Mathematica
Balkanica

New Series Vol. 24, 2010, Fasc. 1-2

The Rupture Degree of Some Graphs

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Communication networks have been characterized by high levels of service reliability. In a communication networks, requiring greater degrees of stability or less vulnerability. The vulnerability of communication network measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. If we think of a graph as modeling a network, then the rupture degree of a noncomplete connected graph G is defined by

$$r(G) = max\{\ \omega(G-S) - |S| - m(G-S) : S \subset V(G),\ \omega(G-S) \geq 2\ \}$$

where $\omega(G-S)$ denotes the number of components in the graph $G-S$ and $m(G-S)$ is the order of the largest component of $G-S$. This parameter can be used to measure the vulnerability of a graph. In this paper, we consider general results on the rupture degree of a graph. Firstly, some bounds on the rupture degree are given. Further, the rupture degree of thorn graphs G^* and E_p^t are calculated. Finally, we give formulas for the rupture degree of the corona operation of some special graphs.

Keywords: Network design and communication, Vulnerability, Connectivity, and Rupture Degree

AMS subject classification (2000): 05C40, 94C15, 90C27, 68R10,.

1. Introduction

The stability of a communication network, which consists of processing nodes and communication links, is very important for network designers. When the network begins losing its nodes or links, at last there is damage in its effectiveness. Communication networks must be constructed to be as stable as possible, since they depend on some initial damage and some possibility of their reconstruction. The vulnerability value shows the resistance of the network until communication breakdown after the disruption of certain stations or communication links. In a communication networks, requiring greater degrees of stability or less vulnerability. The problem of quantifying the vulnerability of networks

has received much attention nowadays. In an analysis of the vulnerability of networks to disruption, three important quantities (there may be others) are (1) the number of elements that are not functioning, (2) the number of remaining connected subnetworks and (3) the size of a largest remaining group within which mutual communication can still occur.
If we think of a graph as modeling a network, then we have some graph parameters to measure the vulnerability. Many graph theoretical parameters have been used to describe the stability of communication networks. Most studied and best-known parameters in graph theory are including connectivity, integrity, toughness, binding number, and scattering number [1-4, 12, 13].
A graph G, it is denoted by $G = (V(G), E(G))$, where $V(G)$ is vertices set of G and $E(G)$ is edges set of G. The number of vertices and the number of edges of the graph G are denoted by $|V(G)|$, $|E(G)|$ respectively [6, 7].
Throughout this paper for any graph G, we denote the number of components by $\omega(G)$ and the order of the largest component by *m(G)*.
The *tenacity* of an incomplete connected graph G is defined as

$$T(G) = \left\{ \frac{|S|+m(G-S)}{w(G-S)} : S \subset V(G) \ and \ \omega(G-S) > 1 \right\}$$

and the tenacity of K_n is defined as n. Clearly, tenacity is the most appropriate for measuring the vulnerability of networks [4].
It is natural to consider the additive dual of tenacity. We call this parameter the rupture degree of graphs. Formally, the *rupture degree* of an incomplete connected graph G is defined by

$$r(G) = max\{ \ \omega(G-S) - |S| - m(G-S) : S \subset V(G), \ \omega(G-S) \geq 2 \ \}$$

and the rupture degree of K_n is defined as $1 - n$ [10,11].
We consider that two graphs have same tenacity. Then, these two graphs must be different in respect to stability. How can we measure that property? This idea offers the concept of rupture degree that is different from tenacity.

Definition 1.1. (see [6, 9, 12]) Two vertices are said to cover each other in a graph G if they are incident in G. A vertex cover in G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the vertex covering number of G and is denoted by $\alpha(G)$.

Definition 1.2. (see [6, 9, 12]) An independent set of vertices of a graph G is a set of vertices of G whose elements are pairwise nonadjacent. The independence number $\beta(G)$ of G is the maximum cardinality among all independent sets of vertices of G.

Theorem 1.1. *(see [6, 9, 12]) If G is a graph of order n, then*

$$\alpha(G) + \beta(G) = n.$$

Definition 1.3. (see [6, 7]) The *complement* $\overline{G}$ of a graph G has $V(G)$ as its node sets, but two nodes are adjacent in $\overline{G}$ if only if they are not adjacent in G.

In this paper, at first, we give some of the known results. Finally, we give some bounds consisted of the relationships between the rupture degree and some vulnerability parameters on the rupture degree of a graph.

2. Basic Results

In this section we give some of the known results about the rupture degree. The $\delta(G)$ parameter will be used in this paper as the minimum vertex degree. The graphs that will be given the results in this section are in the following:
P_n: the path graph of order n
C_n: the cycle graph of order n vertices
$K_{1,n-1}$: the star graph of order n vertices

Theorem 2.1. *(see [10]) The rupture degrees of the path $P_n(n \geq 3)$, the star $K_{1,n-1}(n \geq 3)$ and the cycle C_n are given in the following.*
a) The rupture degree of the path $P_n(n \geq 3)$ is

$$r(P_n) = \begin{cases} -1, & n \ is \ even \\ 0, & n \ is \ odd \end{cases}$$

b)The rupture degree of the star $K_{1,n-1}(n \geq 3)$ is $n-3$.

c)The rupture degree of the cycle C_n is

$$r(C_n) = \begin{cases} -1, & n \ is \ even \\ -2, & n \ is \ odd \end{cases}$$

Theorem 2.2. *(see [10]) Let G be an incomplete connected graph of order n. Then*

a) $2\alpha(G) - n - 1 \leq r(G) \leq \frac{[\alpha(G)]^2 - \kappa(G)[\alpha(G)-1]-n}{\alpha(G)}$.

b) $3 - n \leq r(G) \leq n - 3$.

c) $r(G) \leq 2\delta(G) - 1$.

Theorem 2.3. *(see [10]) Let* G_1 *and* G_2 *be two connected graphs of order* n_1 *and* n_2*, respectively. Then*

$$r(G_1 + G_2) = max\{\ r(G_1) - n_2, r(G_2) - n_1\}.$$

3. The Rupture Degree of G^* Thorn Graph

In this section, we give some results on the rupture degree of G^* thorn graph are calculated. Finally, we give formulas for the rupture degree of the corona operation of some special graphs.

Definition 3.1. (see [8]) Let $p_1, p_2, ..., p_n$ be non-negative integers and G be such a graph, $V(G) = n$. The *thorn graph* of the graph G, with parameters $p_1, p_2, ..., p_n$ is obtained by attaching p_i new vertices of degree one to the vertex u_i of the graph G, i=1, 2, ... , n.

The thorn graph of the graph G will be denoted by G^*, or if the respective parameters need to specified, by $G^*(\ p_1, p_2, ..., p_n)$.

Definition 3.2. (see [7]) The corona $G_1 \circ G_2$ was defined by Frucht and Harary as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i'th node of G_1 to every node in the i'th copy of G_2.

Theorem 3.1. *Let G be a graph of order n. If* G^* *is a thorn graph with every* $p_i = 1$*, then*

$$r(G^*) = \beta(G) - 2$$

P r o o f. Let $S^* \subset V(G^*)$ and S^* be satisfying the rupture degree of G^* . Thus we have two cases according to S^*.

Case1: If S^* is any covering set of G, then we have $\omega(G^* - S^*) = \beta(G) + \alpha(G)$ and $m(G^* - S^*) = 2$. If we write these values in rupture definition, then

$$\begin{aligned} r(G^*) &= max\{\ \omega(G^* - S^*) - |S^*| - m(G^* - S^*)\} \\ &= max\{\ \beta(G) + \alpha(G) - \alpha(G) - 2\} \\ &= max\{\ \beta(G) - 2\} \end{aligned}$$

$$= \beta(G) - 2 \tag{3.1}$$

Case2: If S^* is not any covering set of G and $|S^*| \neq n$, then we have $\omega(G^* - S^*) = \omega(G - S^*) + |S^*|$ and $m(G^* - S^*) > 2$. If we write these values in rupture definition, then

$$\begin{aligned} r(G^*) &= max\{\ \omega(G^* - S^*) - |S^*| - m(G^* - S^*)\} \\ &< max\{\ \omega(G - S^*) + |S^*| - |S^*| - 2\} \\ &< max\{\ \omega(G - S^*) - 2\} \end{aligned}$$

Since $\omega(G - S^*) < \beta(G)$ for any graph G. Then we have

$$r(G^*) \leq max\{\beta(G) - 2\}$$

$$\leq \beta(G) - 2 \tag{3.2}$$

If $|S^*| = n$, then we have $\omega(G^* - S^*) = n$ and $m(G^* - S^*) = 1$. Thus the rupture degree of G^*, $r(G^*) = \max\{n - n - 1\} = \max\{-1\ \} = -1$. We claim that $-1 \leq \beta(G) - 2$. Let we prove this claim. We know that $1 \leq \beta(G)$. If we subtract 2 from both side of this inequality, then we have $-1 \leq \beta(G) - 2$. Thus we complete the proof of this claim. It is easy to see that $r(G^*) = \beta(G) - 2$ from **(3.1)** and **(3.2)**. Hence

$$r(G^*) = \beta(G) - 2.$$

The proof is completed. ∎

Theorem 3.2. *Let G be a graph of order n. If G^* is a thorn graph of G with $p_i \geq 2$, then*

$$r(G^*) = \beta(G^*) - \alpha(G^*) - 1.$$

P r o o f. The independence number and covering number of G^* are $\beta(G^*) = \sum_{i=1}^{n} p_i$ and $\alpha(G^*) = n$, respectively. If $S^* \subset V(G^*)$, then $\omega(G^* - S^*) = \sum_{i=1}^{k} p_i + \omega(G - S^*)$ where $|S^*| = k$. Firstly we can assume that S^* is a cover set of G^*. Therefore, we have $k = \alpha(G^*) = n$, $\omega(G^* - S^*) = \sum_{i=1}^{n} p_i$ and $m(G^* - S^*) = 1$. If we write these values in rupture definition, then

$$\begin{aligned} r(G^*) &= max\{ \omega(G^* - S^*) - |S^*| - m(G^* - S^*)\} \\ &= max\{ \sum_{i=1}^{n} p_i - \alpha(G^*) - 1\} \\ &= \sum_{i=1}^{n} p_i - \alpha(G^*) - 1 \\ &= \beta(G^*) - \alpha(G^*) - 1 \qquad (3.3) \end{aligned}$$

If S^* is not a cover set of G^*, then we have three cases for S^* set.

Case1: If $|S^*| < \alpha(G)$, then $\omega(G - S^*) < \beta(G)$, $\omega(G^* - S^*) < \sum_{i=1}^{k} p_i + \beta(G)$ where $|S^*| = k$, and $m(G^* - S^*) \geq 6$. If we remove k vertices from G, then one of the remaining connected components has at least 2 vertices. So the graph G will have a subgraph K_2 (see Figure 1).

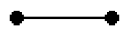

Figure 1: K_2 Graph

If we write these values in rupture definition, then

$$\begin{aligned} r(G^*) &= max\{ \omega(G^* - S^*) - |S^*| - m(G^* - S^*)\} \\ &\leq max\{ \sum_{i=1}^{k} p_i + \beta(G) - k - 6\} \\ &\leq \sum_{i=1}^{k} p_i + \beta(G) - k - 6 \qquad (3.4) \end{aligned}$$

We claim that the value which we found in **(3.3)** is bigger than the value which we found in **(3.4)**, that is, the claim is $\sum_{i=1}^{k} p_i + \beta(G) - k - 6 \leq \beta(G^*) - \alpha(G^*) - 1$. Now let we prove this claim. We know that

$2\ \beta(G) \leq \sum_{i=\alpha(G)+1}^{n} p_i$ and $\alpha(G) - k \leq \sum_{i=k+1}^{\alpha(G)} p_i$ for any G graph. So,

$2\ \beta(G) + \alpha(G) - k \leq \sum_{i=k+1}^{\alpha(G)} p_i + \sum_{i=\alpha(G)+1}^{n} p_i.$

Let we add $\sum_{i=1}^{k} p_i$ both side. Then we have

$\sum_{i=1}^{k} p_i + 2\ \beta(G) + \alpha(G) - k \leq \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{\alpha(G)} p_i + \sum_{i=\alpha(G)+1}^{n} p_i$

$\sum_{i=1}^{k} p_i + \ \beta(G) + \ (\beta(G) + \alpha(G)) - k \leq \sum_{i=1}^{n} p_i$

$\sum_{i=1}^{k} p_i + \ \beta(G) + n - k \leq \sum_{i=1}^{n} p_i$

$\sum_{i=1}^{k} p_i + \ \beta(G) - k \leq \sum_{i=1}^{n} p_i - n$

We also know that $-6 < -1$. Hence,

$\sum_{i=1}^{k} p_i + \ \beta(G) - k - 6 \leq \sum_{i=1}^{n} p_i - n - 1$

$\sum_{i=1}^{k} p_i + \ \beta(G) - k - 6 \leq \beta(G^*) - \alpha(G^*) - 1$

Thus the proof of claim is completed.

Case2: If $|S^*| = \alpha(G)$, then $\omega(G - S^*) = \beta(G)$, $\omega(G^* - S^*) = \sum_{i=1}^{\alpha(G)} p_i + \beta(G)$, and $m(G^* - S^*) \geq 3$. If we write these values in rupture definition, then

$$r(G^*) = max\{\ \omega(G^* - S^*) - |S^*| - m(G^* - S^*)\}$$

$$\leq max\{\ \sum_{i=1}^{\alpha(G)} p_i + \beta(G) - \alpha(G) - 3\}$$

$$\leq \sum_{i=1}^{\alpha(G)} p_i + \beta(G) - \alpha(G) - 3 \tag{3.5}$$

Now we claim that the value which we found in **(3.3)** is bigger than the value which we found in **(3.5)**, that is, the claim is
$\sum_{i=1}^{\alpha(G)} p_i + \beta(G) - \alpha(G) - 3 \leq \beta(G^*) - \alpha(G^*) - 1$. Let we prove this claim. We know that $2\ \beta(G) \leq \sum_{i=\alpha(G)+1}^{n} p_i$ for any G graph. Since $\sum_{i=1}^{\alpha(G)} p_i > 0$, then we have
$\sum_{i=1}^{\alpha(G)} p_i + 2\ \beta(G) \leq \sum_{i=1}^{\alpha(G)} p_i + \sum_{i=\alpha(G)+1}^{n} p_i$
$\sum_{i=1}^{\alpha(G)} p_i + 2\ \beta(G) \leq \sum_{i=1}^{n} p_i$
$\sum_{i=1}^{\alpha(G)} p_i + 2\ \beta(G) - \alpha(G) \leq \sum_{i=1}^{n} p_i - \alpha(G)$
$\sum_{i=1}^{\alpha(G)} p_i + \beta(G) - \alpha(G) \leq \sum_{i=1}^{n} p_i - (\alpha(G) + \beta(G))$
$\sum_{i=1}^{\alpha(G)} p_i + \beta(G) - \alpha(G) \leq \sum_{i=1}^{n} p_i - n$
We also know that $-3 < -1$.Hence,

$\sum_{i=1}^{\alpha(G)} p_i + \beta(G) - \alpha(G) - 3 \leq \sum_{i=1}^{n} p_i - n - 1$
$\sum_{i=1}^{\alpha(G)} p_i + \beta(G) - \alpha(G) - 3 \leq \beta(G^*) - \alpha(G^*) - 1$
Thus the proof of claim is completed.

Case3: If $|S^*| > \alpha(G)$, then $S^* = S_1 \cup S_2$. Let S_1 be a cover set of G and $|S_2| < \beta(G)$. Thus we have

$\omega(G^* - S^*) = \sum_{i=1}^{k} p_i + |V(G)| - |S^*|$ and $m(G^* - S^*) \geq 3$.
If we write these values in rupture definition, then

$$r(G^*) = max\{\ \omega(G^* - S^*) - |S^*| - m(G^* - S^*)\}$$

$$\leq max\{\ \sum_{i=1}^{k} p_i + |V(G)| - |S^*| - |S^*| - 3\}$$

$$\leq max\{\ \sum_{i=1}^{k} p_i + |V(G)| - 2|S^*| - 3\}$$

We also know that $\sum_{i=1}^{k} p_i < \beta(G^*)$, $|V(G)| = \alpha(G) + \beta(G)$ and $|S^*| = |S_1| + |S_2| = \alpha(G) + |S_2|$. If we write these values in rupture definition, then we have

$$r(G^*) \leq max\{\ \beta(G^*) + \alpha(G) + \beta(G) - 2(\alpha(G) + |S_2|) - 3\}$$

$$\leq max\{\ \beta(G^*) + \alpha(G) + \beta(G) - 2(\alpha(G) + \beta(G)) - 3\}$$

$$\leq max\{\ \beta(G^*) - (\alpha(G) + \beta(G)) - 3\}$$

$$\leq max\{\ \beta(G^*) - n - 3\}$$

$$\leq\ \beta(G^*) - n - 3$$

$$\leq\ \beta(G^*) - \alpha(G^*) - 3 \tag{3.6}$$

We claim that the value which we found in **(3.3)** is bigger than the value which we found in **(3.6)**. It's easy to see that
$\beta(G^*) - \alpha(G^*) - 3 < \beta(G^*) - \alpha(G^*) - 1$.Hence

$$r(G^*) = \beta(G^*) - \alpha(G^*) - 1$$

Thus the proof is completed. ■

Theorem 3.3. *Let G be a graph of order n. If G^* is a thorn graph of G with $p_i \geq 1$, then*

$$r(G^* \circ P_2) = \beta(G^*) - 3.$$

P r o o f. The covering number of G^* is $\alpha(G^*) = n$. Let $S^* \subset V(G^* \circ P_2)$. Then firstly we can assume that S^* is a cover set of G^*. If $|S^*| = \alpha(G^*) = n$, then we have $\omega((G^* \circ P_2) - S^*) = n + \sum_{i=1}^{n} p_i$ and $m((G^* \circ P_2) - S^*) = 3$. If we write these values in rupture definition, then we have

$$r(G^* \circ P_2) = max\{\ \omega((G^* \circ P_2) - S^*) - |S^*| - m((G^* \circ P_2) - S^*)\}$$

$$= max\{\ n + \sum_{i=1}^{n} p_i - n - 3\}$$

$$= max\{\ \sum_{i=1}^{n} p_i - 3\}$$

$$= max\{\ \beta(G^*) - 3\}$$

$$= \ \beta(G^*) - 3 \tag{3.7}$$

If S^* is not a cover set of G^*, then we have three cases for S^* set.

Case1: If $|S^*| < \alpha(G)$, then $|S^*| = k$. Then we have $\omega((G^* \circ P_2) - S^*) = \sum_{i=1}^{k} p_i + k + \omega(G - S^*)$ and $\omega(G - S^*) \leq \beta(G)$. Therefore $\omega((G^* \circ P_2) - S^*) \leq k + \sum_{i=1}^{k} p_i + \beta(G)$. It's easy to see that $m((G^* \circ P_2) - S^*) \geq 12$. If we remove k vertices from G, then one of the remaining connected components has at least 2 vertices. So the graph G will have a subgraph K_2 (see Figure 2).

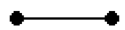

Figure 2: K_2 Graph

If we write these values in rupture definition, then

$$r(G^* \circ P_2) = max\{\ \omega((G^* \circ P_2) - S^*) - |S^*| - m((G^* \circ P_2) - S^*)\}$$

$$\leq max\{k + \sum_{i=1}^{k} p_i + \beta(G) - k - 12\}$$

$$\leq max\{\ \sum_{i=1}^{k} p_i + \beta(G) - 12\} \tag{3.8}$$

We claim that the value which we found in **(3.7)** is bigger than the value which we found in **(3.8)**, that is, the claim is $\sum_{i=1}^{k} p_i + \beta(G) - 12 < \beta(G^*) - 3$. Now let we prove this claim. It's easy to see that $\sum_{i=1}^{k} p_i - 12 \leq \sum_{i=1}^{k} p_i - 3$. Since $\beta(G) = n - \alpha(G)$ and $k = |S^*| < \alpha(G) \Longrightarrow -\alpha(G) < -k$, then we have $n - \alpha(G) < n - k \leq \sum_{i=k+1}^{n} p_i$ and we know that $\beta(G) \leq \sum_{i=k+1}^{n} p_i$. Hence

$$\sum_{i=1}^{k} p_i - 12 + \beta(G) \leq \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{n} p_i - 3$$

$$\sum_{i=1}^{k} p_i + \beta(G) - 12 \leq \sum_{i=1}^{n} p_i - 3$$

$$\sum_{i=1}^{k} p_i + \beta(G) - 12 \leq \beta(G^*) - 3$$

Thus the proof of claim is completed.

Case2: If $k = |S^*| = \alpha(G)$, then we have
$\omega((G^* \circ P_2) - S^*) = k + \sum_{i=1}^{k} p_i + \beta(G)$ and $m((G^* \circ P_2) - S^*) \geq 6$
If we write these values in rupture definition, then we have

$$r(G^* \circ P_2) = max\{\ \omega((G^* \circ P_2) - S^*) - |S^*| - m((G^* \circ P_2) - S^*)\}$$

$$\leq max\{k + \sum_{i=1}^{k} p_i + \beta(G) - k - 6\}$$

$$\leq max\{\ \sum_{i=1}^{k} p_i + \beta(G) - 6\}$$

$$\leq \sum_{i=1}^{k} p_i + \beta(G) - 6 \tag{3.9}$$

We claim that the value which we found in **(3.7)** is bigger than the value which we found in **(3.9)**, that is, the claim is $\sum_{i=1}^{k} p_i + \beta(G) - 6 < \beta(G^*) - 3$. Now let we prove this claim. We know that $\beta(G) \leq \sum_{i=k+1}^{n} p_i$ and $-6 < -3$. Therefore we obtain $\beta(G) - 6 \leq \sum_{i=k+1}^{n} p_i - 3$. Since $\sum_{i=1}^{k} p_i > 0$, then we have

$$\sum_{i=1}^{k} p_i + \beta(G) - 6 \leq \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{n} p_i - 3$$

$$\sum_{i=1}^{k} p_i + \beta(G) - 6 \leq \beta(G^*) - 3.$$

Thus the proof of claim is completed.

Case3: If $k = |S^*| > \alpha(G)$ and $k \neq n$, then $S^* = S_1 \cup S_2$ and let S_1 be a cover set of G. Since $|S_1| = \alpha(G)$ and $|S_2| < \beta(G)$, we have $\omega((G^* \circ P_2) - S^*) = k + \sum_{i=1}^{k} p_i + \omega(G - S^*)$ and $\omega(G - S^*) = n - k$. Thus $\omega((G^* \circ P_2) - S^*) = k + \sum_{i=1}^{k} p_i + n - k$ and $m((G^* \circ P_2) - S^*) \geq 6$. If we write these values in rupture definition, then we have

$$r(G^* \circ P_2) = max\{\ \omega((G^* \circ P_2) - S^*) - |S^*| - m((G^* \circ P_2) - S^*)\}$$

$$\leq max\{\ \sum_{i=1}^{k} p_i + n - k - 6$$

$$\leq max\{\ \sum_{i=1}^{k} p_i + n - (|S_1| + |S_2|) - 6\}$$

(for $|S_1| = \alpha(G)$ and $|S_2| < \beta(G)$)

$$\leq max\{\ \sum_{i=1}^{k} p_i + n - (\alpha(G) + \beta(G)) - 6\}$$

$$\leq max\{\ \sum_{i=1}^{k} p_i + n - n - 6$$

$$\leq max\{\ \sum_{i=1}^{k} p_i - 6$$

$$\leq \sum_{i=1}^{k} p_i - 6 \tag{3.10}$$

We claim that the value which we found in **(3.7)** is bigger than the value which we found in **(3.10)**, that is, the claim is $\sum_{i=1}^{k} p_i - 6 < \beta(G^*) - 3$. It's easy to see that. Then we have

$$r(G^* \circ P_2) = \beta(G^*) - 3$$

Thus the proof is completed. ■

4. The Rupture Degree of Graph E_p^t

In this section we give some results on the rupture degree of graph E_p^t.

Definition 4.1. (see [5]) The graph E_p^t has t legs and each leg has p vertices (Figure 3). Thus E_p^t has $n = pt + 2$ vertices.

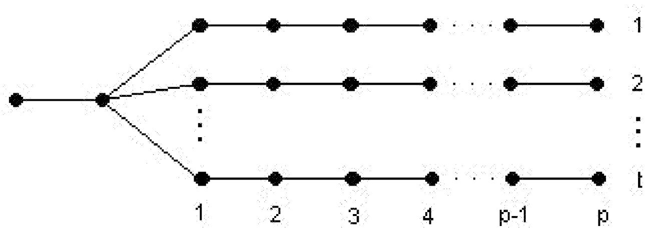

Figure 3: E_p^t *graph with* $pt + 2$ *vertices*

Theorem 4.1. *Let* t *and* p *be an integer* $(t \geq 2, p \geq 2)$. *Then*

$$r(E_p^t) = \begin{cases} t-2, & p \ is \ even \\ t-1, & p \ is \ odd \end{cases}$$

P r o o f. $u \in V(E_p^t)$ and $\deg(u)$ is the maximum degree of E_p^t graph. If we remove the vertex u from E_p^t graph, then we have two components one of which is trivial graph of order one vertex and the other is path graphs of order p. Now we have two cases for p to calculate the rupture degree of path graph.

Case1: If $p = 2n-1$, then we remove cover set of P_p path graph from E_p^t graph. Then we have $\omega(E_p^t - S) = tn+1$ and $m(E_p^t - S) = 1$, $|S| = 1+t(n-1)$. If we write these values in rupture definition, then

$$r(E_p^t) = max\{\ \omega(E_p^t - S) - |S| - m(E_p^t - S)\}$$

$$= max\{\ tn + 1 - 1 - tn + t - 1\}$$

$$= max\{\ t - 1\} = t - 1$$

Case2: If $p = 2n$, then we remove cover set cover set except any vertex of P_p path graph from E_p^t graph. Then we have $\omega(E_p^t - S) = tn + 1$ and $m(E_p^t - S) = 2, |S| = 1 + t(n-1)$. If we write these values in rupture definition, then

$$r(E_p^t) = max\{\ \omega(E_p^t - S) - |S| - m(E_p^t - S)\}$$

$$= max\{\ tn + 1 - 1 - tn + t - 2\}$$
$$= max\{\ t - 2\} = t - 2$$

Thus the proof is completed. ■

Conclusion 4.1. Let t and p be positive integer $(t \geq 2, p \geq 2)$. Then

$$r(E_p^t) = (t - 1) + r(P_p)$$

Theorem 4.2. *Let t and p be an integer $(t \geq 2, p \geq 2)$. Then*

$$r(\overline{E_p^t}) = 1 - tp.$$

Proof. $u \in V(E_p^t)$ and $\deg(u)$ is the maximum degree of E_p^t graph.Then vertex u has been minimum degree in E_p^t graph. If we remove $t(p-1)$ vertices connected with vertex u from the $\overline{E_p^t}$ graph, then we have two components one of which is trivial graph of order one vertex and the other is complete graph of order $t + 1$ (see Figure 4).

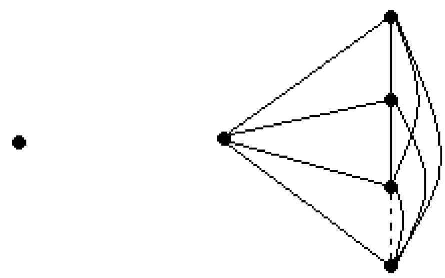

Figure 4: A vertex and complete graph of order $t + 1$

If we make the complete graph unconnected, then $|S|$ will increase but $\omega(\overline{E_p^t} - S)$ will not change. So the rupture degree will decrease. Thus we won't make the complete graph unconnected. Therefore

$$r(\overline{E_p^t}) = max\{\ \omega(\overline{E_p^t} - S) - |S| - m(\overline{E_p^t} - S)\}$$
$$= max\{\ 2 - t(p-1) - (t + 1)\}$$

$$= max\{\ 1 - tp\}$$
$$= 1 - tp$$

Thus the proof is completed. ■

Theorem 4.3. *Let t, p and k be an integer* $(t \geq 2, p \geq 2)$*. Then the rupture degree of* $E_p^t \circ P_k$ *is*

$$r(E_p^t \circ P_k) = \begin{cases} t\beta(P_p) - k & , \quad for\ \ k \leq 4 \\ \begin{cases} -2, & k\ \ is\ \ even \\ -1, & k\ \ is\ \ odd \end{cases} & , \quad otherwise \end{cases}$$

Proof. $u \in V(E_p^t \circ P_k)$ and $\deg(u)$ is the maximum degree of $E_p^t \circ P_k$ graph. If we remove the vertex u from $E_p^t \circ P_k$ graph for $k \leq 4$, then we have $t+1$ components one of which is trivial graph of order one vertex and the others is $P_p \circ P_k$ graphs of order $pk + p$. If we delete the vertex of the cover set of P_p from $P_p \circ P_k$, then we have $\omega((E_p^t \circ P_k) - S) = tp + 2$, $m((E_p^t \circ P_k) - S) = k + 1$ and $|S| = t\alpha(P_p) + 1$. When we write these values in rupture definition, we obtain

$$r(E_p^t \circ P_k) = max\{\ \omega((E_p^t \circ P_k) - S) - |S| - m((E_p^t \circ P_k) - S)\}$$
$$= max\{\ tp + 2 - t\alpha(P_p) - 1 - k - 1\}$$
$$= max\{\ t(p - \alpha(P_p)) - k\}$$
$$= max\{\ t\beta(P_p) - k\}$$
$$= \ t\beta(P_p) - k$$

If we remove all vertices of E_p^t graph from $E_p^t \circ P_k$ graph for $k \geq 5$, then we have $pt + 2$ components all of which are P_k path graph. We make these graphs unconnected like the rupture degree of path graph definition. Then we have two cases for k.

Case1: Let k be even. If we remove $\alpha(P_k) - 1$ vertices from P_k path graph, then we have $\beta(P_k)$ components and $m((E_p^t \circ P_k) - S) = 2$. Since the number of the P_k path graph is $pt + 2$, $|S| = (pt + 2)(\alpha(P_k) - 1) + (pt + 2) = \alpha(P_k)(pt + 2)$, $\omega((E_p^t \circ P_k) - S) = (pt + 2)\beta(P_k)$ and $m((E_p^t \circ P_k) - S) = 2$. When we write these values in rupture definition, we obtain

$$r(E_p^t \circ P_k) = max\{\ \omega((E_p^t \circ P_k) - S) - |S| - m((E_p^t \circ P_k) - S)\}$$

$$= max\{\ (pt+2)\beta(P_k) - \alpha(P_k)(pt+2) - 2\}$$

$(for\ \alpha(P_k) = k - \beta(P_k))$

$$= max\{\ (pt+2)(\beta(P_k) - (k - \beta(P_k))) - 2\}$$

$$= max\{\ (pt+2)(2\beta(P_k) - k) - 2\}$$

$(for\ \beta(P_k) = \frac{k}{2})$

$$= max\{\ (pt+2)(2\ \frac{k}{2} - k) - 2\}$$

$$= max\{\ 0 - 2\} = -2$$

Thus if k is even, then the rupture degree of $E_p^t \circ P_k$ is -2.

Case2: Let k be odd. In this case we remove $\alpha(P_k)$ vertices from P_k path graph. The proof is similar to that of ***Case1***. So we obtain $r(E_p^t \circ P_k) = -1$. The proof is completed. ■

Conclusion 4.2. Let t, p and k be positive integer $(t \geq 2, p \geq 2)$. Then

$$r(E_p^t \circ P_k) = r(P_k) - 1$$

5. Conclusion

In this article, we have studied the stability of various topologies to vertex failure. We give comparisons between popular interconnection networks and thorn graphs of these below. These networks are complete graph K_{100}, path graph P_{100}, cycle graph C_{100}, star graph $K_{1,99}$ and E_{49}^2. The thorn graphs of these networks were considered to improve the graph stability. The rupture degree of the above graphs are shown in Table 1.

By using Table 1, we say that the stability of graphs G^* is more powerful than the stability of graphs G. Moreover, the results in Table 1 represent a trade-off between the amount of work done to damage the network and how badly the network is damaged. Therefore, designers for choosing the appropriate topology can use these results.

G graph	r(G)	Thorn graph $(p_i = 1)$	$r(G^*)$
P_{100}	-1	P_{50}^*	23
C_{100}	-1	C_{50}^*	23
$K_{1,99}$	97	$K_{1,49}^*$	47
K_{100}	-99	K_{50}^*	-1
E_{49}^2	-1	$(E_6^8)^*$	23

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Received 11.05.2009

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