (resp. l.m.c.) and G is u.a.n.c. (resp. l.a.n.c.), then $G \circ F : (X, m_X) \to (Z, \theta)$ is u.a.n.m.c. (resp. l.a.n.m.c.)*

Proof. Let V be any regular open set of Z having N-closed complement. Since G is *u.a.n.c.* (resp. *l.a.n.c.*), by Theorem 3 of [10] $F^+(V)$ (resp. $F^-(V)$) is an open set of Y. Since F is *u.m.c.* (resp. *l.m.c.*), by Lemma 4.1 (resp. Lemma 4.2) $(G \circ F)^+(V) = F^+(G^+(V)) = \mathrm{mInt}(F^+(G^+(V))) = \mathrm{mInt}((G \circ F)^+(V))$ (resp. $(G \circ F)^-(V) = F^-(G^-(V)) = \mathrm{mInt}(F^-(G^-(V))) = \mathrm{mInt}((G \circ F)^-(V))$). By Theorem 3.3 (resp. Theorem 3.4) F is *u.a.n.m.c.* (resp. *l.a.n.m.c.*) ■

Remark 4.2. If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction and $m_X = \tau$, then by Theorem 4.4 we obtain the result established in Theorem 17 of [10].

5. Some properties

Definition 5.1. A topological space (Y, σ) is said to be *N-normal* [9] if for each disjoint closed sets K and H of Y, there exist open sets U and V having N-closed complement such that $K \subset U, H \subset V$ and $U \cap V = \emptyset$.

Definition 5.2. An m-space (X, m_X) is said to be m-T_2 [29] if for each distinct points $x, y \in X$, there exist $U, V \in m_X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Theorem 5.1. *If $F : (X, m_X) \to (Y, \sigma)$ is an u.a.n.m.c. multifunction satisfying the following conditions:*

(1) F(x) is closed in Y for each $x \in X$,

(2) $F(x) \cap F(y) = \emptyset$ for each distinct points $x, y \in X$,

(3) (Y, σ) is an N-normal space, and

(4) m_X has property $\mathcal{B}$,

then (X, m_X) is m-T_2.

Proof. Let x and y be distinct points of X. Then, we have $F(x) \cap F(y) = \emptyset$. Since $F(x)$ and $F(y)$ are closed and Y is N-normal, there exist disjoint open sets U and V of Y having N-closed complement such that $F(x) \subset U$ and $F(y) \subset V$. By Corollary 3.1, we obtain $x \in F^+(\mathrm{Int}(\mathrm{Cl}(U))) \in m_X, y \in F^+(\mathrm{Int}(\mathrm{Cl}(V))) \in m_X$ and $F^+(\mathrm{Int}(\mathrm{Cl}(U))) \cap F^+(\mathrm{Int}(\mathrm{Cl}(V))) = \emptyset$. This shows that X is m-T_2. ■

Remark 5.1. If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction and $m_X = \tau$, then by Theorem 5.1 we obtain the result established in Theorem 15 of [10].

Definition 5.3. A subset A of an m-space (X, m_X) is said to be *m-dense* on X [26] if $\mathrm{mCl}(A) = X$.

Theorem 5.2. *Let X be a nonempty set with two minimal structures m_1 and m_2 such that $U \cap V \in m_2$ whenever $U \in m_1$ and $V \in m_2$ and (Y, σ) be an N-normal space. If the following four conditions are satisfied:*

(1) $F : (X, m_1) \to (Y, \sigma)$ is u.a.n.m.c.,

(2) $G : (X, m_2) \to (Y, \sigma)$ is u.a.n.m.c.,

(3) F(x) and G(x) are closed in Y for each $x \in X$, and

(4) $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$,

then $A = \mathrm{m_2Cl}(A)$. If $F(x) \cap G(x) \neq \emptyset$ for each point x in an m-dense set D of (X, m_2), then $F(x) \cap G(x) \neq \emptyset$ for each point $x \in X$.

Proof. Suppose that $x \in X - A$. Then $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are closed sets and Y is N-normal, there exist open sets V and W in Y having N-closed complement such that $F(x) \subset V$, $G(x) \subset W$, and $V \cap W = \emptyset$; hence $\mathrm{Int}(\mathrm{Cl}(V)) \cap \mathrm{Int}(\mathrm{Cl}(W)) = \emptyset$. Since F is *u.a.n.m.c.* at x, there exists $U_1 \in m_1$ containing x such that $F(U_1) \subset V$. Since G is *u.a.n.m.c.* at x, there exists $U_2 \in m_2$ containing x such that $G(U_2) \subset W$. Now set $U = U_1 \cap U_2$, then $U \in m_2$ and $U \cap A = \emptyset$. Therefore, by Lemma 3.2 we have $x \in X - \mathrm{m_2Cl}(A)$ and hence $A = \mathrm{m_2Cl}(A)$. On the other hand, if $F(x) \cap G(x) \neq \emptyset$ on an m-dense set D of an m-space (X, m_2), then we have $X = \mathrm{m_2Cl}(D) \subset \mathrm{m_2Cl}(A) = A$. Therefore, $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$. ■

Theorem 5.3. *Let (X, m_X) be an m-space. If for each pair of distinct points x_1 and x_2 in X, there exists a multifunction F from (X, m_X) into an N-normal space (Y, σ) satisfying the following conditions:*

(1) $F(x_1)$ and $F(x_2)$ are closed in Y,

(2) F is u.a.n.m.c. at x_1 and x_2, and

(3) $F(x_1) \cap F(x_2) = \emptyset$,

then (X, m_X) is m-T_2.

Proof. Let x_1 and x_2 be distinct points of X. Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction satisfying the conditions in this theorem. Then, we have $F(x_1) \cap F(x_2) = \emptyset$. Since $F(x_1)$ and $F(_2)$ are closed and Y is N-normal, there exist disjoint open sets V_1 and V_2 having N-closed complement such that $F(x_1) \subset V_1$ and $F(x_2) \subset V_2$. Since F is *u.a.n.m.c.* at x_1 and x_2, there exist $U_1 \in m_X$ and $U_2 \in m_X$ containing x_1 and x_2, respectively, such that $F(U_1) \subset \mathrm{Int}(\mathrm{Cl}(V_1))$ and $F(U_2) \subset \mathrm{Int}(\mathrm{Cl}(V_2))$. Therefore, $F(U_1) \cap F(U_2) = \emptyset$ because $\mathrm{Int}(\mathrm{Cl}(V_1)) \cap \mathrm{Int}(\mathrm{Cl}(V_2)) = \emptyset$. This implies that $U_1 \cap U_2 = \emptyset$. Hence (X, m_X) is an m-T_2-space. ■

Definition 5.4. A topological space (X, τ) is said to be *N-connected* [10] if X cannot be written as the union of two disjoint nonempty open sets having N-closed complements.

Definition 5.5. An m-space (X, m_X) is said to be *m-connected* [24] if X cannot be written as the union of two disjoint nonempty m_X-open sets.

Theorem 5.4. *Let (X, m_X) be an m-space, where m_X has property $\mathcal{B}$. If $F : (X, m_X) \to (Y, \sigma)$ is an u.a.n.m.c. or l.a.n.m.c. surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m_X) is m-connected, then (Y, σ) is N-connected.*

Proof. Suppose that (Y, σ) is not N-connected. There exist nonempty open sets U and V of Y having N-closed complement such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is connected for each $x \in X$, either $F(x) \subset U$ or

$F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$,
(2) $F^+(U) \cap F^+(V) = \emptyset$,
(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ and $F^+(V)$ are m_X-open sets.

(i) Let F be *u.a.n.m.c.* Since U and V are clopen in Y, $\mathrm{Int}(\mathrm{Cl}(U)) = U$ and $\mathrm{Int}(\mathrm{Cl}(V)) = V$ and hence U and V are regular open sets having N-closed complements. Since F is *u.a.n.m.c.*, by Corollary 3.1 $F^+(U)$ and $F^+(V)$ are m_X-open sets.

(ii) Let F be *l.a.n.m.c.* By Corollary 3.2, $F^+(U)$ is m_X-closed because U is clopen in (Y, σ). Therefore, $F^+(V)$ is m_X-open. Similarly $F^+(U)$ is m_X-open.

Therefore, (X, m_X) is not m-connected. ■

Remark 5.2. If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction and $m_X = \tau$, then by Theorem 5.4 we obtain the result established in Theorem 14 of [10].

Definition 5.6. Let (X, m_X) be an m-space and A a subset of X. The *m-frontier* of A [30], denoted by $\mathrm{mFr}(A)$, is defined as follows:

$$\mathrm{mFr}(A) = \mathrm{mCl}(A) \cap \mathrm{mCl}(X - A) = \mathrm{mCl}(A) - \mathrm{mInt}(A).$$

Theorem 5.5. *The set of all points $x \in X$ at which a multifunction $F : (X, m_X) \to (Y, \sigma)$ is not u.a.n.m.c. (resp. l.a.n.m.c.) is identical with the union of the m-frontiers of the upper (resp. lower) inverse images of regular open sets containing (resp. meeting) F(x) and having N-closed complement.*

Proof. Let x be a point of X at which F is not *u.a.n.m.c.* Then, by Theorem 3.1 there exists a regular open set V of Y containing $F(x)$ and having N-closed complement such that $U \cap (X - F^+(V)) \neq \emptyset$ for every m_X-open set U containing x. Hence, by Lemma 3.2 we have $x \in \mathrm{mCl}(X - F^+(V))$ and hence $x \in \mathrm{mFr}(F^+(V))$ since $x \in F^+(V) \subset \mathrm{mCl}(F^+(V))$.

Conversely, suppose that V is a regular open set of Y containing $F(x)$ and having N-closed complement such that $x \in \mathrm{mFr}(F^+(V))$. If F is *u.a.n.m.c.* at x. then by Theorem 3.1 we have $x \in \mathrm{mInt}(F^+(V))$. This is a contradiction and hence F is not *u.a.n.m.c.*

In case F is *l.a.n.m.c.*, the proof is similar. ■

6. New forms of *u.a.n.c./l.a.n.c.* multifunctions

For modifications of open sets defined in Definition 2.1, the following relationships are known:

$$\begin{array}{ccccccc} \text{open} & \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} & & \\ & & \Downarrow & & \Downarrow & & \\ & & \text{semi-open} & \Rightarrow & b\text{-open} & \Rightarrow & \text{semi-preopen} \end{array}$$

First, we can define the following modifications of upper/lower almost nearly continuous multifunctions.

Definition 6.1. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper almost nearly α-continuous* (resp. *upper almost nearly precontinuous*, *upper almost nearly b-continuous*, *upper almost nearly sp-continuous*) at a point $x \in X$ if for each regular open set V containing $F(x)$ and having N-closed complement, there exists an α-open (resp. preopen, b-open, semi-preopen) set U containing x such that $F(U) \subset V$,

(2) *lower almost nearly α-continuous* (resp. *lower almost nearly precontinuous*, *lower almost nearly b-continuous*, *lower almost nearly sp-continuous*) at a point $x \in X$ if for each regular open set V meeting $F(x)$ and having N-closed complement, there exists an α-open (resp. preopen, b-open, semi-preopen) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower almost nearly α-continuous* (resp. *upper/lower almost nearly precontinuous*, *upper/lower almost nearly b-continuous*, *upper/lower almost nearly sp-continuous*) on X if it has this property at each $x \in X$.

For multifunctions defined in Definition 5.1, the following relationships hold:

$$\begin{array}{ccccccc} \text{upper } a.n. \text{ con.} & \Rightarrow & \text{upper } a.n.\ \alpha\text{-con.} & \Rightarrow & \text{upper } a.n. \text{ precon.} & & \\ & & \Downarrow & & \Downarrow & & \\ & & \text{upper } a.n. \text{ quasi-con.} & \Rightarrow & \text{upper } a.n.\ b\text{-con.} & \Rightarrow & \text{upper } a.n.\ sp\text{-con.} \end{array}$$

Remark 6.1. In the diagram above, *a.*, *n.* and "con." means almost, near and continuity, respectively. And also the analogous diagram holds for the case "lower".

Let define the further modifications of upper/lower almost nearly continuous multifunctions. For the purpose, we recall the definitions of the θ-closure due to Veličko [36]. Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a θ-cluster point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set

V containing x. The set of all θ-cluster points of A is called the *θ-closure* of A and is denoted by $\mathrm{Cl}_\theta(A)$. A subset A is said to be *θ-closed* if $\mathrm{Cl}_\theta(A) = A$. The complement of a θ-closed set is said to be *θ-open*. The union of all θ-open sets contained in the subset A is called the θ-interior of A and is denoted by $\mathrm{Int}_\theta(A)$.

Definition 6.2. A subset A of a topological space (X, τ) is said to be
(1) *δ-semiopen* [27] (resp. *θ-semiopen* [7]) if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$
(resp. $A \subset \mathrm{Cl}(\mathrm{Int}_\theta(A))$),
(2) *δ-preopen* [34] (resp. *θ-preopen* [24]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$
(resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$),
(3) *δ-sp-open* [12] (resp. *θ-sp-open* [24]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$
(resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))$).

By $\delta\mathrm{SO}(X)$ (resp. $\delta\mathrm{PO}(X)$, $\delta\mathrm{SPO}(X)$, $\theta\mathrm{SO}(X)$, $\theta\mathrm{PO}(X)$, $\theta\mathrm{SPO}(X)$), we denote the collection of all δ-semiopen (resp. δ-preopen, δ-sp-open, θ-semiopen, θ-preopen, θ-sp-open) sets of a topological space (X, τ). These six collections are all m-structures with property $\mathcal{B}$. It is known that the families of all θ-open sets and δ-open sets of (X, τ) are topologies for X, respectively. In [24] and [7], the following relationships are known:

$$\begin{array}{ccccccccccc}
\theta\text{-open} & \Rightarrow & \delta\text{-open} & \Rightarrow & \text{open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\theta\text{-semiopen} & \Rightarrow & \delta\text{-semiopen} & \Rightarrow & \text{semi-open} & \Rightarrow & sp\text{-open} & \Rightarrow & \delta\text{-}sp\text{-open} & \Rightarrow & \theta\text{-}sp\text{-open}
\end{array}$$

Definition 6.3. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be
(1) *upper almost nearly θ-continuous* (resp. *upper almost nearly θ-precontinuous*, *upper almost nearly θ-semi-continuous*, *upper almost nearly θ-sp-continuous*) at a point $x \in X$ if for each regular open set V containing $F(x)$ and having N-closed complement, there exists a θ-open (resp. θ-preopen, θ-semiopen, θ-*sp*-open) set U containing x such that $F(U) \subset V$,
(2) *lower almost nearly θ-continuous* (resp. *lower almost nearly θ-precontinuous*, *lower almost nearly θ-semi-continuous*, *lower almost nearly θ-sp-continuous*) at a point $x \in X$ if for each regular open set V meeting $F(x)$ and having N-closed complement, there exists a θ-open (resp. θ-preopen, θ-semiopen, θ-*sp*-open) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,
(3) *upper/lower almost nearly θ-continuous*(resp. *upper/lower almost nearly θ-precontinuous*, *upper/lower almost nearly θ-semi-continuous*, *upper/lower almost nearly θ-sp-continuous*) on X if it has this property at each $x \in X$.

Definition 6.4. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be
(1) *upper almost nearly δ-continuous* (resp. *upper almost nearly*

δ-precontinuous, *upper almost nearly δ-semi-continuous*, *upper almost nearly δ-sp-continuous*) at a point $x \in X$ if for each regular open set V containing $F(x)$ and having N-closed complement, there exists a δ-open (resp. δ-preopen, δ-semiopen, δ-*sp*-open) set U containing x such that $F(U) \subset V$,

(2) *lower almost nearly δ-continuous* (resp. *lower almost nearly δ-precontinuous*, *lower almost nearly δ-semi-continuous*, *lower almost nearly δ-sp-continuous*) at a point $x \in X$ if for each regular open set V meeting $F(x)$ and having N-closed complement, there exists a δ-open (resp. δ-preopen, δ-semiopen, δ-*sp*-open) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower almost nearly δ-continuous* (resp. *upper/lower almost nearly δ-precontinuous*, *upper/lower almost nearly δ-semi-continuous*, *upper/lower almost nearly δ-sp-continuous*) on X if it has this property at each $x \in X$.

For the multifunctions defined above, the following diagram hold, where u., a., n. and c. mean upper, almost, near and continuity, respectively. And also the analogous diagram holds for the case "lower".

$$\begin{array}{ccccccccccc}
\text{u.a.n.}\theta\text{-c.} & \Rightarrow & \text{u.a.n.}\delta\text{-c.} & \Rightarrow & \text{u.a.n.c.} & \Rightarrow & \text{u.a.n.p.c.} & \Rightarrow & \text{u.a.n.}\delta\text{-p.c.} & \Rightarrow & \text{u.a.n.}\theta\text{-p.c.} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{u.a.n.}\theta\text{-s.c.} & \Rightarrow & \text{u.a.n.}\delta\text{-s.c.} & \Rightarrow & \text{u.a.n.q.c.} & \Rightarrow & \text{u.a.n.sp.c.} & \Rightarrow & \text{u.a.n.}\delta\text{-sp.c.} & \Rightarrow & \text{u.a.n.}\theta\text{-sp.c.}
\end{array}$$

Conclusion. We can apply the results established in Sections 3, 4 and 5 to all multifunctions defined in Definitions 6.1, 6.3 and 6.4.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assiut Univ.*, **12**, 1983, 77–90.

[2] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, β-closure and β-interior, *J. Fac. Ed. Ain Shans Univ.*, **10**, 1986, 235–245.

[3] D. Andrijević, Semi-preopen sets, *Mat. Vesnik*, **38**, 1986, 24–32.

[4] D. Andrijević, On b-open sets, *Mat. Vesnik*, **48**, 1996, 59–64.

[5] T. Bânzaru, Multifunctions and M-product spaces (Romanian), *Bul. St. Tehn. Inst. Politehn. "Traian. Vuia", Timişoara, Mat. Fiz. Mec. Teor. Appl.*, **17**(**31**), 1972, 17–23.

[6] D. Carnahan, Locally nearly compact spaces, *Boll. Un. Mat. Ital. (4)*, **6**, 1972, 143–153.

[7] M. Caldas, M. Ganster, D. N. Georgiou, S. Jafari and T. Noiri, On θ-semiopen sets and separation axioms in topological spaces, *Carpathian J. Math.*, **24**, 2008, 13–22.

[8] S. G. Crossley and S. K. Hildebrand, Semi-closure, *Texas J. Sci.*, **22**, 1971, 99–112.

[9] E. Ekici, Nearly continuous multifunctions, *Acta Math. Univ. Comenianae*, **72**, 2003, 229–235.

[10] E. Ekici, Almost nearly continuous multifunctions, *Acta Math. Univ. Comenianae*, **73**, 2004, 175–186.

[11] N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, On p-regular spaces, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **27**(**75**), 1983, 311–315.

[12] E. Hatir and T. Noiri, Decompositions of continuity and complete continuity, *Acta Math. Hungar.*, **113**, 2006, 281–287.

[13] J. E. Joseph, On characterizations of nearly compact spaces, *Boll. Un. Mat. Ital. (5)*, **B-13**, 1976, 311–321.

[14] I. Kovačević, Subsets and paracompactness, *Univ. u Novom Sadu Zb. Rad. Priod.-Mat. Fak. Ser. Mat.*, **14**, 1984, 79–87.

[15] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly* **70**, 1963, 36–41.

[16] H. Maki, K. C. Rao and A. Nagoor Gani, On generalizing semi-open and preopen sets, *Pure Appl. Math. Sci.*, **49**, 1999, 17–29.

[17] S. R. Malghan and V. V. Hanchinamani, N-continuous functions, *Ann. Soc. Sci. Bruxelles*, **98**, 1984, 69–79.

[18] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt.*, **53**, 1982, 47–53.

[19] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, α-continuous and α-open mappings, *Acta Math. Hungar.*, **41**, 1983, 213–218.

[20] O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.*, **15**, 1965, 961–970.

[21] T. Noiri, N-closed sets and some separation axioms, *Ann. Soc. Sci. Bruxelles*, **88**, 1974, 195–199.

[22] T. Noiri, Almost quasi-continuous functions, *Bull. Inst. Math. Acad. Sinica*, **18**, 1990, 321–332.

[23]T. Noiri and N. Ergun, Notes on N-continuous functions, *Res. Rep. Yatsushiro Coll. Tech.*, **11**, 1989, 65–68.

[24] T. Noiri and V. Popa, On upper and lower M-continuous multifunctions, *Filomat (Nis)*, **14**, 2000, 73–86.

[25] T. Noiri and V. Popa, A unified theory of weak continuity for multifunctions, *Stud. Cerc. St. Ser. Mat. Univ. Bacău*, **16**, 2006, 167–200.

[26] T. Noiri and V. Popa, A unified theory of almost continuous multifunctions (submitted).

[27] J. H. Park, B. Y. Lee and M. J. Son, On δ-semi-open sets in topological spaces, *J. Indian Acad. Math.*, **19**, 1997, 59–67.

[28] V. Popa, Almost continuous multifunctions, *Mat. Vesnik*, **6**(**9**)(**34**), 1982, 75–84.

[29] V. Popa and T. Noiri, On M-continuous functions, *Anal. Univ. "Dunărea de Jos" Galaţi, Ser. Mat. Fiz. Mec. Teor. (2)*, **18**(**23**), 2000, 31–41.

[30] V. Popa and T. Noiri, A unified theorey for S-continuity of multifunctions, *J. Math. Fac. Sci. Univ. Istanbul*, **59**, 2000, 1–15.

[31] V. Popa and T. Noiri, On m-continuous multifunctions, *Bull. St. Univ. Politeh. Timişoara, Ser. Mat. Fiz.*, **46**(**50**), 2001, 1–12.

[32] V. Popa and T. Noiri, A unified theory of weak continuity for functions, *Rend. Circ. Mat. Palermo (2)*, **51**, 2002, 439–464.

[33] V. Popa and T. Noiri, On upper and lower almost quasi continuous multifunctions, *Bull. Inst. Math. Acad. Sinica*, **21**, 1993, 337–349.

[34] S. Raychaudhuri and M. N. Mukherjee, On δ-almost continuity and δ-preopen sets, *Bull. Inst. Math. Acad. Sinica*, **21**, 1993, 357–366.

[35] A. Rychlewicz, On almost nearly continuity with reference to multifunctions, *Acta Math. Univ. Comenianae* (to appear).

[36] N. V. Veličko, H-closed topological spaces, *Amer. Math. Soc. Transl. (2)*, **78**, 1968, 103–118.

[37] J. D. Wine, Locally paracompact spaces, *Glasnik Mat.*, **10**(**30**), 1975, 351–357.

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