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Power Series Description of the Commutant of Powers of the Dunkl Operator

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In this paper the commutant of an arbitrarily fixed power n of the Dunkl operator $D_k f(z) = \frac{df(z)}{dz} + k\frac{f(z)-f(-z)}{z}$ with parameter $k \geq 0$ is described as power series in the space A_R of the analytic functions in the disk $D_R = \{z \in \mathbb{C} : |z| < R\}$.

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1. Introduction

The Dunkl operator D_k was defined in [3], in 1989, and since then many mathematicians have studied its different properties and applications.

Here we start with some definitions.

Denote by A_R the space of the analytic functions in the disk $D_R = \{z \in \mathbb{C} : |z| < R\}$.

Definition 1. For $f \in A_R$, the operator $D_k : A_R \to A_R$ defined by

$$D_k f(z) = \frac{df(z)}{dz} + k\frac{f(z)-f(-z)}{z} \tag{1}$$

is called the *Dunkl operator* with parameter $k \geq 0$.

Through this paper the parameter k will be arbitrary, but fixed, therefore we will write only "Dunkl operator" without mentioning the parameter k.

Definition 2. It is said that a continuous linear operator M *commutes with a fixed operator* L, if $ML = LM$. The set of all such operators is called the *commutant of* L and will be denoted by CL.

In a previous paper [7] we described the commutant of the Dunkl operator. Here our goal is to extend this description to the case of an arbitrarily fixed power D_k^n of the Dunkl operator D_k.

2. Representation of the commutant

Theorem 1. *Let $f \in A_R$ be an analytic function in D_R with a Taylor series $f(z) = \sum_{m=0}^{\infty} a_m z^m$. Then every continuous linear operator $M : A_R \to A_R$ commutes with an arbitrarily fixed power n of the Dunkl operator D_k, i.e. $M \in CD_k^n$, if and only if it can be represented in a power series form as*

$$
\begin{aligned}
(2) \qquad Mf(z) &= \sum_{\mu=0}^{n-1} \sum_{m=0}^{\infty} a_m b_{m,\mu} z^{\mu} + \\
&+ \sum_{\mu=n}^{\infty} \sum_{m=[\frac{\mu}{n}]n}^{\infty} a_m \frac{c_m \dots c_{m-[\frac{\mu}{n}]n+1}}{c_\mu \dots c_{\mu-[\frac{\mu}{n}]n+1}} b_{m-[\frac{\mu}{n}]n,\mu-[\frac{\mu}{n}]n} \cdot z^{\mu},
\end{aligned}
$$

where

$$(3) \qquad c_m = m + k(1-(-1)^m), \quad m \geq 0,$$

$k \geq 0$ is the parameter of the Dunkl operator, $[A]$ denotes the integer part of the number A, and $b_{m,\mu}$, $0 \leq \mu \leq n-1$, $m = 0, 1, 2, \dots$, are arbitrary complex numbers such that the series in the representation (0.2) *are convergent.*

Proof. First, let us consider the action of the Dunkl operator D_k on a single power z^m of the variable $z \in \mathbb{C}$. If the power is even, i.e. $m = 2s$, then

$$D_k z^{2s} = \begin{cases} \dfrac{dz^{2s}}{dz} + k\dfrac{z^{2s} - (-z)^{2s}}{z} = 2sz^{2s-1} & \text{for } s \geq 1, \\ 0 & \text{for } s = 0. \end{cases}$$

If the power is odd, i.e. $m = 2s+1$, then

$$D_k z^{2s+1} = \frac{dz^{2s+1}}{dz} + k\frac{z^{2s+1} - (-z)^{2s+1}}{z} = (2s+1)z^{2s} + 2kz^{2s} = (2s+1+2k)z^{2s}.$$

The two representations can be combined in one formula:

$$(4) \qquad D_k z^m = \begin{cases} c_m z^{m-1}, \ c_m = m + k[1-(-1)^m] & \text{for } m \geq 1, \\ 0 & \text{for } m = 0. \end{cases}$$

Next, if $n \geq 1$ is an arbitrarily fixed natural number, (0.4) implies that

$$(5) \qquad D_k^n z^m = \begin{cases} c_m c_{m-1} \dots c_{m-n+1} z^{m-n} & \text{for } m \geq n, \\ 0 & \text{for } 0 \leq m \leq n-1. \end{cases}$$

Now consider an arbitrary operator M from the commutant CD_k. Let us represent its action on any fixed power m of z by the power series

$$(6) \qquad Mz^m = \sum_{\mu=0}^{\infty} d_{m\mu} z^{\mu}, \quad m = 0, 1, 2, \dots$$

with unknown coefficients $d_{m\mu}$ which will be determined.

Next step is to find expressions for $MD_k^n z^m$ and $D_k^n M z^m$:

$$(7) \qquad MD_k^n z^m = \begin{cases} Mc_m \dots c_{m-n+1} z^{m-n} = \sum_{\mu=0}^{\infty} c_m \dots c_{m-n+1} d_{m-n,\mu} z^{\mu}, & m \geq n, \\ 0, & 0 \leq m \leq n-1. \end{cases}$$

$$(8) \qquad \begin{aligned} D_k^n M z^m &= D_k^n \sum_{\mu=0}^{\infty} d_{m\mu} z^{\mu} = \sum_{\mu=0}^{\infty} d_{m\mu} D_k^n z^{\mu} \\ &= \sum_{\mu=n}^{\infty} d_{m\mu} c_{\mu} \dots c_{\mu-n+1} z^{\mu-n} = \sum_{\mu=0}^{\infty} d_{m,\mu+n} c_{\mu+n} \dots c_{\mu+1} z^{\mu}. \end{aligned}$$

In the last formula $\mu - n$ was replaced by a single letter μ for convenience.

We want to have $MD_k^n f = D_k^n M f$ for every $f \in A(R)$. By the uniqueness theorem for analytic functions this will be true if and only if for every $m \geq 0$ one has $MD_k^n z^m = D_k^n M z^m$, i.e. if the expressions in (0.7) and (0.8) coincide.

Our first observation will be in the case when $0 \leq m \leq n-1$. Then one must have

$$0 = \sum_{\mu=0}^{\infty} d_{m,\mu+n} c_{\mu+n} \dots c_{\mu+1} z^{\mu}.$$

By the uniqueness theorem the power series on the right is zero if and only if all its coefficients are equal to zero, i.e. $d_{m,\mu+n} c_{\mu+n} \dots c_{\mu+1} = 0$ for every $\mu = 0, 1, 2, \dots$. But all c_s, $s \geq 1$, are different from zero and hence it is necessary to have

$$d_{m,n+\mu} = 0, \quad 0 \leq m \leq n-1, \ \mu = 0, 1, 2, \dots.$$

This can be written in a better way if $n + \mu$ is replaced by a single index μ:

$$(9) \qquad d_{m,\mu} = 0, \quad 0 \leq m \leq n-1, \ \mu \geq n.$$

The next step is to find a recurrent formula for arbitrary $m \geq n$. Comparing the first line in (0.7) with (0.8), we get by the uniqueness theorem that

$$(10) \qquad c_m \dots c_{m-n+1} d_{m-n,\mu} = d_{m,\mu+n} c_{\mu+n} \dots c_{\mu+1}, \quad m \geq n, \mu \geq 0.$$

Replacing μ by $\mu - n$ we have

$$(11) \qquad c_m \dots c_{m-n+1} d_{m-n,\mu-n} = c_\mu \dots c_{\mu-n+1} d_{m,\mu}, \quad m \geq n, \mu \geq n.$$

But all constants c_s, $s \geq 1$, are different from zero and we obtain the desired recurrent formula

$$(12) \qquad d_{m,\mu} = \frac{c_m \dots c_{m-n+1}}{c_\mu \dots c_{\mu-n+1}} \, d_{m-n,\mu-n}, \quad m \geq n, \mu \geq n.$$

The next step is to use (0.12) for expressing arbitrary coefficient $d_{m,\mu}$, $m \geq n$, $\mu \geq n$, by coefficient $d_{p,q}$, where either $0 \leq p \leq n-1$ or $0 \leq q \leq n-1$.

In the sequel $[A]$ will denote the integer part of a number A.

In the case $\left[\dfrac{m}{n}\right] < \left[\dfrac{\mu}{n}\right]$ one can apply $\left[\dfrac{m}{n}\right]$ times the recurrent formula (0.12), and then

$$(13) \qquad \begin{aligned} d_{m,\mu} &= \frac{c_m \dots c_{m-n+1}}{c_\mu \dots c_{\mu-n+1}} \, d_{m-n,\mu-n} = \frac{c_m \dots c_{m-2n+1}}{c_\mu \dots c_{\mu-2n+1}} \, d_{m-2n,\mu-2n} \\ &= \dots = \frac{c_m \dots c_{m-\left[\frac{m}{n}\right]n+1}}{c_\mu \dots c_{\mu-\left[\frac{m}{n}\right]n+1}} \, d_{m-\left[\frac{m}{n}\right]n,\mu-\left[\frac{m}{n}\right]n}. \end{aligned}$$

But $m - \left[\frac{m}{n}\right] n \leq n-1$ and $\mu - \left[\frac{m}{n}\right] n \geq n$, i.e. the first index is the remainder when m is divided by n. Then by our first observation (0.9) the coefficient $d_{m-\left[\frac{m}{n}\right]n,\mu-\left[\frac{m}{n}\right]n}$ must be zero. Therefore, using (0.13), one has

$$(14) \qquad d_{m,\mu} = 0, \quad \text{for } \left[\frac{m}{n}\right] < \left[\frac{\mu}{n}\right].$$

In the other case, when $\left[\dfrac{m}{n}\right] \geq \left[\dfrac{\mu}{n}\right]$, one can apply $\left[\dfrac{\mu}{n}\right]$ times the recurrent formula (0.12) to get

$$(15) \qquad \begin{aligned} d_{m,\mu} &= \frac{c_m \dots c_{m-n+1}}{c_\mu \dots c_{\mu-n+1}} d_{m-n,\mu-n} = \frac{c_m \dots c_{m-2n+1}}{c_\mu \dots c_{\mu-2n+1}} d_{m-2n,\mu-2n} \\ &= \dots = \frac{c_m \dots c_{m-\left[\frac{\mu}{n}\right]n+1}}{c_\mu \dots c_{\mu-\left[\frac{\mu}{n}\right]n+1}} d_{m-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n}. \end{aligned}$$

Now the second index $\mu - \left[\frac{\mu}{n}\right] n$ is the remainder when μ is divided by n.

Let us combine (0.14) and (0.15) as

$$d_{m,\mu} = \begin{cases} 0 & \text{for } \left[\frac{m}{n}\right] < \left[\frac{\mu}{n}\right], \\ \dfrac{c_m \cdots c_{m-\left[\frac{\mu}{n}\right]n+1}}{c_\mu \cdots c_{\mu-\left[\frac{\mu}{n}\right]n+1}} d_{m-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n} & \text{for } \left[\frac{m}{n}\right] \geq \left[\frac{\mu}{n}\right]. \end{cases} \tag{16}$$

From this important formula we can conclude that:

- *all coefficients $d_{m,\mu}$ with $0 \leq \mu \leq n-1$ can be chosen arbitrarily,*
  and then,
- *all other coefficients $d_{m,\mu}$ with $\mu \geq n$ are either equal to zero or can be expressed by some of the arbitrarily chosen $d_{\nu,\varkappa}$ with $\varkappa \leq n-1$.*

The recurrent relation (0.16) allows a representation of Mz^m as a polynomial of degree at most $\left(\left[\frac{m}{n}\right]+1\right)n-1$:

$$Mz^m = \sum_{\mu=0}^{n-1} d_{m,\mu} z^\mu + \sum_{\mu=n}^{\left(\left[\frac{m}{n}\right]+1\right)n-1} \frac{c_m \cdots c_{m-\left[\frac{\mu}{n}\right]n+1}}{c_\mu \cdots c_{\mu-\left[\frac{\mu}{n}\right]n+1}} d_{m-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n} \cdot z^\mu. \tag{17}$$

Finally, the action of an operator $M \in CD_k$ on some analytic function $f(z) = \sum_{m=0}^{\infty} a_m z^m$ is

$$Mf(z) = M \sum_{m=0}^{\infty} a_m z^m = \sum_{m=0}^{\infty} a_m M z^m = \tag{18}$$

$$= \sum_{m=0}^{\infty} a_m \left(\sum_{\mu=0}^{n-1} d_{m,\mu} z^\mu + \sum_{\mu=n}^{\left(\left[\frac{m}{n}\right]+1\right)n-1} \frac{c_m \cdots c_{m-\left[\frac{\mu}{n}\right]n+1}}{c_\mu \cdots c_{\mu-\left[\frac{\mu}{n}\right]n+1}} d_{m-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n} \cdot z^\mu \right),$$

which can be written also as

$$\begin{aligned} Mf(z) &= \sum_{\mu=0}^{n-1} \sum_{m=0}^{\infty} a_m d_{m,\mu} z^\mu + \\ &+ \sum_{\mu=n}^{\infty} \sum_{m=\left[\frac{\mu}{n}\right]n}^{\infty} a_m \frac{c_m \cdots c_{m-\left[\frac{\mu}{n}\right]n+1}}{c_\mu \cdots c_{\mu-\left[\frac{\mu}{n}\right]n+1}} d_{m-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n} \cdot z^\mu. \end{aligned} \tag{19}$$

This is in fact the desired representation (0.2) if one denotes the arbitrarily chosen constants by a different letter b,

$$b_{m,\nu} = d_{m,\nu}, \quad 0 \leq \nu \leq n-1, \; m = 0, 1, 2, \ldots. \tag{20}$$

Thus, we proved the *necessity*, i.e. if $M \in CD_k^n$, then the operator M must be of the form (0.2).

Now, let us check the *sufficiency*, i.e. if an operator M has the form (0.2), then it commutes with the n-th power of the Dunkl operator D_k^n, i.e. $MD_k^n = D_k^n M$. It is enough to verify this for all powers z^s, $s = 0, 1, 2, \ldots$, since they form a basis of the space of the analytic functions A_R. In fact, for arbitrarily fixed s we can use the representation (0.17) with $m = s$ and $d_{m,\mu} = b_{s,\mu}$ instead of the general expression (0.2).

In the case $0 \le s \le n-1$ the representation (0.17) reduces to the first sum and $Mz^s = \sum_{m=0}^{n-1} b_{s,\mu} z^\mu$. Now we calculate $D_k^n M z^s$ and $MD_k^n z^s$:

$$D_k^n(Mz^s) = D_k^n \sum_{m=0}^{n-1} b_{s,\mu} z^\mu = \sum_{m=0}^{n-1} b_{s,\mu} D_k^n z^\mu = \sum_{m=0}^{n-1} b_{s,\mu}.0 = 0;$$

$$M(D_k^n z^s) = M0 = 0,$$

i.e. $D_k^n M z^s = MD_k^n z^s = 0$. Here we used the second case in (0.5).

In the case $s \ge n$, if we apply D_k^n to

$$(21) \quad Mz^s = \sum_{\mu=0}^{n-1} b_{s,\mu} z^\mu + \sum_{\mu=n}^{([\frac{s}{n}]+1)n-1} \frac{c_s \ldots c_{s-[\frac{\mu}{n}]n+1}}{c_\mu \ldots c_{\mu-[\frac{\mu}{n}]n+1}} \, b_{s-[\frac{\mu}{n}]n, \mu-[\frac{\mu}{n}]n} \cdot z^\mu,$$

then the first sum will vanish because (0.5) gives $D_k^n z^\mu = 0$ for $0 \le \mu \le n-1$. Again (0.5) used for $\mu \ge n$ gives

$$(22) \; D_k^n M z^s = \sum_{\mu=n}^{([\frac{s}{n}]+1)n-1} \frac{c_s \ldots c_{s-[\frac{\mu}{n}]n+1}}{c_\mu \ldots c_{\mu-[\frac{\mu}{n}]n+1}} \, b_{s-[\frac{\mu}{n}]n, \mu-[\frac{\mu}{n}]n} c_\mu \ldots c_{\mu-n+1} z^{\mu-n},$$

It is suitable to separate the sum as $\sum_{\mu=n}^{2n-1} + \sum_{\mu=2n}^{([\frac{s}{n}]+1)n-1}$. In the first one the whole denominator will be canceled, but in the second sum, after canceling $c_\mu \ldots c_{\mu-n+1}$, the denominator will have n numbers less than the numerator:

$$\begin{aligned}(23) \quad D_k^n M z^s &= \sum_{\mu=n}^{2n-1} c_s \ldots c_{s-n+1} b_{s-n,\mu-n} z^{\mu-n} + \\ &+ \sum_{\mu=2n}^{([\frac{s}{n}]+1)n-1} \frac{c_s \ldots c_{s-[\frac{\mu}{n}]n+1}}{c_{\mu-n} \ldots c_{\mu-[\frac{\mu}{n}]n+1}} \, b_{s-[\frac{\mu}{n}]n, \mu-[\frac{\mu}{n}]n} z^{\mu-n}.\end{aligned}$$

It is also suitable to replace $\mu - n$ by only one number ν for future comparison:

$$(24) \qquad D_k^n M z^s = \sum_{\nu=0}^{n-1} c_s \ldots c_{s-n+1} b_{s-n,\nu} z^\nu +$$

$$+\sum_{\nu=n}^{\left[\frac{s}{n}\right]n-1} \frac{c_s \dots c_{s-\left[\frac{\nu}{n}\right]n-n+1}}{c_\nu \dots c_{\nu-\left[\frac{\nu}{n}\right]n+1}} \, b_{s-\left[\frac{\nu}{n}\right]n-n,\nu-\left[\frac{\nu}{n}\right]n} z^\nu.$$

Let us now represent $MD_k^n z^s$ in the same case $s \geq n$:

$$(25) \quad MD_k^n z^s = Mc_s \dots c_{s-n+1} z^{s-n} = c_s \dots c_{s-n+1} M z^{s-n} = c_s \dots c_{s-n+1} \times$$

$$\times \left(\sum_{\mu=0}^{n-1} b_{s-n,\mu} z^\mu + \sum_{\mu=n}^{\left(\left[\frac{s-n}{n}\right]+1\right)n-1} \frac{c_{s-n} \dots c_{s-n-\left[\frac{\mu}{n}\right]n+1}}{c_\mu \dots c_{\mu-\left[\frac{\mu}{n}\right]n+1}} \, b_{s-n-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n} \cdot z^\mu \right).$$

In fact, the right hand sides of (0.24) and (0.25) are one and the same but with different indices ν and μ. Thus $D_k^n M z^s = MD_k^n z^s$ and the sufficiency of (0.2) is also proved. ■

3. Corrolaries

Let us note that as a simplest particular case of the Dunkl operator, when the parameter k is taken to be 0, one can have the classical differentiation operator $D_0 f(z) = Df(z) = \dfrac{df(z)}{dz}$. Then $c_m = m$ and Theorem 1 describes the commutant of D in the case $n = 1$ as:

$$(26) \qquad Mf(z) = \sum_{m=0}^{\infty} a_m b_{m,0} + \sum_{\mu=1}^{\infty} \sum_{m=\mu}^{\infty} a_m \frac{m \dots (m-\mu+1)}{\mu \dots 1} b_{m-\mu,0} z^\mu.$$

In the case $n \geq 2$, the representation becomes

$$(27) \quad Mf(z) = \sum_{\mu=0}^{n-1} \sum_{m=0}^{\infty} a_m b_{m,\mu} z^\mu +$$

$$+ \sum_{\mu=n}^{\infty} \sum_{m=\left[\frac{\mu}{n}\right]n}^{\infty} a_m \frac{m \dots (m - \left[\frac{\mu}{n}\right]n + 1)}{\mu \dots (\mu - \left[\frac{\mu}{n}\right]n + 1)} b_{m-\left[\frac{\mu}{n}\right]n,\mu-\left[\frac{\mu}{n}\right]n} \cdot z^\mu.$$

Similar results for D, its powers, and generalizations of D are given by some Russian mathematicians. In particular in [4], §5.1, one can find such theorem and also additional bibliography.

If we take the parameter $k > 0$, but $n = 1$, then the representation of CD_k from our previous paper [7] is obtained:

$$(28) \qquad Mf(z) = \sum_{m=0}^{\infty} a_m b_m z^\mu + \sum_{\mu=1}^{\infty} \sum_{m=\mu}^{\infty} a_m \frac{c_m \dots c_{m-\mu+1}}{c_\mu \dots c_1} b_{m-\mu} \, z^\mu.$$

Final notes: A different description of the commutant CD_k of the first power of the Dunkl operator in the space of the continuous functions on the real line $\mathbb{R}$ is given in [2], based on the convolutional approach (see Dimovski [1]). It depends on an arbitrary continuous linear functional $\Phi : C(\mathbb{R}) \to \mathbb{C}$. If we try to make a comparison with our description in Theorem 1, here we also have an arbitrary choice. Namely, it is possible to vary arbitrarily the constants $b_{m,\mu}$, $0 \leq \mu \leq n-1$, $m = 0, 1, 2, \ldots$.

Power series have been used by the author also for descriptions of the commutants of other operators and studying of their different properties, e.g. a general linear operator of differentiation type [5], or the generalized Hardy-Littlewood operator [6], etc.

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