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Cyclic parallelisms of $PG(5,2)$

Stela D. Zhelezova

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A spread is a set of lines of $PG(d,q)$, which partition the point set. A parallelism of $PG(d,q)$ is a partition of the set of lines by spreads. A parallelism is cyclic if there is an automorphism which permutes its spreads in one cycle. In the present paper a classification by computer search of all cyclic parallelisms of $PG(5,2)$ is presented. It is established that there are 1090494 nonisomorphic cyclic parallelisms, among which 286 ones with full automorphism group of order 155 (this result coincides with a result of Stinson and Vanstone [15]), and the rest with full automorphism group of order 31.

1. Introduction

For the basic concepts and notations concerning combinatorial designs, projective spaces, spreads and parallelisms, refer, for instance, to [1], [3], [4], [7], [8], or [18].

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ a finite collection of k-element subsets of V, called *blocks*. $D = (V,\mathcal{B})$ is a 2-(v,k,λ) design if any 2-subset of V is contained in exactly λ blocks of $\mathcal{B}$.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence. An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points which maps blocks into blocks.

A *parallel class* is a partition of the point set by blocks. A *resolution* of the design is a partition of the collection of blocks by parallel classes. Two resolutions are *isomorphic* if there is an automorphism of the design mapping the first one into the second. An *automorphism of a resolution* is an automorphism of the design, which maps parallel classes into parallel classes.

A *t-spread* in $PG(d,q)$ is a set of t-dimensional subspaces which partition the point set. A *t-parallelism* is a partition of the set of t-dimensional subspaces by t-spreads. Usually 1-spreads and 1-parallelisms are called just spreads and parallelisms or line spreads and line parallelisms, since the lines are the 1-dimensional subspaces. There can be line spreads and parallelisms if d is odd.

The incidence of the points and t-dimensional subspaces of $PG(d,q)$ defines a 2-design (see for instance [17, sect.2.35-2.36]), i.e. the points of this design correspond to the points of the projective space, and the blocks to the t-dimensional subspaces. There is a one-to-one correspondence between parallelisms and the resolutions of the related point-line design. Isomorphism and automorphisms of parallelisms are defined as for resolutions. An automorphism of $PG(d,q)$ is a bijective map on the point set that preserves collinearity, i.e. maps lines into lines, and thus t-dimensional subspaces into t-dimensional subspaces. Therefore all related designs have the full automorphism group of the projective space. A parallelism is cyclic if there is an automorphism, which permutes its spreads in one cycle.

A construction of 1-parallelisms in $PG(d,2)$ was first presented by Zaicev, Zinoviev and Semakov [21] and independently by Baker [2]. Denniston has constructed cyclic parallelisms of $PG(3,2)$ [5] and $PG(8,q)$ [6], Penttila and Williams found two cyclic parallelisms of $PG(3,q)$ for $q \equiv 2 \ (mod\ 3)$ [13] and Prince classified cyclic parallelisms of $PG(3,5)$ [14].

There are quite many constructions of t-spreads and t-parallelisms in $PG(5,2)$. Looking for affine 2-(64,16,5) designs of small rank, Mavron, McDonough and Tonchev constructed more than 30000 1-spreads of $PG(5,2)$ [12]. Mateva and Topalova classified up to isomorphism the 1-spreads of $PG(5,2)$ [11] and Topalova and Zhelezova the 2-spreads [19]. In a recent work Topalova and Zhelezova constructed 2-parallelisms with automorphisms of order 31 [20] and found among them the first examples of transitive t-parallelisms of $PG(d,q)$ for $t > 2$. Sarmiento classified parallelisms of $PG(5,2)$ with a point-transitive cyclic group of order 63 [16], and Stinson and Vanstone cyclic parallelisms of $PG(5,2)$ with a full automorphism group of order 155 [15].

The present work completes the classification of cyclic parallelisms of $PG(5,2)$ and coincides with the classification of cyclic parallelisms with an automorphism group of order 155 by Stinson and Vanstone.

There are 63 points and 651 lines in $PG(5,2)$. Its parallelisms have 31 spreads of 21 lines each. In order to be cyclic a parallelism of $PG(5,2)$ must possess automorphisms of order 31. We use design approach to the problem. We actually make all the computations on the related to $PG(5,2)$ designs, namely:

- instead of a spread, we construct a parallel class of 21 of the 651 blocks of the related 2-$(63,3,1)$ point-line design, instead of a parallelism – a resolution of this design;
- we find a generating set of the automorphism group of $PG(5,2)$, as well as a subgroup of order 31 and its normalizer as automorphism groups of the related 2-(63,31,15) point-hyperplane design. This design if symmetric, i.e. with the least number of blocks among the related designs and therefore computation of the automorphism groups is fastest on it.

The programme performing the computer computations, is based on the exhaustive back track search techniques (see for instance [9, chapter 4]). To filter away isomorphic parallelisms, the normalizer of the subgroup of order 31 in the automorphism group of the projective space is found.

2. Construction and results

Denote the full automorphism group of $PG(5,2)$ by G, it is of order 20158709760.

We use the most popular enumeration and incidence of the points and lines of $PG(5,2)$. The points of $PG(5,2)$ correspond to the nonzero vectors of the 6-dimensional vector space $V_6(2)$ over $GF(2)$ in the following way:

$$\begin{array}{c} \text{point } 1 \longleftrightarrow 000001 \\ \text{point } 2 \longleftrightarrow 000010 \\ \text{point } 3 \longleftrightarrow 000011 \\ \dots \\ \text{point } 63 \longleftrightarrow 111111 \end{array}$$

i.e. the number of the point is the decimal value of the binary number corresponding to the vector. t-dimensional subspaces of $V_6(2)$ are $t+1$-dimensional subspaces of $PG(5,2)$. We then construct the related designs and find the generators of their full automorphism group G.

Each subgroup of order 31 is a Silow p-subgroup of G (for $p = 31$) because

$$|G| = 20158709760 = 31.7^2.5.3^4.2^{15}.$$

By Silow's Theorems (see, for instance [10, sect.7.2.4]) all p-subgroups of G of the same order are conjugate. To construct all cyclic parallelisms with automorphisms of order 31 we can choose an arbitrary one among these subgroups.

Denote it G_{31}. It fixes one point, while the other 62 points are in 2 orbits of length 31.

We sort the 651 lines in lexicographic order defined on the numbers of the points they contain and then assign to each of them a number according to this order. The first point is then in the first 31 lines.

G_{31} partitions the lines into 21 orbits of length 31. We sort each line orbit with respect to the contained line numbers. A resolution has 31 spreads. We assume invariance under G_{31}, which permutes them in one cycle. Therefore one spread is enough to determine the whole resolution. If we know it, we can use its orbit under G_{31} to obtain the other 30 spreads. Without loss of generality the spread we construct can contain the first line. It is in the first orbit. We have to add one line from each of the remaining 20 orbits. A spread is a partition of the point set, so from each of these orbits we can delete the lines incident with any point of the first line. In this way we obtain shortened orbits. We perform a backtrack search on them. If there are already n lines in the parallel class, we choose the $n+1$-st one among the lines with points, which are in none of the n blocks.

This way we construct 5451326 parallelisms. Our next task is to filter away isomorphic ones. Let $\varphi \in G$. Let $\mathcal{P}_1$ be a parallelism with automorphism group $G_{\mathcal{P}_1}$, and let $\mathcal{P}_2 = \varphi\mathcal{P}_1$. Denote by $G_{\mathcal{P}_2}$ the automorphism group of the parallelism $\mathcal{P}_2$. Let $\alpha \in G_{\mathcal{P}_1}$ and $\beta \in G_{\mathcal{P}_2}$. Then $\beta\varphi\mathcal{P}_1 = \varphi\alpha\mathcal{P}_1$ and thus $\beta = \varphi\alpha\varphi^{-1}$ and $G_{\mathcal{P}_2} = \varphi G_{\mathcal{P}_1}\varphi^{-1}$. In our case $G_{\mathcal{P}_2} = G_{\mathcal{P}_1} = G_{31}$ and therefore we are interested in the normalizer $N(G_{31})$ of G_{31} in G, which is defined as $N(G_{31}) = \{g \in G \mid gG_{31}g^{-1} = G_{31}\}$. If an automorphism $\varphi \in G$ transforms one of the constructed parallelisms into another one, then $\varphi \in N(G_{31})$.

The normalizer $N(G_{31})$ is a group of order 155. Let $G_5 = N(G_{31})/G_{31}$. For each parallelism we obtain, we check if an automorphism of G_5 transforms it into a parallelism with a lexicographically smaller orbit leader, and drop it if so. It turns out that all 5451040 parallelisms are in 1090208 orbits of length 5 and 286 orbits of length 1.

The latter correspond to parallelisms with full automorphism group of order 155 (this result coincides with [15]). All nonisomorphic cyclic parallelisms of $PG(5,2)$ invariant under the chosen automorphism group of order 31 are 1090494.

Figure 1: Orbits of cyclic parallelisms of $PG(5,2)$ under G_5

5451326

↙ G_5 ↘

1090208 orbits of length 5 ↘ | 286 orbits of length 1 ↙

1090494
nonisomorphic cyclic parallelisms of $PG(5,2)$

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Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences, BULGARIA
V.Tarnovo 5000, p.b.323
E-mail: stela@math.bas.bg

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