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φ_0-Integral Stability in Terms of Two Measures for Differential Equations

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This paper investigates one generalization of integral stability in terms of two measures for systems of nonlinear differential equations. Cone-valued Lyapunov functions and comparison results for scalar differential equations are employed.

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1. Introduction

One of the main problems in the qualitative theory of differential equations is stability of their solutions. Many various types of stability have been introduced and studied. One of the most useful ones of stability is connected with two different measures. Theory of stability in terms of two measures was introduced in 1960 by Movchan [10]. It has been further developed and applied to different classes of equations by many researchers ([2], [3], [4], [5], [6], [7], [8], [9], [11]).

In the present paper we study the φ_0-integral stability in terms of two different measures of systems of differential equations. An appropriate definition for φ_0-integral stability in terms of two measures of differential equations is given. Two types of sufficient conditions for φ_0-equi-integral stability in two measures are obtained. Second method of Lyapunov and comparison method are the base of the investigations.

2. Preliminary notes and definitions

Consider the initial value problem for the system of nonlinear differential equations

(1) $$x' = F(t, x(t)) \quad \text{for} \quad t \geq t_0,$$

(2) $$x(t_0) = x_0,$$

where $x \in \mathbb{R}^n$, $F : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

We denote the solution of the initial value problem (1), (2) by $x(t; t_0, x_0)$ and $J(t_0, x_0)$ - the maximal interval of the type $[t_0, \beta)$ in which $x(t; t_0, x_0)$ is defined.

Consider the perturbed system of nonlinear differential equations (1)

(3) $$x' = F(t, x) + G(t, x) \quad \text{for} \quad t \geq t_0,$$

where $x \in \mathbb{R}^n$, $F, G : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 1. ([1]). A proper set $\mathcal{K} \subset \mathbb{R}^n$ is called a cone if:
(i) $\lambda\mathcal{K} \subset \mathcal{K}$, $\lambda > 0$, (ii) $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$, (iii) $\bar{\mathcal{K}} = \mathcal{K}$,
$(iv) \mathcal{K}^o \neq \emptyset$ (v) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.
The set
$$\mathcal{K}^* = \{\varphi \in \mathbb{R}^n : \ (\varphi, x) \geq 0, \ x \in \mathcal{K}\}$$
is called a joint cone, if $\mathcal{K}^*$ is a cone.

We will define the following set of cone-valued vector functions:

Definition 2. We will say that the function $V(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathcal{K}$ belongs to the class Λ if:

1. $V(t, x)$ is a continuous function for any $t \geq 0$ and $x \in \mathcal{K}$;
2. The function $V(t, x)$ is componentwisely Lipschitz in x relatively to $\mathcal{K}$.

Consider the following sets

$$K = \{a \in C[[0, \infty), [0, \infty)] \ : \ a(s) \text{ is strictly increasing and } a(0) = 0\};$$

$$CK = \{b \in C[[0, \infty) \times [0, \infty), [0, \infty)] \ : \ b(t, .) \in K \text{ for any fixed } t \geq 0\};$$

$$\Gamma = \{h \in C[[0, \infty) \times [0, \infty), [0, \infty)] \ : \ \inf_{s \in [0, \infty)} h(t, s) = 0 \text{ for each } t \geq 0\}.$$

Let ρ, t, T be positive numbers, $\varphi_0 \in \mathcal{K}^*$, $h \in \Gamma$. We define sets:

$$S(h,\rho,\varphi_0) = \{(t,x) \in [0,\infty) \times \mathcal{K}: \quad h(t,(\varphi_0,x)) < \rho\};$$

$$S^C(h,\rho,\varphi_0) = \{(t,x) \in [0,\infty) \times \mathcal{K}: \quad h(t,(\varphi_0,x)) \geq \rho\};$$

$$\Omega(t,T,\rho,\varphi_0,h) = \{x \in \mathcal{K}: \quad h(s,(\varphi_0,x)) < \rho \text{ for } s \in [t,t+T]\}.$$

We note that integral stability in terms of two measures for ordinary differential equations is studied by Lyapunov functions in [9] and φ_0-stability is studied in [1], [12]. Employing both ideas we will introduce the definition for φ_0-integral stability in terms of two measures for differential equations.

Definition 3. . Let $\varphi_0 \in \mathcal{K}^*$, $h, h_0 \in \Gamma$. System of differential equations (1) is said to be (h_0,h)*-equi-integrally* φ_0*-stable* if for every $\alpha > 0$ and for any $t_0 \geq 0$ there exists a positive function $\beta = \beta(t_0,\alpha) \in CK$ which is continuous in t_0 for each α and the inequality

$$h(t,(\varphi_0,y(t))) < \beta, \quad t \geq t_0,$$

holds, provided that

$$h_0(t,(\varphi_0,x_0)) < \alpha,$$

and for every $T > 0$

$$\int_{t_0}^{t_0+T} sup_{x\in\Omega(t_0,T,\beta,\varphi_0,h)} ||G(s,x)||ds < \alpha,$$

where $y(t) = y(t;t_0,x_0)$ is the maximal solution of the perturbed system of differential equations (3) through the point (t_0,x_0).

Definition 4. Let $\varphi_0 \in \mathcal{K}^*$, $h, h_0 \in \Gamma$. A system of differential equations (1) is said to be (h_0,h)*-uniform-integrally* φ_0*-stable* if in Definition 3 for every $\alpha > 0$ and for any $t_0 \geq 0$ there exists a positive function $\beta = \beta(\alpha) \in K$.

Remark 1. We note that in the case $h_0(t,u) \equiv u$ and $h(t,u) \equiv u$ the (h_0,h)-equi-integral (uniform-integral) φ_0-stability reduces to equi-integral (uniform-integral) φ_0-stability.

In our further investigations we will use the following comparison scalar differential equations

$$u' = g_1(t,u), \quad t \geq t_0, \tag{4}$$

and

(5) $$w' = g_2(t, w), \quad t \geq t_0,$$

and the perturbed scalar differential equation

(6) $$w' = g_2(t, w) + \xi(t), \quad t \geq t_0,$$

where $u, w \in \mathbb{R}$.

We will use the following definition for stability of scalar differential equations:

Definition 5. ([7]). Differential equation (5) is said to be *equi-integrally stable* if for every $\alpha > 0$ and for any $t_0 \geq 0$ a positive function $\beta = \beta(t_0, \alpha) \in CK$ exists such that the inequality

$$|w(t)| < \beta, \quad t \geq t_0$$

holds, provided that

$$|w_0| < \alpha,$$

and for every $T > 0$

$$\int_{t_0}^{t_0+T} |\xi(s)| ds < \alpha,$$

where $w(t) = w(t; t_0, w_0)$ is the maximal solution of the perturbed differential equation (6) through the point (t_0, w_0).

Definition 6. ([7]). Differential equation (5) is said to be *uniform-integrally stable*, if in Definition 5 there exists a positive function $\beta = \beta(\alpha) \in K$.

We will introduce the following properties for functions from the class Λ, given by the definition:

Definition 7. . Let $\varphi_0 \in \mathcal{K}^*$, $h \in \Gamma$. Function $V(t, x) \in \Lambda$ is said to be *φ_0-weakly h-decreascent*, if there exists a constant $\delta > 0$ and a function $a \in CK$ such that inequality $h(t, (\varphi_0, x)) < \delta$ implies $(\varphi_0, V(t, x)) \leq a(t, h(t, (\varphi_0, x)))$.

Definition 8. . Let $\varphi_0 \in \mathcal{K}^*$, $h \in \Gamma$. Function $V(t, x) \in \Lambda$ is said to be *φ_0-strongly h-decreascent*, if there exists a constant $\delta > 0$ and a function $a \in K$ such that the inequality $h(t, (\varphi_0, x)) < \delta$ implies $(\varphi_0, V(t, x)) \leq a(h(t, (\varphi_0, x)))$.

Definition 9. ([9]). Let $h, h_0 \in \Gamma$. The function $h_0(t, u)$ is said to be *uniformly finer* than $h(t, u)$, if there exists a constant $\delta > 0$ and a function $a \in K$ such that $h_0(t, u) < \delta$ implies $h(t, u) \leq a(h_0(t, u))$.

Let $t \geq 0,\ x \in \mathbb{R}^n,\ \ V(t,x) \in \Lambda$. We define the derivative of the cone-valued vector function $V(t,x)$ along the trajectory of solution of (1) as follows

$$D^+V_{(1)}(t,x) = \limsup_{\epsilon \to 0+}(1/\epsilon)\{V(t+\epsilon, x+\epsilon F(t,x)) - V(t,x)\}.$$

In the further investigations we will use the following comparison result:

Lemma 1. *(Theorem 1.4.1 [7]). Let $E \subset \mathbb{R} \times \mathbb{R}$ be an open set and*
1. Function $g_1 \in C[E, \mathbb{R}]$.
2. Function $m \in C[[t_0, t_0 + a) \times \mathbb{R} \bigcap E, \mathbb{R}]$ satisfies the inequalities

$$m' \leq g_1(t,m), \quad t \in [t_0, t_0 + a), \quad m(t_0) \leq u_0.$$

3. Function $r^*(t) = r^*(t; t_0, x_0)$ is the maximal solution of (4) through the point (t_0, u_0), defined for $t \in [t_0, t_0 + a)$.
Then

$$m(t) \leq r^*(t), \qquad t \in [t_0, t_0 + a).$$

3. Main results

We will obtain sufficient conditions for (h_0, h)-integral φ_0-stability of systems of differential equations. We will apply two different types of cone-valued Lyapunov functions and comparison results, employing scalar comparison differential equations.

Theorem 1. *Let the following conditions be fulfilled:*
1. Functions $F, G \in C[[0,\infty) \times \mathbb{R}^n, \mathcal{K}]$.
2. Functions $h_0, h \in \Gamma$, h_0 is uniformly finer than h.
3. Function $\varphi_0 \in \mathcal{K}^*$.
4. There exists a function $V_1 \in \Lambda$ with a Lipshitz constant $M_1 = (M_1^1, M_1^2, \ldots, M_1^n)$, it is φ_0-weakly h_0-decreascent, and
(i) $(\varphi_0, D^+_{(1)}V_1(t,x)) \leq g_1(t, (\varphi_0, V_1(t,x)))$ for $(t,x) \in S(h, \rho, \varphi_0)$,
where $g_1(t,u) \in C([0,\infty) \times \mathbb{R}, \mathbb{R}]$, $g_1(t,0) \equiv 0$, and $\rho > 0$ is a constant.
5. For any number $\eta > 0$ there exists a function $V_2^{(\eta)} \in \Lambda$ with a Lipshitz constant $M_2 = M_2(\eta) > 0$, $M_2 = (M_2^1, M_2^2, \ldots, M_2^n)$ and for $(t,x) \in [0,\infty) \times \mathbb{R}^n$ the inequality
(ii) $b(h(t, (\varphi_0, x))) \leq (\varphi_0, V_2^{(\eta)}(t,x)) \leq a(h_0(t, (\varphi_0, x)))$
holds, where $a, b \in K$ and $lim_{u \to \infty} b(u) = \infty$.

6. For $(t,x)\in S(h,\rho,\varphi_0,)\bigcap S^C(h_0,\eta,\varphi_0,)$ *the inequality*
$(iii)\Big(\varphi_0,\{D^+_{(1)}V_1(t,x)+D^+_{(1)}V_2^{(\eta)}(t,x)\}\Big)\le g_2\Big(t,(\varphi_0,V_1(t,x)+V_2^{(\eta)}(t,x))\Big)$
holds, where $g_2(t,x)\in C([0,\infty)\times\mathbb{R},\mathbb{R}]$.

7. For any initial conditions the solutions of systems (1) and (3) and the solutions of scalar equations (4), (5) and (6) exist on $[t_0,\infty)$.

8. Zero solution of the scalar differential equation (4) is equi-stable.

9. Scalar differential equation (5) is equi-integrally stable.

Then the system of differential equations (1) is (h_0,h)*- equi-integrally* φ_0*-stable.*

Proof. Since function $V_1(t,x)$ is φ_0-weakly h_0-decreascent, there exists a constant $\rho_1\in(0,\rho)$ and a function $\psi_1\in CK$ such that $h_0(t,(\varphi_0,x))<\rho_1$ implies

$$(\varphi_0,V_1(t,x))\le\psi_1(t,h_0(t,(\varphi_0,x))). \tag{7}$$

Since $h_0(t,u)$ is uniformly finer than $h(t,u)$, there exists a constant $\rho_0\in(0,\rho_1)$ and a function $\psi_2\in K$ such that $h_0(t,u)<\rho_0$ implies

$$h(t,u)\le\psi_2(h_0(t,u)), \tag{8}$$

where $\psi_2(\rho_0)<\rho_1$.

Let $\alpha>0$ be a positive enough small number, $t_0\ge0$ be a fixed number, and $\alpha_1=max\Big(2a(\alpha),||\varphi_0||.||M_1+M_2||\ \alpha\Big)$.

Since scalar differential equation (5) is equi-integrally stable, for the given $\alpha_1>0$ there exists a positive function $\beta_1=\beta_1(t_0,\alpha_1)=\beta_1(t_0,\alpha)\in CK$ such that the maximal solution $w(t)=w(t;t_0,w_0)$ of the perturbed differential equation (6) satisfies the inequality

$$|w(t)|<\beta_1,\quad t\ge t_0, \tag{9}$$

provided that

$$|w_0|<\alpha_1 \tag{10}$$

and for every $T>0$

$$\int_{t_0}^{t_0+T}|\xi(s)|ds<\alpha_1. \tag{11}$$

We note that $\alpha_1\le\beta_1$.

Since the function $b \in K$ and $lim_{s\to\infty}b(s) = \infty$, we could choose a constant $\beta = \beta(\beta_1) = \beta(t_0, \alpha) > 0$ such that

$$b(\beta) \geq \beta_1. \tag{12}$$

Now we choose $x_0 \in \mathbb{R}^n$ such that

$$h_0(t_0, (\varphi_0, x_0)) < \alpha \tag{13}$$

and let for every $T > 0$

$$\int_{t_0}^{t_0+T} sup_{x\in\Omega(t_0,T,\beta,\varphi_0,h)}||G(s,x)||ds \leq \alpha. \tag{14}$$

Since the functions $a \in K$ and $\psi_2 \in K$ we can find a $\delta_1 = \delta_1(\alpha_1, \beta_1) = \delta_1(t_0, \alpha) > 0$, $\delta_1 < \rho_0$ such that the inequalities

$$a(\delta_1) < \frac{\alpha_1}{2}, \quad \psi_2(\delta_1) < \beta \tag{15}$$

hold.

From the choice of β_1 follows that we can choose $\delta_1 > \alpha$.

Consider the function $V_2^{(\eta)}(t, x)$, defined in the condition 5 of Theorem 1, where $\eta = \delta_1$.

Since zero solution of scalar differential equation (4) is equi-stable, there exists a positive function $\delta_2 = \delta_2(t_0, \alpha_1) = \delta_2(t_0, \alpha)$ which is continuous in t_0 for each α and inequality $|u_0| < \delta_2$ implies

$$|u(t; t_0, u_0)| < \frac{\alpha_1}{2}, \quad t \geq t_0, \tag{16}$$

where $u(t; t_0, u_0)$ is the maximal solution of (4) through (t_0, u_0).

Since function $\psi_1 \in CK$, there exists $\delta_3 = \delta_3(\delta_2) = \delta_3(t_0, \alpha) > 0$ such that for $|s| < \delta_3$ the inequality

$$\psi_1(t_0, s) < \delta_2 \tag{17}$$

holds.

From inequalities (7) and (17) follows that there exists a function $\delta_6 = \delta_6(t_0, \alpha) < min(\delta_3, \rho_1)$ such that $h_0(t_0, (\varphi_0, x_0)) < \delta_6$ implies

$$(\varphi_0, V_1(t_0, x_0)) \leq \psi_1(t_0, h_0(t_0, (\varphi_0, x_0))) < \delta_2. \tag{18}$$

We note that from the choice of the point x_0 it follows that $b(h(t_0, (\varphi_0, x_0))) \leq (\varphi_0, V_2^{(\eta)}(t_0, x_0)) \leq a(h_0(t_0, (\varphi_0, x_0))) < a(\alpha) < a(\delta_1) < \frac{\alpha_1}{2} < \alpha_1 \leq \beta_1 \leq b(\beta)$, i.e. $h(t_0, (\varphi_0, x_0)) < \beta$.

We will prove that if the inequalities (13) and (14) are satisfied, then

$$h(t,(\varphi_0, y(t;t_0,x_0))) < \beta, \quad t \geq t_0, \tag{19}$$

where $y(t;t_0,x_0)$ is the maximal solution of (3) through the point (t_0,x_0).

Suppose it is not true. Therefore there exists a point $t^* > t_0$, such that

$$h(t^*,(\varphi_0, y(t^*;t_0,x_0))) = \beta, \quad h(t,(\varphi_0, y(t;t_0,x_0))) < \beta, \ \ t \in [t_0,t^*). \tag{20}$$

If we assume that $h_0(t^*,(\varphi_0, y(t^*;t_0,x_0))) < \delta_1$, then from the choice of δ_1 and inequality (8) it follows

$$h(t^*,(\varphi_0, y(t^*;t_0,x_0))) \leq \psi_2(h_0(t^*,(\varphi_0, y(t^*;t_0,x_0))) < \beta,$$

that contradicts (20).

Therefore,

$$h_0(t^*,(\varphi_0, y(t^*;t_0,x_0))) \geq \delta_1, \quad h_0(t_0,(\varphi_0,x_0)) < \delta_1. \tag{21}$$

If $h_0(t,(\varphi_0, y(t;t_0,x_0)) \leq \delta_1 < \rho_0$ on an interval $[t_0,t_0+T]$ for some T, then $h(t,(\varphi_0, y(t;t_0,x_0)) < \psi(h_0(t,(\varphi_0, y(t;t_0,x_0))) \leq \psi(\delta_1) < \beta$ for $t \in [t_0,t_0+T]$. According to the assumption and the conclusion above there exists a point $t_0^* \in (t_0,t^*)$ such that

$$\delta_1 = h_0(t_0^*,(\varphi_0, y(t_0^*;t_0,x_0)))$$

and

$$(t, y(t;t_0,x_0)) \in S(h,\beta,\varphi_0) \bigcap S^c(h_0,\delta_1,\varphi_0), \quad t \in [t_0^*,t^*). \tag{22}$$

Let $r_1(t;t_0,u_0)$ be the maximal solution of the scalar equation (4) through point (t_0,u_0), where $u_0 = (\varphi_0, V_1(t_0,x_0))$.

Define function $p(t,x) = (\varphi_0, V_1(t,x))$ for $(t,x) \in [0,\infty) \times \mathbb{R}^n$.

Let $(t,x) \in S(h,\beta,\varphi_0) \bigcap S^c(h_0,\delta_1,\varphi_0)$. Then

$$\begin{aligned} D^+_{(1)} p(t,x) &= \limsup_{\epsilon\to 0+}(1/\epsilon)\big\{p(t+\epsilon, x+\epsilon F(t,x)) - p(t,x)\big\} \\ &= \limsup_{\epsilon\to 0+}(1/\epsilon)\Big\{\Big(\varphi_0, V_1(t+\epsilon, x+\epsilon F(t,x))\Big) - \Big(\varphi_0, V_1(t,x)\Big)\Big\} \\ &= \Big(\varphi_0, \limsup_{\epsilon\to 0+}(1/\epsilon)\big\{V_1(t+\epsilon, x+\epsilon F(t,x)) - V_1(t,x)\big\}\Big) \\ &= (\varphi_0, D^+_{(1)} V_1(t,x)) \leq g_1(t,(\varphi_0, V_1(t,x))) = g_1(t,p(t,x)). \end{aligned} \tag{23}$$

Let $t \in [t_0^*, t^*)$. From (22) and (23) we obtain $q'(t) \le g_1(t, q(t))$ for $t \in [t_0^*, t^*)$, where $q(t) = p(t, y(t; t_0, x_0))$. According to Lemma 1 $q(t) \le r_1(t_0^*; t_0, u_0)$ for $t \in [t_0^*, t^*)$, or

$$(\varphi_0, V_1(t_0^*, x_0^*)) \le r_1(t_0^*; t_0, u_0), \tag{24}$$

where $x_0^* = y(t_0^*; t_0, x_0)$.

Let $\alpha < \delta_6$. If conditions (13) and (14) are satisfied, then from inequality (18) follows that $|u_0| < \delta_2$. From (16) we obtain $|r_1(t; t_0, u_0)| < \frac{\alpha_1}{2}$. Applying (24) we obtain

$$(\varphi_0, V_1(t_0^*, x_0^*)) < \frac{\alpha_1}{2}. \tag{25}$$

From inequalities (15), (25) and condition (*ii*) of Theorem 1, it follows that

$$(\varphi_0, V_2^{(\eta)}(t_0^*, x_0^*)) < a(h_0(t_0^*, (\varphi_0, x_0^*))) = a(\delta_1) < \frac{\alpha_1}{2}. \tag{26}$$

Consider function $m: \ [0, \infty) \times \mathbb{R}^n \to \mathcal{K}$:

$$m(t, x) = V_1(t, x) + V_2^{(\eta)}(t, x), \qquad t \ge t_0. \tag{27}$$

From inequalities (25) and (26) we obtain

$$(\varphi_0, m(t_0^*, x_0^*)) < \alpha_1. \tag{28}$$

Let $(t, x) \in S(h, \beta, \varphi_0) \bigcap S^c(h_0, \delta_1, \varphi_0)$. Applying Lipshitz condition for functions $V_1(t, x)$ and $V_2^{(\eta)}(t, x)$, we obtain

$$\begin{aligned}
D^+_{(3)} m(t, x) &= \limsup_{\epsilon \to 0+} \tfrac{1}{\epsilon} \Big\{ \big\{ V_1(t + \epsilon, x + \epsilon[F(t, x) + G(t, x)]) - V_1(t, x) \big\} \\
&\quad + \big\{ V_2^{(\eta)}(t + \epsilon, x + \epsilon[F(t, x) + G(t, x)]) - V_2^{(\eta)}(t, x) \big\} \Big\} \\
&\le \limsup_{\epsilon \to 0+} \tfrac{1}{\epsilon} \Big\{ \big\{ V_1(t + \epsilon, x + \epsilon F(t, x)) - V_1(t, x) \big\} \\
&\quad + \big\{ V_2^{(\eta)}(t + \epsilon, x + \epsilon F(t, x)) - V_2^{(\eta)}(t, x) \big\} \Big\} \\
&\quad + \limsup_{\epsilon \to 0+} \tfrac{1}{\epsilon} \Big\{ \big\{ V_1(t + \epsilon, x + \epsilon[F(t, x) + G(t, x)]) - V_1(t + \epsilon, x + \epsilon F(t, x)) \big\} \\
&\quad + \big\{ V_2^{(\eta)}(t + \epsilon, x + \epsilon[F(t, x) + G(t, x)]) - V_2^{(\eta)}(t + \epsilon, x + \epsilon F(t, x)) \big\} \Big\} \\
&\le D^+_{(1)} \Big(V_1(t, x) + V_2^{(\eta)}(t, x) \Big) + (M_1 + M_2) ||G(t, x)||.
\end{aligned} \tag{29}$$

Let $t \in [t_0^*, t^*)$. From (22), (29) and condition (*iii*) of Theorem 1, we obtain

$$p'(t) \le g_2(t, p(t)) + \eta(t) \tag{30}$$

where function $p(t) = (\varphi_0, m(t, y(t; t_0, x_0))$ and function $\eta(t) = (\varphi_0, (M_1 + M_2) sup_{x\in\Omega(t_0^*,T,\beta,\varphi_0,h)} ||G(t,x)||)$.

Applying Lemma 1 to inequality (30) proves the validity of inequality

$$(\varphi_0, m(t, y(t; t_0, x_0)) \le r^*(t; t_0^*, w_0^*), \quad t \in \Xi, \tag{31}$$

where $r^*(t; t_0^*, w_0^*)$ is the maximal solution of (6) through the point (t_0^*, w_0^*), $w_0^* = (\varphi_0, m(t_0^*, x_0^*))$, $\xi(t) = \eta(t)$ and Ξ is the common interval of existence of $r^*(t; t_0^*, w_0^*)$ and $y(t; t_0, x_0)$. We note that $\Xi \supset [t_0^*, t^*)$.

Inequality (14) for $T = t^* - t_0^*$ implies

$$\int_{t_0^*}^{t^*} |\eta(s)| ds \le ||\varphi_0||.||M_1 + M_2|| \int_{t_0^*}^{t^*} sup_{x\in\Omega(t_0^*,T,\beta,\varphi_0,h)} ||G(s,x)|| ds \Big\}$$
$$< ||\varphi_0||.||M_1 + M_2|| \alpha \le \alpha_1.$$

Choose a point $T^* > t^*$ such that

$$\int_{t_0^*}^{t^*} |\eta(s)| ds + \frac{1}{2}(T^* - t^*)|\eta(t^*)| < \alpha_1.$$

Now define the continuous function $\psi^*(t): \ [t_0^*, \infty) \to \mathbb{R}:$

$$\psi^*(t) = \begin{cases} \eta(t) & \text{for } \ t \in [t_0^*, t^*] \\ \frac{\eta(t^*)}{t^* - T^*}(t - T^*) & \text{for } \ t \in [t^*, T^*] \\ 0 & \text{for } \ t \ge T^*. \end{cases}$$

We note that if (14) is satisfied then for every $T > 0$

$$\int_{t_0^*}^{t_0^*+T} |\psi^*(s)| ds < \alpha_1. \tag{32}$$

Let $r^{**}(t; t_0^*, w_0^*)$ be the maximal solution of equation (6) through point (t_0^*, w_0^*), where $\xi(t) = \psi^*(t)$. Function $r^{**}(t; t_0^*, w_0^*)$ is defined for $t \ge t_0^*$ and

$$r^{**}(t; t_0^*, w_0^*) = r^*(t; t_0^*, w_0^*), \quad t \in [t_0^*, t^*].$$

From inequalities (28) and (32) follows that $(\varphi_0, w_0^*) < \beta_1$ and inequality (9) holds, i.e.

$$|r^{**}(t; t_0^*, w_0^*)| < \beta_1, \quad t \ge t_0^*. \tag{33}$$

From inequalities (33), (31), choice of point t^*, and condition *(ii)* of Theorem 1 we obtain

$$b(\beta) \geq \beta_1 > |r^{**}(t^*; t_0^*, w_0^*)| \geq (\varphi_0, m(t^*, y(t^*; t_0, x_0)))$$
$$\geq (\varphi_0, V_2^{(\eta)}(t^*, y(t^*; t_0, x_0))) \geq b(h(t^*, (\varphi_0, y(t^*; t_0, x_0)))) = b(\beta).$$

The obtained contradiction proves the validity of inequality (19) and (h_0, h)-equi-integral φ_0-stability of the system of differential equations (1). ∎

Theorem 2. *Let the following conditions be fulfilled:*

1. Conditions 1, 2, 3, 4, 5, 6, and 7 are satisfied, where the function $V_1 \in \Lambda$ *is* φ_0*-strongly* h_0*-decreascent.*

2. Zero solution of the scalar differential equation (4) is uniformly stable.

3. Scalar differential equation (5) is uniform-integrally stable.

Then the system of differential equations (1) is (h_0, h)*- uniform-integrally* φ_0*-stable.*

The proof of Theorem 2 is similar to the proof of Theorem 1.

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