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The Internal Bisector's Slope and External Bisector's Slope on Poincaré Upper Half Plane

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In this paper the internal bisector's slope and external bisector's slope of the Poincaré angle between Poincaré lines on the Poincaré upper half plane are defined.

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Key Words: Poincaré lines, Poincaré angle, Poincaré tangent.

1. Introduction

The Poincaré upper half plane geometry has been introduced by Henri Poincaré. Denote this plane by H. The plane H is the upper half plane of the Euclidean analytical plane $\mathbb{R}^2$. Although the points in the plane H are the same as the points in the upper half plane of the Euclidean analytical plane $\mathbb{R}^2$, the lines and the distance function between any two points are different. The lines in the plane H are defined by

$$_aL = \{(x,y) \in \mathbb{R}^2 \mid x = a, \ \ y > 0, \ \ a \in \mathbb{R}, \ a \ constant\} \qquad half\ lines$$

and

$$_cL_r = \{(x,y) \in \mathbb{R}^2 \mid (x-c)^2 + y^2 = r^2, \ \ y > 0 \ , \ c, r \in \mathbb{R}, \ c, r \ constant \ , \ \ r > 0\} \quad half\ circles\ .$$

The geometry of the half plane H is non-Euclidean, since it fails to satisfy the parallel postulate but satisfies all the remaining twelve axioms of the Euclidean plane geometry [1,2,3,4]. In this half plane geometry, the lines and the function of distance are different, therefore, it seems interesting to study the Poincaré analogues of the topics that include the concept of distance in the

Euclidean geometry. A few of such topics have been studied by some authors [3,5,6]. In this paper the internal bisector's slope and external bisector's slope of the Poincaré angle between Poincaré lines on the Poincaré upper half plane are defined.

Definition 1. If $\overrightarrow{BA}$ is a ray in the Poincaré upper half plane where $B = (x_B, y_B)$ and $A = (x_A, y_A)$, then the Euclidean tangent to $\overrightarrow{BA}$ at B is

$$T_{BA}=\begin{cases} (0, y_A - y_B) & \text{, if } \overleftrightarrow{AB} \text{ is a type I line} \\ (y_B, c - x_B) & \text{, if } \overleftrightarrow{AB} \text{ is a type II line } {}_cL_r,\ x_B < x_A \\ \text{-}(y_B, c - x_B) & \text{, if } \overleftrightarrow{AB} \text{ is a type II line } {}_cL_r,\ x_B > x_A \end{cases}$$

The Euclidean tangent ray to $\overrightarrow{BA}$ is the Euclidean ray $\overrightarrow{BA'}$ where $A' = B + T_{BA}$.

Definition 2. The measure of the Poincaré $\langle$ ABC in H is

$$m_H(\langle ABC) = m_E(\langle A'BC') = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\|T_{BC}\|}\right) \tag{1}$$

where $A' = B + T_{BA}$ and $C' = B + T_{BC}$ and $m_E(\langle A'BC')$ is the Euclidean measure of the Euclidean angle $\langle A'BC'$ [2]. *(See Figure 1)*

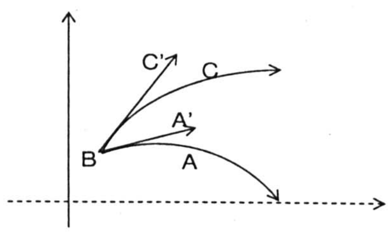

Figure 1:

Proposition 1. *The tangent of the Poincaré angle* $\theta = m_H(\langle ABC)$ *between Poincaré lines is*

$$\tan\theta = \begin{cases} \left|\frac{m_1-m_2}{1+m_1m_2}\right| & \textit{if } \theta \textit{ is between } {}_cL_r \textit{ and } {}_dL_s \\ \left|m_1^{-1}\right| & \textit{if } \theta \textit{ is between } {}_cL_r \textit{ and } {}_aL \ (x_B \neq c) \\ \textit{Infinity} & \textit{Other cases} \end{cases}$$

where

$$m_1 = \frac{c-x_B}{y_B} \ or \ \text{-}\frac{c-x_B}{y_B} \quad m_2 = \frac{d-x_B}{y_B} \ or \ \text{-}\frac{d-x_B}{y_B} \quad (y_B \rangle\ 0)$$

Proof.

(2)
$$\cos\theta = \left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\|T_{BC}\|}\right)$$

(See Definition 2)

Since $\sin\theta = \pm\sqrt{1-\cos^2\theta}$, *for* $\tan\theta$ it follows that

(3)
$$\tan\theta = \pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$$
$$\tan^2\theta = \frac{1}{\cos^2\theta} - 1$$

Now, there are seven cases:

i) If $x_B < x_A$ and $x_B < x_C$ (see figure 2),

From equation (2) and definition (1) we obtain that

$$\cos\theta \quad = \frac{y_B^2+(c-x_B)(d-x_B)}{\sqrt{\left[y_B^2+(c-x_B)^2\right]\left[y_B^2+(d-x_B)^2\right]}}$$

If we substitute $\cos\theta$ from the above equality in the equation (3) then

$$\tan^2\theta = \frac{\left[y_B^2+(c-x_B)^2\right]\left[y_B^2+(d-x_B)^2\right]}{\left[y_B^2+(c-x_B)(d-x_B)\right]^2} - 1$$

$$= \frac{\left[y_B^2\left(1+\left[(c-x_B)/\ y_B\right]^2\right)\right]\left[y_B^2\left(1+\left[(d-x_B)/\ y_B\right]^2\right)\right]}{\left[y_B^2\left\{1+\left(\left[(c-x_B)/\ y_B\right]\left[(d-x_B)/\ y_B\right]\right)\right\}\right]^2} - 1$$

Since the slope of tangents drawn from the point B to the lines ${}_cL_r$ and ${}_dL_s$, respectively, it is following that [2],

$$m_1 = \frac{c-x_B}{y_B}, \quad m_2 = \frac{d-x_B}{y_B}$$

Then the end equation is

$$\tan^2\theta = \frac{y_B^4(1+m_1^2)(1+m_2^2)}{y_B^4(1+m_1m_2)^2} - 1$$

$$\tan\theta = \sqrt{\frac{(1+m_1^2)(1+m_2^2)}{(1+m_1m_2)^2} - 1}$$

$$tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

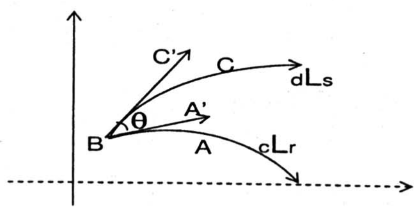

Figure 2:

ii)If $x_B > x_A$ and $x_B > x_C$,

Clearly, since $m_1 = -\frac{c-x_B}{y_B}$, $m_2 = -\frac{d-x_B}{y_B}$

this case is the same as (i).

iii)If $x_B < x_A$ and $x_B > x_C$,

Since $m_1 = \frac{c-x_B}{y_B}$, $m_2 = -\frac{d-x_B}{y_B}$,

in this case,

$$\tan\theta = -\sqrt{\frac{(1+m_1^2)(1+m_2^2)}{(1+m_1m_2)^2} - 1}$$

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|.$$

iv) If $x_B > x_A$ and $x_B < x_C$,

Since $m_1 = -\frac{c-x_B}{y_B}, \quad m_2 = \frac{d-x_B}{y_B}$,

in this case,

$$\tan\theta = -\sqrt{\frac{(1+m_1^2)(1+m_2^2)}{(1+m_1 m_2)^2} - 1}$$

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1 m_2}\right|.$$

Obviously, since $m_1 = -\frac{c-x_B}{y_B}, \quad m_2 = -\frac{d-x_B}{y_B}$ this case is the same as (*iii*)

v)If $x_B < x_A$ and $x_B = x_C$ and $(x_B \neq c)$ (see figure 3)

$$\cos\theta = \frac{(c-x_B)(y_A-y_B)}{\sqrt{[y_B^2+(c-x_B)^2]}|y_A-y_B|}$$

Hence,

$$\tan^2\theta = \frac{(y_A-y_B)^2[y_B^2+(c-x_B)^2]}{(c-x_B)^2(y_A-y_B)^2} - 1$$

$$= \frac{y_B^2\left(1+[(c-x_B)/\, y_B]^2\right)}{(c-x_B)^2} - 1$$

$$= \left(\frac{y_B}{c-x_B}\right)^2\left(1+[(c-x_B)/\, y_B]^2\right) - 1$$

$$= m_1^{-2}\left(1+m_1^2\right) - 1 = m_1^{-2}$$

$$\tan\theta = \left|m_1^{-1}\right|$$

$$\left(m_1 = \frac{c-x_B}{y_B} \right)$$

vi)If $x_B > x_A$ and $x_B = x_C$ and $(x_B \neq c)$,

Clearly, since $m_1 = -\frac{c-x_B}{y_B}$ this case is the same as (*v*) *vii*) Other cases

1) If $x_B < x_A$ or $x_B > x_A$ and $x_B = x_C$ and $(x_B = c)$

Since the Type I line that is passing from its center of the Type II line in the Poincaré upper half plane is right to circle[2,3] , for tan θ it follows that

$\tan\theta = \tan 90 = \infty$

Also in case the Type II lines are right or the Euclidean tangents are right, then for tan θ it follows that

$\tan\theta = \tan 90 = \infty$ ■

Now, let us find the internal bisector's slope of lines whose slopes are m_1 and m_2.

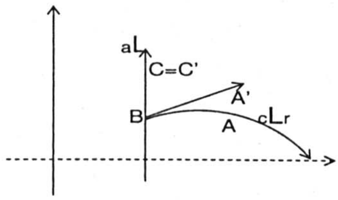

Figure 3:

2. Internal bisector's slope

On the Poincaré upper half plane, let the point D be the intersection of the line BC with the internal bisector belonging to the vertex A of a triangle ABC. If m_1 ,m_2 and m_3 are the slopes of the edges AC , AB and AD , respectively then

$\frac{(m_1m_2-1)\pm\sqrt{1+m_1^2+m_2^2+(m_1m_2)^2}}{m_1+m_2}$ $\frac{-(m_1m_2-1)\pm\sqrt{1+m_1^2+m_2^2+(m_1m_2)^2}}{m_1+m_2}$	If $\overleftrightarrow{AB}, \overleftrightarrow{AC}and\overleftrightarrow{AD}$ are type II lines
$m_1 \pm \sqrt{1+m_1^2}$ $-m_1 \pm \sqrt{1+m_1^2}$	If $\overleftrightarrow{AB}$ is a type I and $\overleftrightarrow{AC}, \overleftrightarrow{AD}$ are type II lines
∞	If $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are type II lines and $\overleftrightarrow{AD}$ is a type I line

Proof.

$$\tan\alpha = \left|\frac{m_2-m_3}{1+m_2m_3}\right| \quad \text{and} \quad \tan\alpha = \left|\frac{m_3-m_1}{1+m_3m_1}\right|$$

where (see figure 4)

$m_1 = \frac{c-x_A}{y_A}$ $(y_A \rangle 0)$ The center of the line AC is $(c, 0)$

$m_2 = -\frac{d-x_A}{y_A}$ $(y_A \rangle 0)$ The center of the line AB is $(d, 0)$

$m_3 = \frac{e-x_A}{y_A}$ $(y_A \rangle 0)$ The center of the line AD is $(e, 0)$

...

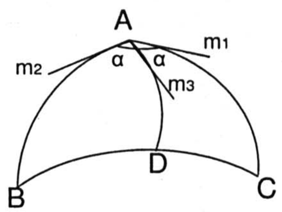

Figure 4

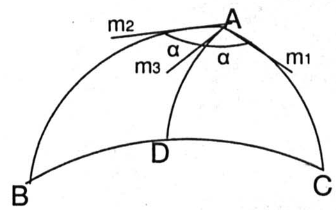

Figure 5

Equalize the squares of both sides.

(4)
$$\frac{m_2-m_3}{1+m_2m_3} = \frac{m_3-m_1}{1+m_3m_1}$$
$$(m_1+m_2)\,m_3^2 - 2(m_1m_2-1)m_3 - (m_1+m_2) = 0$$

This equation is a quadratic equation in terms of m_3. Its roots give the bisector's slope. Hence, the bisector's slope is as follows

$$\frac{(m_1m_2-1)-\sqrt{1+m_1^2+m_2^2+(m_1m_2)^2}}{m_1+m_2}$$

$$\frac{(m_1m_2-1)+\sqrt{1+m_1^2+m_2^2+(m_1m_2)^2}}{m_1+m_2}$$

$$\tan\alpha = \left|\frac{m_2-m_3}{1+m_2m_3}\right| \quad \text{and} \quad \tan\alpha = \left|\frac{m_3-m_1}{1+m_3m_1}\right|$$

where (see Figure 5)

$m_1 = \frac{c-x_A}{y_A}$ $(y_A \rangle 0)$ The center of the line AC is $(c,0)$
$m_2 = -\frac{d-x_A}{y_A}$ $(y_A \rangle 0)$ The center of the line AB is $(d,0)$
$m_3 = -\frac{e-x_A}{y_A}$ $(y_A \rangle 0)$ The center of the line AD is $(e,0)$

Hence the equation (4) gets the following form:

$$\frac{m_2-(-m_3)}{1+m_2(-m_3)} = \frac{-m_3-m_1}{1+(-m_3)m_1}$$

(5)

$$(m_1+m_2)\,m_3^2 + 2(m_1m_2-1)m_3 - (m_1+m_2) = 0$$

Hence, the bisector's slope is as follows

$$\frac{-(m_1m_2-1)-\sqrt{1+m_1^2+m_2^2+(m_1m_2)^2}}{m_1+m_2}$$

$$\frac{-(m_1m_2-1)+\sqrt{1+m_1^2+m_2^2+(m_1m_2)^2}}{m_1+m_2}$$

$\tan\alpha = \pm m_3^{-1}$ and $\tan\alpha = \frac{\pm m_3 - m_1}{1+(\pm m_3)m_1}$

where (see figure 6)

$m_1 = \frac{c-x_A}{y_A}$ or $m_1 = -\frac{c-x_A}{y_A}$ $(y_A \rangle 0)$ The center of the line AC is $(c,0)$

$m_3 = \frac{e-x_A}{y_A}$ or $m_3 = -\frac{e-x_A}{y_A}$ $(y_A \rangle 0)$ The center of the line AD is $(e,0)$

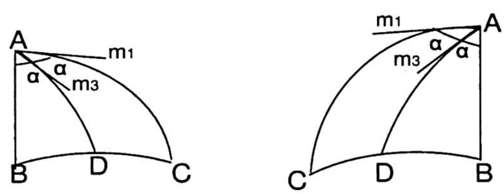

Figure 6:

Consider the squares of both sides equal.

$$m_3^{-1} = \frac{m_3-m_1}{1+m_3m_1}$$

$$m_3^2 - 2m_1m_3 - 1 = 0$$

This equation is a quadratic equation in terms of m_3. Its roots give the bisector's slope. Hence, the bisector's slope is the following

$$(6)\qquad \begin{array}{l} m_1 + \sqrt{1+m_1^2} \\ m_1 - \sqrt{1+m_1^2} \end{array}$$

In the other case,

$-m_3^{-1} = \frac{-m_3-m_1}{1+(-m_3)m_1}$

$m_3^2 - 2m_1m_3 - 1 = 0$

$m_3^{-1} = \frac{m_3-m_1}{1+m_3m_1}$

$m_3^2 + 2m_1m_3 - 1 = 0$

Hence, the bisector's slope is

$$(7)\qquad \begin{array}{l} -m_1 - \sqrt{1+m_1^2} \\ -m_1 + \sqrt{1+m_1^2} \end{array}$$

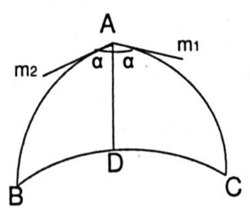

Figure 7:

$m_1 = \frac{c-x_A}{y_A} \ (y_A \rangle 0)$ The center of the line AC is $(c,0)$

$m_2 = -\frac{d-x_A}{y_A} \ (y_A \rangle 0)$ The center of the line AB is $(d,0)$

$m_3 = \infty$ ∎

Note: If $m_1 = -m_2$, there is a root of the equations (4) and (5). It is zero. In this case, we say that one of bisectors is the x-axis. This case is not valid on the half plane H. The other bisector is the positive y-axis.

Now, consider the external bisector's slope.

3. External bisector's slope

On the Poincaré upper half plane, let the point D be the intersection of the line BC with the external bisector belonging to the vertex A of a triangle ABC. Let m_1 ,m_2 and m_3 be the slopes of the edges AC , AB and AD , respectively.

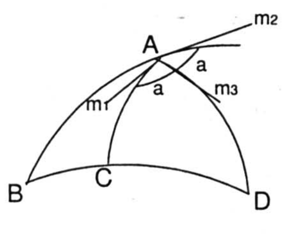

Figure 8

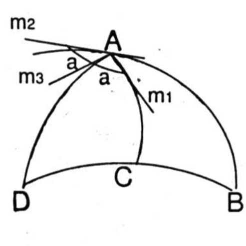

Figure 9

Since the cases in Figure 8 and Figure 9 are the same as those in Figure 4 and Figure 5, the values m_3 are the same as in (4) and (5).

Since the cases in Figure 10 are the same as in Figure 6, the values m_3 are the same as in (6).

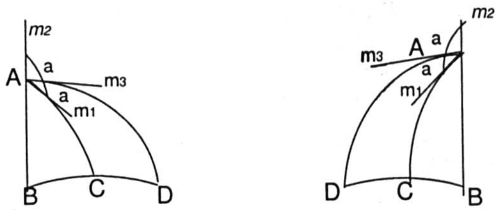

Figure 10

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