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A Counterexample for Boundedness of Some Pseudo-Differential Operators on Pointwise Multipliers Triebel-Lizorkin Space

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We study the boundedness of some pseudo-differential operators of order 0 on pointwise multipliers Triebel-Lizorkin space $M(F^{\eta}_{p,q}(\mathbb{R}^n))$, where $F^{\eta}_{p,q}(\mathbb{R}^n)$ is defined from a positive function $\eta : [0, +\infty) \to [0, +\infty)$ satisfies the condition:

$$\sup_{t>0} t^{-a} \sup_{k=0,1,2...} \frac{\eta(2^{-k})}{\eta(2^{-k}t)} < +\infty \,, \qquad (a \in \mathbb{R}).$$

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1. Introduction

Several authors studied the continuity of the pseudo-differential operators (ps.d.o.) on certain functional spaces, in particular on Besov space $B^s_{p,q}(\mathbb{R}^n)$, or on Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$, can be found in the different works, as [1, 4, 5, 13]. In this work we consider the study of the boundedness on pointwise multipliers Triebel-Lizorkin space $M\big(F^{\eta}_{p\,,\,q}(\mathbb{R}^n)\big)$ of the ps.d.o. $\varrho\,(\cdot, D)$ defined as

$$\varrho\,(\cdot, D)\, f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\cdot\xi} \varrho\,(x,\xi)\, \widehat{f}\,(\xi)\, d\xi \,, \qquad (\forall f \in \mathcal{S}\,, \quad \forall x \in \mathbb{R}^n)\,.$$

We denote by $OP^m_{\rho,\delta}$ the set of all $\varrho\,(\cdot, D)$ such that its symbol $\varrho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ satisfies

$$\left|\partial^{\alpha}_{\xi}\partial^{\beta}_{x}\varrho(x,\xi)\right| \le c\,(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

where α, $\beta \in \mathbb{N}^n$ and δ, ρ, $m \in [0,\infty)$.

The main result of this paper has been proved in the case of Besov space $B_{p,q}^{\eta_a}(\mathbb{R}^n)$ by Moussai [5], where the function $\eta_a : [0,+\infty) \to [0,+\infty)$ satisfies the condition:

$$\sup_{t>0} t^{-a} \sup_{k=0,1,2\dots} \frac{\eta_a\left(2^{-k}\right)}{\eta_a\left(2^{-k}t\right)} < +\infty\,, \qquad (a \in \mathbb{R}). \tag{1}$$

Our contribution is the following result: we shall prove that $B_{p,q}^{\eta_{\frac{n}{p}}} \cap L^\infty$ is included in the pointwise multipliers space $M\big(F_{p,q}^{\eta_a}\big)$ if $-\frac{n}{p} + \max(0, \frac{2n}{p} - n) < a < \frac{n}{p}$.

Theorem 1. *Let* $1 < p < \infty$, $1 < q \leq \infty$, $-\frac{n}{p} + \max(0, \frac{2n}{p} - n) < a < \frac{n}{p}$, *and* η_a *be a positive function satisfies (1), then*

$$B_{p,\infty}^{\eta_{\frac{n}{p}}} \cap L^\infty \hookrightarrow M\big(F_{p,q}^{\eta_a}\big).$$

On the other hand as it is well-known that for all $\psi \in \mathcal{D}$ the function $|\cdot|^\tau \psi$ belongs to $B_{p,\infty}^{\tau+\frac{n}{p}}$ $(\tau > -\frac{n}{p})$, we will give an assertion in this context for $F_{p,q}^{\eta_a}$ which leads to prove that all ps.d.o. of $OP_{1,1}^0$ is not bounded on $M\big(F_{p,q}^{\eta_a}\big)$.

Theorem 2. *Let* $0 < a < \frac{n}{p}$, $1 < p < \infty$, $1 < q \leq \infty$, *and* η_a *be a positive function satisfies (1). Then there exists a ps.d.o.* $\varrho(\cdot, D)$ *in* $OP_{1,0}^0$ *and a function* $h \in M\big(F_{p,q}^{\eta_a}\big)$ *such that* $\varrho(\cdot, D)\, h \notin M\big(F_{p,q}^{\eta_a}\big)$.

2. Preliminaries

2.1. Notation

We will work in the Euclidean space $\mathbb{R}^n$, then all spaces and all functions are defined on $\mathbb{R}^n$. The norm of the Lebesgue space L^p is $\|\cdot\|_p$. We note by $\partial_x^\beta f = \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n} f$ the derivative of f of order a multi-indices β. The notations $\mathcal{F}f = \widehat{f}$ and $\mathcal{F}^{-1}f = \overset{\vee}{f}$ are used for the Fourier transform of f and its inverse. If $g \in L^1_{loc}$, the Hardy-Littlewood maximal function is

$$M\,(g)\,(x) = \sup_{r>0} \frac{1}{|B\,(x,r)|} \int_{B(x,r)} |g\,(y)|\, \mathrm{dy},$$

where $B(x,r)$ is the ball of center x and radius r. The difference operators are

$$\Delta_h^N f = \sum_{k=0}^{N} \binom{N}{k} (-1)^k f(\cdot + (N-k)h) \qquad (N \in \mathbb{N}, h \in \mathbb{R}^n).$$

If $p \in [1,\infty]$, its conjugate p' is $p/(p-1)$.If $x \in \mathbb{R}$, the number $[x]$ denotes the greatest integer less than or equal to x. As usual, constants $c, c_1, \ldots$ are strictly positive and depend only on the fixed parameters n, s, p, q and their values may vary from line to line.

2.2. Series of Littlewood-Paley

Consider a partition of unity

$$\varphi(\xi) + \sum_{j=1}^{\infty} \phi(2^{-j}\xi) = 1 \qquad (\forall \xi \in \mathbb{R}^n),$$

where φ, $\phi \in \mathcal{C}^\infty$ are positive functions such that $\operatorname{supp} \varphi \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\}$ and $\operatorname{supp} \phi \subset \{\xi \in \mathbb{R}^n : 2^{-1} \le |\xi| \le 2\}$. We define the convolution operators S_j and Q_k by the following

$$\begin{aligned} S_j f &= \left(\phi\left(2^{-j}\cdot\right)\widehat{f}\right)^{\vee} \qquad (j = 1, 2, \ \ldots) \\ Q_k f &= \left(\varphi\left(2^{-k}\cdot\right)\widehat{f}\right)^{\vee} \qquad (k = 0, 1, \ \ldots), \end{aligned}$$

and we set $Q_0 = S_0$. Thus we obtain the Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} S_j f$ (convergence in $\mathcal{S}'$). The product $f \cdot g$ is defined bay

$$f \cdot g = \lim_{k\to\infty} Q_k f \cdot Q_k g, \qquad (\forall f, \ g \in \mathcal{S}'),$$

whenever the limit on the right hand side exists in $\mathcal{S}'$(see [8, Section 4.2]). The support of $\mathcal{F} S_k (S_j f \cdot \Delta_\ell g)$ is not empty in one the following cases

$$\begin{array}{lll} \ell \le k+1 & \text{and} & k-2 \le j \le k+4, \\ j \le k+1 & \text{and} & k-2 \le \ell \le k+4, \\ \ell, j \ge k & \text{and} & |\ell - 1| \le 1, \end{array}$$

and we have

$$S_k(f \cdot g) = \sum_{j,\ell=0}^{\infty} S_k(S_j f \cdot S_\ell g) = \left(\Pi_k^1 + \Pi_k^2 + \Pi_k^3\right)(f \cdot g),$$

where

$$\begin{aligned}\Pi_k^1(f\cdot g) &= S_k(Q_{k+1}g\cdot \widetilde{S}_k f),\\ \Pi_k^2(f\cdot g) &= S_k(\widetilde{S}_k g\cdot Q_{k+1}f),\\ \Pi_k^3(f\cdot g) &= \sum_{j=k}^{\infty} S_k(\overline{S}_j g\cdot S_j f),\end{aligned}$$

with $\widetilde{S}_k=\sum_{j=k-2}^{k+4}S_j$ and $\overline{S}_k=\sum_{j=k-1}^{k+1}S_j$.

2.3. Triebel-Lizorkin and Besov spaces.

Let us now recall the definition of $F_{p,q}^{\eta_a}$ and $B_{p,q}^{\eta_a}$. The classical different properties of $B_{p,q}^s$ and $F_{p,q}^s$ (obtained here by taking $\eta_s(t)=t^{-s}$), as equivalent norms and embeddings, can be found in [3, 8, 10, 11, 12].

Definition 1. Let $\gamma>0$, $a\in\mathbb{R}$, $1\le p<\infty$, (*resp.*, $1\le p\le\infty$) and $1\le q\le\infty$. Let η_a be a positive function satisfies (1). The space $L^p(\ell_q^{\eta_a})$ (*resp.*, $\ell_q^{\eta_a}(L^p)$) is the set of the sequences $\{f_k\}_{k\in\mathbb{N}}\subset\mathcal{S}'$, such that $\operatorname{supp}\widehat{f_k}\subset\left\{\xi\in\mathbb{R}^n:|\xi|\le\gamma 2^k\right\}$ *and*

$$\left\|\{f_k\}_{k\in\mathbb{N}}\right\|_{L^p(\ell_q^{\eta_a})}=\left\|\left\{\eta_a(2^{-k})f_k\right\}_{k\in\mathbb{N}}\right\|_{L^p(\ell^q)}<\infty,$$

$$\left(resp.,\ \left\|\{f_k\}_{k\in\mathbb{N}}\right\|_{\ell_q^{\eta_a}(L^p)}=\left\|\left\{\eta_a(2^{-k})f_k\right\}_{k\in\mathbb{N}}\right\|_{\ell^q(L^p)}<\infty\right).$$

Definition 2. (i) Let $a\in\mathbb{R}$, $1\le p<\infty$, $1\le q\le\infty$, and η_a be a positive function satisfies (1), then the Triebel-Lizorkin space is

$$F_{p,q}^{\eta_a}=\left\{f\in\mathcal{S}':\|f\|_{F_{p,q}^{\eta_a}}=\left\|\left\{\eta_a(2^{-k})S_k f\right\}_{k\in\mathbb{N}}\right\|_{L^p(\ell^q)}<\infty\right\}.$$

(ii) Let $a\in\mathbb{R}$, $1\le p,q\le\infty$, and η_a be a positive function satisfies (1), then the Besov space is

$$B_{p,q}^{\eta_a}=\left\{f\in\mathcal{S}':\|f\|_{B_{p,q}^{\eta_a}}=\left\|\left\{\eta_a(2^{-k})S_k f\right\}_{k\in\mathbb{N}}\right\|_{\ell^q(L^p)}<\infty\right\}.$$

Remark 1. We introduce the maximal function

$$S_k^{*,d}f(x)=\sup_{y\in\mathbb{R}^n}\frac{S_k f(x-y)}{1+(2^k|y|)^d},\qquad (\forall x\in\mathbb{R}^n,\ f\in\mathcal{S}',\ d>0,\ k\in\mathbb{N}).$$

Then we can replace $S_k f$ by $S_k^{*,d}f$ with $d>n/\min(p,q)$ (resp. $d>n/p$) in Definition /(i) (resp. (ii)), cf [11, Theorem 2.3.2].

Lemma 1. *(i) Let* $a \in \mathbb{R},\ 0 < p < \infty,\ 0 < u, v, q \leq \infty,$ *and* η_a *be a positive function satisfies (1), then*

$$B^{\eta_a}_{p,u} \hookrightarrow F^{\eta_a}_{p,q} \hookrightarrow B^{\eta_a}_{p,v} \qquad \textit{and} \qquad \ell^{\eta_a}_u(L^p) \hookrightarrow L^p(\ell^{\eta_a}_q) \hookrightarrow \ell^{\eta_a}_v(L^p),$$

for all u *and* v *such that* $0 < u \leq \min(p,q)$ *and* $\max(p,q) \leq v \leq \infty$.

(ii) Let $0 < p_0 < p < p_1 \leq \infty$ *and* $\alpha - (n/p_0) = \beta - (n/p) = a - (n/p_1)$ *and* $0 < u, v, q \leq \infty,$ *then*

$$B^{\eta_\alpha}_{p_0,u} \hookrightarrow F^{\eta_\beta}_{p,q} \hookrightarrow B^{\eta_a}_{p_1,v} \qquad \textit{and} \qquad \ell^{\eta_\alpha}_u(L^{p_0}) \hookrightarrow L^p(\ell^{\eta_\beta}_q) \hookrightarrow \ell^{\eta_a}_v(L^{p_1}),$$

for all u *and* v *such that* $0 < u \leq p \leq v \leq \infty$.

Lemma 2. *Let* $0 < p \leq \infty$ *and* $\gamma > 0$. *Let* $\{f_j\}_{j\in\mathbb{N}} \subset L^p$ *be a sequence of functions such that* $\operatorname{supp} \widehat{f_j} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma 2^j\}$. Then the estimate

$$\|S_k f_j\|_p \leq c\, 2^{(j-k)\rho}\, \|f_j\|_p$$

holds for $\rho = \max(0, \frac{n}{p} - n)$ and $k \leq j < \infty$, where the constant c depends only on n, p and γ.

For Lemma 1. we can see [10, Section 2.3 and 2.8] and [11, Section 2.7], the proof of Lemma is given in [3, Section 2.4].

Proposition 1. *Let* $1 < p < \infty,\ 1 < q \leq \infty,\ a \in \mathbb{R}$ *and* $\gamma > 1$. *Let* η_a *be a positive function satisfies (1). There exists a constant* $c > 0$ *such that*

$$\Big\| \sum_{j=0}^{\infty} g_j \Big\|_{F^{\eta_a}_{p,q}} \leq c \Big\| \Big(\sum_{j=0}^{\infty} \left(\eta_a\left(2^{-j}\right) |g_j|\right)^q \Big)^{1/q} \Big\|_p$$

holds for all $\{g_j\}_{j\in\mathbb{N}} \subset \mathcal{S}'$ *with* $\operatorname{supp} \mathcal{F} g_j \subset \{\xi \in \mathbb{R}^n : \gamma^{-1} 2^j \leq |\xi| \leq \gamma 2^j\}$.

Before the proof of Proposition 1. we shall discuss an estimates in ℓ^q and a maximal inequality.

Lemma 3. *Let* $\{\varepsilon_j\}_{j\in\mathbb{N}} \in \ell^q(\mathbb{R}^+)$, $0 < b < 1$ *and* $1 \leq q \leq \infty$. *Then we have*

$$\Big\| \{ \sum_{j=0}^{k} b^{(k-j)} \varepsilon_j \}_{k\in\mathbb{N}} \Big\|_{\ell^q} + \Big\| \{ \sum_{j=k}^{\infty} b^{(j-k)} \varepsilon_j \}_{k\in\mathbb{N}} \Big\|_{\ell^q} \leq \frac{2}{1-b} \Big\| \{\varepsilon_k\}_{k\in\mathbb{N}} \Big\|_{\ell^q}.$$

Lemma 4. *Let $1 < p < \infty$ and $1 < q \leq \infty$.*
(i) *If $\theta \in L_1$ and $g \in L_1^{loc}$, then*

$$\left|\left(t^{-n}\theta\left(\cdot/t\right)\right) * g(x)\right| \leq \|\theta\|_1 Mg(x), \qquad (\forall t > 0,\ \forall x \in \mathbb{R}^n)\,.$$

(ii) *There exists a constant $c > 0$ such that the following inequality*

$$\Big\|\Big(\sum_{j=0}^{\infty}(Mg_j)^q\Big)^{1/q}\Big\|_p \leq c\,\Big\|\Big(\sum_{j=0}^{\infty}|g_j|^q\Big)^{1/q}\Big\|_p$$

holds, for all sequences $\{g_j\}_{j\in\mathbb{N}}$ of locally Lebesgue-integrable functions.

The proof of Lemma 3. is immediate by using Young's inequality in ℓ^q. However the proof of Lemma 4. can be found in [9, pp. 55] and [8, p. 21].

P r o o f. of P r o p o s i t i o n 1.
(i) It is easy to see that

$$S_k g_j = 0 \quad \text{if} \quad j \geq k+L \quad \text{or} \quad j \leq k-L, \quad (L = 2 + [\log\gamma/\log 2])\,. \tag{2}$$

One has

$$\Big\|\sum_{j=0}^{\infty} g_j\Big\|_{F_{p,q}^{\eta_a}} = \Big\|\Big(\sum_{k=0}^{\infty}\Big|\eta_a(2^{-k})\sum_{j=k-L}^{k+L} S_k g_j\Big|^q\Big)^{1/q}\Big\|_p.$$

One makes the decomposition of $S_k g_j\,(x)$ into $I_1\,(x) + I_2\,(x)$ where

$$I_1\,(x) = 2^{kn}\int\limits_{|x-y|\leq 2^{-k}} \mathcal{F}^{-1}\phi(2^k(x-y))g_j\,(y)\,\mathrm{d}y,$$

$$I_2\,(x) = 2^{kn}\int\limits_{|x-y|\geq 2^{-k}} \mathcal{F}^{-1}\phi(2^k(x-y))g_j\,(y)\,\mathrm{d}y.$$

Clearly from Lemma /(i) we have $|I_1\,(x)| \leq c\,Mg_j\,(x)$. Also we have

$$\begin{aligned} |I_2\,(x)| &\leq 2^{kn}\sum_{\ell\geq 0}\int\limits_{2^{-k+\ell}\leq|x-y|\leq 2^{-k+\ell+1}}\Big|\mathcal{F}^{-1}\phi(2^k(x-y))\Big|\,|g_j\,(y)|\,\mathrm{d}y \\ &\leq c\,Mg_j\,(x)\sum_{\ell\geq 0}2^{\ell(n-N)}. \end{aligned}$$

We choose $N > n$, then $|I_2(x)| \le c\, Mg_j(x)$, where c is independent of j and k. Now according to sign of a we separate the cases, indeed by (1) we have

$$\eta_a(2^{-k}) = \left(2^{-(k-j)a}\frac{\eta_a\left(2^{-k}\right)}{\eta_a\left(2^{-k}2^{k-j}\right)}\right)2^{(k-j)a}\eta_a\left(2^{-j}\right) \le c\,2^{a(k-j)}\eta_a\left(2^{-j}\right),$$

then, in the case $a > 0$, we obtain

$$\begin{aligned}\eta_a(2^{-k})\sum_{j=k-L}^{k+L}|S_k g_j| &\le c_1\sum_{j=k}^{\infty}\frac{\eta_a\left(2^{-k}\right)}{\eta_a\left(2^{-k}2^{k-j}\right)}\eta_a\left(2^{-j}\right)|S_{k+L}g_j| \\ &\le c_2\,2^{ka}\sum_{j=k}^{\infty}2^{-aj}\eta_a(2^{-j})Mg_j, \qquad (3)\end{aligned}$$

and in the case $a < 0$, we obtain

$$\begin{aligned}\eta_a(2^{-k})\sum_{j=k-L}^{k+L}|S_k g_j| &\le c_1\sum_{j=0}^{k}\frac{\eta_a\left(2^{-k}\right)}{\eta_a\left(2^{-k}2^{k-j}\right)}\eta_a\left(2^{-j}\right)|S_{k-L}g_j| \\ &\le c_2\,2^{ka}\sum_{j=0}^{k}2^{-aj}\eta_a\left(2^{-j}\right)Mg_j. \qquad (4)\end{aligned}$$

In both (3) and (4) we calculate $L^p(\ell^q)$–norm, we apply Lemma and Lemma /(ii), successively, then we obtain the desired bound.

If $a = 0$, we immediately get

$$\begin{aligned}\eta_a(2^{-k})\sum_{j=k-L}^{k+L}|S_k g_j| &\le c_1\sum_{j=k-L}^{k+L}\eta_0\left(2^{-j}\right)Mg_j \\ &\le c_2\,(2L+1)^{1/q'}\left(\sum_{j=k-L}^{k+L}\left(\eta_0\left(2^{-j}\right)Mg_j\right)^q\right)^{1/q},\end{aligned}$$

which implies

$$\begin{aligned}\left\|\sum_{j=0}^{\infty}g_j\right\|_{F^{\eta_0}_{p,q}} &\le c_3\left\|\left(\sum_{j=0}^{\infty}\left(\eta_0\left(2^{-j}\right)Mg_j\right)^q\sum_{j=k-L}^{k+L}1\right)^{1/q}\right\|_p \\ &\le c_4\left\|\left(\sum_{j=0}^{\infty}\left(\eta_0\left(2^{-j}\right)Mg_j\right)^q\right)^{1/q}\right\|_p.\end{aligned}$$

■

Proposition 2. *Let* $1\leq p<\infty$, $1\leq q\leq\infty$, *and* $0<a<N$ *where* $N\in\mathbb{N}$. *Let* η_a *be a positive function satisfies (1). We set*

$$\|f\|^{(1)}_{F^{\eta_a}_{p,q}}=\|f\|_p+\big\|\Big(\int_0^1\big(\eta_a(t)\sup_{|h|<t}\big|\Delta_h^N f\big|\big)^q\frac{\mathrm{d}t}{t}\Big)^{1/q}\big\|_p.$$

Then $\|\cdot\|_{F^{\eta_a}_{p,q}}$ *and* $\|\cdot\|^{(1)}_{F^{\eta_a}_{p,q}}$ *are equivalent norms in* $F^{\eta_a}_{p,q}$.

Proof. *Step 1.* We prove that there exists a constant $c>0$ such that

$$\|f\|_{F^{\eta_a}_{p,q}}\leq c\,\|f\|^{(1)}_{F^{\eta_a}_{p,q}}. \tag{5}$$

Knowing that $\phi(0)=0$, we derive that

$$S_jf(x)=\int_{\mathbb{R}^n}\mathcal{F}^{-1}\phi(-h)\triangle^1_{2^{-j}h}f(x)\ \mathrm{d}h. \tag{6}$$

We combine (6) with the equality

$$\Delta_h^N f(x)=\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\,\Delta_h^1 f\left(x+(N-k-1)h\right),$$

and we obtain

$$S_jf(x)+A_{j,N}(x)=(-1)^{N-1}\int_{\mathbb{R}^n}\mathcal{F}^{-1}\phi(-h)\triangle^N_{2^{-j}h}f(x)\ \mathrm{d}h, \tag{7}$$

where

$$A_{j,N}(x)=\sum_{k=0}^{N-2}\binom{N-1}{k}(-1)^{k+N-1}\int_{\mathbb{R}^n}\mathcal{F}^{-1}\phi(-h)\times \triangle^1_{2^{-j}h}f\left(x+2^{-j}(N-k-1)h\right)\ \mathrm{d}h,$$

for $N\geq 2$ and $A_{j,1}\equiv 0$.

Next we use the Koch and Sickel's idea [7] (but not in the sense of Nikol'skij [6, 5.2.1]). For that, let $I_0=[0,1)$ and $I_\ell=\left[2^{\ell-1},2^\ell\right)$, $\ell\in\mathbb{N}$. Then we estimate the right hand side of (7) by

$$\begin{aligned}|S_jf(x)+A_{j,N}(x)| &\leq c_1\sum_{\ell=0}^{\infty}\int_{I_\ell}\int_{|h'|=1}t^{n-1}\left|\mathcal{F}^{-1}\phi(-th')\right|\left|\triangle^N_{2^{-j}th'}f(x)\right|\ \mathrm{d}t\mathrm{d}h'\\ &\leq c_2\sum_{\ell=0}^{\infty}2^{-\ell L}\sup_{|u|\leq 2^{\ell-j}}\left|\triangle^N_u f(x)\right|,\end{aligned}$$

where $L > 1$ is an arbitrary natural number. On take the norm of both sides, and passing by the triangle inequality, we get

$$\left\|f\right\|_{F^{\eta_a}_{p,q}} \leq c_1 \Big\| \sum_{\ell=0}^{\infty} 2^{-\ell L} \Big(\sum_{j=0}^{\infty} \Big(\eta_a\left(2^{-j}\right) \sup_{|u|\leq 2^{\ell-j}} \left|\triangle_u^N f\right| \Big)^q \Big)^{1/q} \Big\|_p . \tag{8}$$

The left hand-side in (8) is bounded by

$$\begin{aligned} c_2 \Big\| \sum_{\ell=0}^{\infty} 2^{-\ell L} \Big[\sup_{k=\ell+1,\ldots} \Big(\frac{\eta_a\left(2^{-k}\right)}{\eta_a\left(2^{\ell-k}\right)} \Big)^q \sum_{j=\ell+1}^{\infty} \Big(\eta_a(2^{\ell-j}) \sup_{|u|\leq 2^{\ell-j}} \left|\triangle_u^N f\right| \Big)^q \Big]^{1/q} \Big\|_p \\ \leq c_3 \Big\| \sum_{\ell=0}^{\infty} 2^{\ell(a-L)} \Big(\int_0^1 \Big(\eta_a\left(t\right) \sup_{|u|\leq t} \left|\triangle_u^N f\right| \Big)^q \frac{\mathrm{d}t}{t} \Big)^{1/q} \Big\|_p , \end{aligned}$$

then it suffices to choose $L > 2a$ to get (5).

Step 2. We prove the converse assertion of (5). As above, there exists a constant $c > 0$, such that $\eta_a\left(t\right) \leq c\, 2^{-a\ell}\, t^{-a}\, \eta_a(2^{-\ell})$ for all $0 < t < 1$ and $\forall \ell \in \mathbb{N}$, then we obtain

$$\begin{aligned} \int_0^1 \Big(\eta_a\left(t\right) \sup_{|u|\leq t} \left|\triangle_u^N f\right| \Big)^q \frac{\mathrm{d}t}{t} &\leq c_1 \sum_{\ell=0}^{\infty} \Big(2^{-a\ell} \eta_a(2^{-\ell}) \sup_{|u|\leq 2^{-\ell}} \left|\triangle_u^N f\right| \Big)^q \int_{2^{-\ell}}^{2^{-\ell+1}} t^{-aq-1} \mathrm{d}t \\ &\leq c_2 \sum_{\ell=0}^{\infty} \Big(\eta_a(2^{-\ell}) \sup_{|u|\leq 2^{-\ell}} \left|\triangle_u^N f\right| \Big)^q . \end{aligned}$$

Let $u \in \mathbb{R}^n$ be such that $|u| \leq 2^{-\ell}$ and $j = 0, ..., \ell$, we have

$$\sup_{|u|\leq 2^{-\ell}} \left|\triangle_u^N \left(S_j f\right)\left(x\right)\right| \leq c\, \min(1,\ 2^{(j-\ell)N}) S_j^{*,d} f\left(x\right) . \tag{9}$$

The proof of (9) is given in [10, p. 111]. We split $\triangle_u^N f$ into

$$\Big(\sum_{j=0}^{\ell} + \sum_{j=\ell+1}^{\infty} \Big) \triangle_u^N \left(S_j f\right) = A_\ell + B_\ell$$

and we estimate each of the two terms. Choose two real numbers r and t such that $0 < r < a < t$, and using (9) where we replace d by r and t in the estimate of A_ℓ and B_ℓ respectively, we obtain

$$\left\|\{A_\ell\}_{\ell\in\mathbb{N}}\right\|_{L^p(\ell_q^{\eta_a})} \leq c \Big\| \Big(\sum_{\ell=0}^{\infty} \Big(2^{-(t-a)\ell} \sum_{j=0}^{\ell} 2^{(t-a)j} \eta_a\left(2^{-j}\right) \left|S_j^{*,t} f\right| \Big)^q \Big)^{1/q} \Big\|_p \tag{10}$$

and

(11)

$$\Big\|\{B_\ell\}_{\ell\in\mathbb{N}}\Big\|_{L^p(\ell_q^{\eta_a})} \le c\Big\|\Big(\sum_{\ell=0}^{\infty}\Big(2^{(a-r)\ell}\sum_{j=\ell+1}^{\infty}2^{-(a-r)j}\eta_a\left(2^{-j}\right)\left|S_j^{*,r}f\right|\Big)^q\Big)^{1/q}\Big\|_p .$$

To end the proof, we apply Lemma to (10) and (11). ∎

2.4. Pointwise multipliers Triebel-Lizorkin space

Definition 3. Let $a\in\mathbb{R}$, $1\le p<\infty$, $1\le q\le\infty$, and η_a be a positive function satisfies (1). Then we define

$$M\big(F_{p,q}^{\eta_a}\big)=\Big\{f\in L_1^{loc}\ :\quad f\cdot g\in F_{p,q}^{\eta_a}\qquad\text{for all}\qquad g\in F_{p,q}^{\eta_a}\Big\}$$

and call the elements of $M\big(F_{p,q}^{\eta_a}\big)$ pointwise multipliers of $F_{p,q}^{\eta_a}$.

Now, we shall give some properties of $M\big(F_{p,q}^{\eta_a}\big)$ necessary for us.

Proposition 3. *Let $a>0$, $1\le p<\infty$, $1\le q\le\infty$ and η_a be a positive function satisfies (1), then*

(i) $\big(F_{p,q}^{\eta_a}\big)'=F_{p',q'}^{1/\eta_a}$,

(ii) $M\big(F_{p,q}^{\eta_a}\big)=M\big(F_{p',q'}^{1/\eta_a}\big)$,

(iii) $M\big(F_{p,q}^{\eta_a}\big)\hookrightarrow L^\infty$.

P r o o f. Let $f\in F_{p,q}^{\eta_a}$ and $g\in F_{p',q'}^{1/\eta_a}$. We first prove the case $a>0$: Using the identity

$$\langle f\mid g\rangle=\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}S_jf\left(x\right)\cdot Q_jg\left(x\right)\mathrm{d}x\ +\ \int_{\mathbb{R}^n}\sum_{k=1}^{\infty}S_k\widetilde{f}\left(x\right)\cdot Q_{k-1}\widetilde{g}\left(x\right)\mathrm{d}x, \tag{12}$$

where $\widetilde{f}\left(x\right)=f\left(-x\right)$ and $\widetilde{g}\left(x\right)=g\left(-x\right)$. Then, by Hölder's inequality in ℓ^q, the fact that $(1/\eta_a(2^{-j})\le c\,(2^{a(k-j)}/\eta_a(2^{-k}))$, and take to Lemma , we obtain

$$\Big|\int\sum_{j=0}^{\infty}S_jf\left(x\right)\cdot Q_jg\left(x\right)\,\mathrm{d}x\Big|$$

$$\le c_1\int\big\|\big\{\eta_a\left(2^{-j}\right)\left|S_jf\left(x\right)\right|\big\}_{j\in\mathbb{N}}\big\|_{\ell^q}\Big\|\Big\{2^{-ja}\sum_{k=0}^{j}2^{ak}\frac{1}{\eta_a\left(2^{-k}\right)}\left|S_kg\left(x\right)\right|\Big\}_{j\in\mathbb{N}}\Big\|_{\ell^{q'}}\mathrm{d}x$$

$$\le c_2\left\|f\right\|_{F_{p,q}^{\eta_a}}\ \left\|g\right\|_{F_{p',q'}^{1/\eta_a}} .$$

Similarly, we derive

$$\Big|\int\sum_{k=1}^{\infty}S_k\widetilde{f}\,(x)\cdot Q_{k-1}\widetilde{g}\,(x)\;\mathrm{d}x\Big|\leq c\left\|\widetilde{f}\right\|_{F^{\eta_a}_{p,q}}\left\|\widetilde{g}\right\|_{F^{1/\eta_a}_{p',q'}}\;,$$

and since $\Delta^N_h\widetilde{f}(x)=\Delta^N_{-h}f(x)$, the Proposition 2 gives the correct bound. Next, we prove the case $a<0$: As in (12), we have

$$\langle f\mid g\rangle=\int\sum_{j=0}^{\infty}Q_jf\,(x)\cdot S_jg\,(x)\,\mathrm{d}x\;+\;\int\sum_{k=1}^{\infty}Q_{k-1}\widetilde{f}\,(x)\cdot S_k\widetilde{g}\,(x)\,\mathrm{d}x\;,$$

then we get the estimation

$$\Big|\int\sum_{j=0}^{\infty}Q_jf\,(x)\cdot S_jg\,(x)\;\mathrm{d}x\Big|\leq c\;\|f\|_{F^{\eta_a}_{p,q}}\;\|g\|_{F^{1/\eta_a}_{p',q'}}\,.$$

It remain to prove the last case $a=0$: Remark that, there exists $y_{j,x}\in\mathbb{R}^n$ such that $|x-y_{j,x}|\leq 2^{-j}$ for all j and x. Choosing $d>(n/\min(p,q))$, we obtain

$$\Big|\int\sum_{j=0}^{\infty}S_jf\,(x)\cdot Q_jg\,(x)\,\mathrm{d}x\Big| \tag{13}$$

$$\leq\int\sum_{j=0}^{\infty}|S_jf\,(x)|\,\Big(\sum_{k=0}^{j}\frac{|S_kg\,(x)|}{1+(2^k\,|x-y_{j,x}|)^d}\big(1+(2^k\,|x-y_{j,x}|)^d\big)\Big)\mathrm{d}x\,.$$

Since $\eta_0\left(2^{-j}\right)\leq c\,\eta_0\left(2^{-k}\right)$, the last term of (13) is bounded by

$$c\int\big\|\{\eta_0\left(2^{-j}\right)|S_jf(x)|\,\}_{j\in\mathbb{N}}\big\|_{\ell^q}\Big\|\Big\{\sum_{k=0}^{j}2^{d(k-j)}\;\frac{1}{\eta_0\left(2^{-k}\right)}\big|S^{*,d}_kg(x)\big|\Big\}_{j\in\mathbb{N}}\Big\|_{\ell^{q'}}$$

the we use the Hölder inequality. These cases imply $F^{1/\eta_a}_{p',q'}\hookrightarrow\left(F^{\eta_a}_{p,q}\right)'$.

For the converse embedding, let $g\in\left(F^{\eta_a}_{p,q}\right)'$ and consider the mapping $f\in F^{\eta_a}_{p,q}\to g\,(f)=\langle f\mid g\rangle$. By Hahn-Banach's theorem and [10, Proposition 2.11.1/(i)] we obtain

$$\Big\|\Big(\sum_{k=0}^{\infty}\Big(\frac{1}{\eta_a\left(2^{-k}\right)}\,|S_kg|\Big)^{q'}\Big)^{1/q'}\Big\|_{p'}\;\sim\;\|g\|$$

($\|g\|$ denotes the norm of the continuous linear application g), thus $g\in F^{1/\eta_a}_{p',q'}$, and $\left(F^{\eta_a}_{p,q}\right)'\hookrightarrow F^{1/\eta_a}_{p',q'}$.

In order to show (ii), let $b \in M\big(F^{\eta_a}_{p,q}\big)$, $f \in F^{1/\eta_a}_{p',q'}$ and $g \in F^{\eta_a}_{p,q}$. The product $\langle bf \mid g\rangle$ can be written as

$$\int \sum_{j=0}^{\infty} Q_j f(x) \cdot S_j(bg)(x)\,\mathrm{d}x + \int \sum_{k=1}^{\infty} Q_{k-1}\widetilde{f}(x) \cdot S_k\widetilde{bg}(x)\,\mathrm{d}x \quad (\text{if } a > 0),$$

or

$$\int \sum_{j=0}^{\infty} S_j f(x) \cdot Q_j(bg)(x)\,\mathrm{d}x + \int \sum_{k=1}^{\infty} S_k\widetilde{f}(x) \cdot Q_{k-1}\widetilde{bg}(x)\,\mathrm{d}x \quad (\text{if } a < 0),$$

or

$$\Big|\int \sum_{j=0}^{\infty} Q_j f(x) \cdot S_j(bg)(x)\,\mathrm{d}x\Big| \leq$$

$$c\int \sum_{j=0}^{\infty} \Big(\sum_{k=0}^{j} 2^{d(k-j)} \frac{1}{\eta_0\left(2^{-k}\right)} S_k^{*,d} f(x)\Big)\eta_0\left(2^{-j}\right) |S_j(bg)(x)| \quad (\text{if } a = 0, \text{ see (13)}),$$

and the estimate $|\langle bf \mid g\rangle| \leq c\,\|f\|_{F^{1/\eta_a}_{p',q'}} \|bg\|_{F^{\eta_a}_{p,q}}$ can be obtained in a similar manner to the proof of (i). It follows that,

$$\|bf\|_{F^{1/\eta_a}_{p',q'}} \leq \sup_{\|g\|_{F^{\eta_a}_{p,q}}=1} |\langle bf \mid g\rangle| \leq c\,\|f\|_{F^{1/\eta_a}_{p',q'}}, \quad (\forall a \in \mathbb{R}),$$

which proves that $M\big(F^{\eta_a}_{p,q}\big) \hookrightarrow M\big(F^{1/\eta_a}_{p',q'}\big)$. The converse embedding is obtained by the same technique.

Finally, to prove (iii), let u_s and v_t be two positive functions satisfies (1). The real interpolation see [10, Section 2.4.2] yields

$$\big(F^{u_s}_{p_0,q_0}, F^{v_t}_{p_1,q_1}\big)_{\theta,p} = F^{\eta_r}_{p,q},$$

where

$$\tfrac{1}{p} = \tfrac{\theta}{p_0} + \tfrac{1-\theta}{p_1}, \ \tfrac{1}{q} = \tfrac{\theta}{q_0} + \tfrac{1-\theta}{q_1}, \quad \eta_r = u_s^{\theta} v_t^{1-\theta}, \quad r = \theta s + (1-\theta)t, \quad 0 < \theta < 1.$$

Let $b\ in M\big(F^{\eta_a}_{p,q}\big)$ by duality we have $b \in M\big(F^{1/\eta_a}_{p',q'}\big)$. Remark that $\big(F^{\eta_a}_{2,2}, F^{1/\eta_a}_{2,2}\big)_{1/2,2} = F^{\eta_0}_{2,2} = L^2$, therefore $b \in M\left(L^2\right) = L^{\infty}$ ($M\left(L^2\right)$ denotes the pointwise multipliers space of L^2). This completes the proof of (iii). ∎

3. Proof of Theorem 1

Let $f \in F^{\eta_a}_{p,q}$ and $g \in \left(B^{\eta_{\frac{n}{p}}}_{p,\infty} \cap L^\infty\right)$, according to the decomposition of $f \cdot g$ and Proposition 1., we obtain

$$\|f \cdot g\|_{F^{\eta_a}_{p,q}} \leq c \sum_{i=1}^{3} \left\| \left\{ \eta_a(2^{-k}) \Pi^i_k \left(f \cdot g\right) \right\}_{k\in\mathbb{N}} \right\|_{L^p(\ell^q)}.$$

Then we estimate respectively $\Pi^1_k\left(f \cdot g\right)$, $\Pi^2_k\left(f \cdot g\right)$ and $\Pi^3_k\left(f \cdot g\right)$ in $L^p(\ell^{\eta_a}_q)$-norm.

Estimate of $\left\{\Pi^1_k\left(f \cdot g\right)\right\}_{k\in\mathbb{N}}$. Since

$$\left|\Pi^1_k\left(f \cdot g\right)(x)\right| \leq c \|g\|_\infty \left(\widetilde{S}^{*,b}_k f\right)(x),$$

where $\widetilde{S}^{*,b}_k f$ is defined as in Remark 1 , we derive

$$\left\| \left\{ \eta_a(2^{-k}) \Pi^1_k \left(f \cdot g\right) \right\}_{k\in\mathbb{N}} \right\|_{L^p(\ell^q)} \leq c \|g\|_\infty \left\| \left\{ \eta_a(2^{-k}) \widetilde{S}^{*,b}_k f \right\}_{k\in\mathbb{N}} \right\|_{L^p(\ell^q)}.$$

Choosing $b > n/\min(p,q)$, we get

$$\left\| \left\{ \eta_a(2^{-k}) \Pi^1_k \left(f \cdot g\right) \right\}_{k\in\mathbb{N}} \right\|_{L^p(\ell^q)} \leq c \|g\|_\infty \left\| \left\{ \eta_a(2^{-k}) S_k f \right\}_{k\in\mathbb{N}} \right\|_{L^p(\ell^q)}.$$

Estimate of $\left\{\Pi^2_k\left(f \cdot g\right)\right\}_{k\in\mathbb{N}}$. Let $u \in \mathbb{R}$ such that

(14) $$\max\left(p, n/(n/p - a)\right) < u < \infty.$$

We set

(15) $$\frac{1}{v} = \frac{1}{p} + \frac{1}{u},\ \sigma = a - \frac{n}{p} + \frac{n}{v},\ \text{and } \beta = a - \frac{n}{p} + \frac{n}{u}.$$

Then, the following inclusions hold (see [3, Section 2.3])

(16) $$\ell^{\eta_\sigma}_p(L_v) \hookrightarrow L^p(\ell^{\eta_a}_q) \text{ and } F^{\eta_a}_{p,q} \hookrightarrow B^{\eta_\beta}_{u,p}.$$

On the other hand, the Hölder inequality in L^p-norm yields

$$\eta_\sigma(2^{-k}) \left\|\Pi^2_k(f \cdot g)\right\|_v \leq c\, \eta_{\frac{n}{p}}(2^{-k}) \left\|\widetilde{S}_k g\right\|_p 2^{k\beta} \sum_{j=0}^{k+1} 2^{-j\beta} \left(\eta_\beta\left(2^{-j}\right) \|S_j f\|_u\right).$$

Next the Hölder inequality in ℓ^p-norm and Lemma , gives

$$\left\|\left\{\eta_a(2^{-k})\Pi_k^2(f\cdot g)\right\}_{k\in\mathbb{N}}\right\|_{L^p(\ell^q)} \leq c\,\|g\|_{B^{\eta_{\frac{n}{p}}}_{p,\infty}}\|f\|_{F^{\eta_a}_{p,q}}.$$

Estimate of $\left\{\Pi_k^3\,(f\cdot g)\right\}_{k\in\mathbb{N}}$. We first consider $p\geq 2$. Let $u\in\mathbb{R}$ such that $p<u<\infty$, applying Lemma , we find

$$\eta_\sigma(2^{-k})\left\|\Pi_k^3\,(f\cdot g)\right\|_v \leq c\,2^{k(\beta+\frac{n}{p})}\sum_{j=k}^{\infty}2^{-j(\beta+\frac{n}{p})}\left(\eta_{\frac{n}{p}}\left(2^{-j}\right)\left\|\overline{S_j}g\right\|_p\right)\left(\eta_\beta\left(2^{-j}\right)\|S_jf\|_u\right),$$

where $v,\ \sigma$ and β are defined in (15). Arguing as above we obtain

(17)
$$\begin{aligned}\left\|\left\{\eta_\sigma\left(2^{-k}\right)\Pi_k^3\,(f\cdot g)\right\}_{k\in\mathbb{N}}\right\|_{\ell^p(L^v)} &\leq c\left\|\{2^{k(\beta+\frac{n}{p})}\sum_{j=k}^{\infty}2^{-j(\beta+\frac{n}{p})}(\eta_{\frac{n}{p}}\left(2^{-j}\right)\left\|\overline{S_j}g\right\|_p)\right.\\ &\quad\left.\times\left(\eta_\beta\left(2^{-j}\right)\|S_jf\|_u\right)\}_{k\in\mathbb{N}}\right\|_{\ell^p}.\end{aligned}$$

Thanks to the Hölder inequality, the right hand of (17) is bounded by $\|g\|_{B^{\eta_{\frac{n}{p}}}_{p,\infty}}\|f\|_{B^{\eta_\beta}_{u,p}}$. Thus we conclude the desired estimate by (16).

For the case $p<2$, let $u\in\mathbb{R}^n$ satisfying $p<u<n/\max(0,n-n/p)$, then we set $\frac{1}{v}=\frac{1}{p}+\frac{1}{u},\ (v<1)$, and $\sigma=s-\frac{n}{p}+\frac{n}{v}$. Using Lemma and Hölder's inequality we get

$$\begin{aligned}\eta_\sigma(2^{-k})\left\|\Pi_k^3\,(f\cdot g)\right\|_v &\leq c_1\,\eta_\sigma(2^{-k})\sum_{j=k}^{\infty}2^{(j-k)\left(\frac{n}{p}+\frac{n}{v}-n\right)}\left\|\overline{S_j}g\right\|_p\|S_jf\|_u\\ &\leq c_2\sup_{k\geq 0}\left(\eta_{\frac{n}{p}}\left(2^{-j}\right)\left\|\overline{S_j}g\right\|_p\right)2^{k\mu}\sum_{j=k}^{\infty}2^{-j\mu}\left(\eta_\rho\left(2^{-j}\right)\|S_jf\|_u\right),\end{aligned}$$

where $\rho=s-\frac{n}{p}+\frac{n}{u}$ and $\mu=s-\frac{n}{p}+n>0$ which allow us to apply Lemma , therefore

$$\left\|\left\{\eta_\sigma(2^{-k})\Pi_k^3\,(f\cdot g)\right\}_{k\in\mathbb{N}}\right\|_{\ell^p(L^v)} \leq c\,\|g\|_{B^{\eta_{\frac{n}{p}}}_{p,\infty}}\|f\|_{B^{\eta_\rho}_{u,p}}.$$

Since $\sigma>s,\ v<p$ and $F^{\eta_a}_{p,q}\hookrightarrow B^{\eta_\rho}_{u,p}$, we conclude by using the embeddings in (16). ■

4. Proof of Theorem 2

We set $h(x) = \vartheta(\frac{x}{|x|})\psi(x)$ where $\vartheta(x) > 0$, $\operatorname{supp}\vartheta \subset \{x = (x_1, x') \in S^{n-1} : x_1 \le -1/2\}$, $\psi(x) = 0$ if $|x| \le (1/2)$, $\psi(x) = 1$ if $(1/2) \le |x| \le 1$ and $0 \le \psi(x) \le 1$. It is well-known that $h \in B^{\frac{n}{p}}_{p,\infty}$ (see [5, Proposition 4.3]), then by Theorem , $h \in M(F^{\eta a}_{p,q})$ for $0 < a < (n/p)$. Let

$$\varrho(\xi) = \tfrac{\xi_1}{|\xi|}\big(1 - \widehat{\lambda}(\xi)\big),$$

where $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\widehat{\lambda} \in \mathcal{D}$ which equals 1 near the origin. Since $\left|\partial^\alpha_\xi \varrho(\xi)\right| \le c(1 + |\xi|)^{-|\alpha|}$ we have

$$\varrho(\cdot, D) \in OP^0_{1,0},$$

and, we can verify that

$$\varrho(\cdot, D)\, h \notin M\big(F^{\eta a}_{p,q}\big).$$

Indeed, $\varrho(\cdot, D)\, h = R_1 h - \lambda * R_1 h$ where R_1 is the Riesz transform which is bounded on L^2, (see [9, p. 56] or [2, p. 86]). Thus, we get

$$\|\lambda * R_1 h\|_\infty \le \|h\|_2 \, \|\lambda\|_2 \, .$$

Using the identity

$$\int_{\mathbb{R}^n} \tfrac{\xi_1}{|\xi|}\phi(\xi)\, \mathrm{d}\xi = c_n \int_{\mathbb{R}^n} \tfrac{x_1}{|x|^{n+1}}\widehat{\phi}(x)\, \mathrm{d}x \qquad (\forall \phi \in \mathcal{D}) \tag{18}$$

for $x, y \in \mathbb{R}^n \setminus \{0\}$, such that $x_1 \ge (|x|/2)$ and $y_1 \le (-|y|/2)$ (i.e. $x_1 - y_1 \ge \frac{1}{2}|x - y|$), then

$$\begin{aligned} R_1 h(x) &= c_n \lim_{\varepsilon \to 0} \int_{|x-y| \ge \varepsilon} \tfrac{(x_1 - y_1)}{|x-y|^{n+1}}\, h(y)\, \mathrm{d}y \\ &\ge c \int_{|x| \le |y| \le 1} |y|^{-n}\, \vartheta(\tfrac{y}{|y|})\, \mathrm{d}y \ge c' \,|Log\,|x||\,, \end{aligned}$$

and we deduce the result by Proposition 3/(iii). It remains to prove (18). It suffices to see that the locally integrable function $f(\xi) = \xi_1^k/|\xi|^a$ (for $k = 1, 2, ...$ and $0 < a < n + 2k$), is the Fourier transform in $\mathcal{D}'$, of the function $\mathcal{F}^{-1} f(x) = x_1^k/|x|^{n-a+2k}$, (cf. see [9, ch. 3/(31) p. 73]). Indeed, since the function $(i\xi_1)^k e^{-\frac{1}{2t}|\xi|^2}$is the Fourier transform of $(t/2\pi)^{\frac{n}{2}}\, \partial^k_{x_1}(e^{-\frac{t}{2}|x|^2})$, then

$$(2\pi)^{-\frac{n}{2}}\, t^{k+\frac{n}{2}} \int_{\mathbb{R}^n} \big(-\tfrac{1}{2}x_1\big)^k e^{-\frac{t}{2}|x|^2}\, \widehat{\phi}(x)\, \mathrm{d}x = \int_{\mathbb{R}^n} (i\xi_1)^k e^{-\frac{1}{2t}|\xi|^2}\, \phi(\xi)\, \mathrm{d}\xi,$$

for any $\phi \in \mathcal{D}$. We multiply both sides of this equality by $t^{-1-\frac{a}{2}}$, we integrate on t in $[0,\infty)$ and using the fact that

$$\int_0^\infty t^{\frac{n-a}{2}+k-1} e^{-\frac{t}{2}|x|^2}\,\mathrm{d}t = \frac{2^{k+(n-a)/2}}{|x|^{n-a+2k}} \Gamma\left(k+\frac{n-a}{2}\right),$$

$$\int_0^\infty t^{-1-\frac{a}{2}} e^{-\frac{1}{2t}|\xi|^2}\,\mathrm{d}t = \frac{2^a}{|\xi|^a} \Gamma\left(\frac{a}{2}\right),$$

then we conclude the desired conclusion by Fubini's theorem.

■

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