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On Some Relationships between Groups and Completely Regular Semigroups

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We examine the closeness of groups and completely regular semigroups from a categorical and homotopical viewpoint and prove that the category of groups **Grp** is a reflective subcategory of the category of completely regular semigroups **Crs**. Also we prove that the classifying space of a Clifford monoid is homotopy equivalent to the classifying space of a certain group and derive from this that **Grp** is equivalent to the category **Cscm** of classifying spaces of Clifford monoids and homotopy classes of maps between them.

Keywords: Completely regular semigroups, Clifford monoids, classifying spaces of monoids.

1. Introduction

Semigroups in general are quite different from groups in many aspects and one can not expect to get always interesting results when doing for example homology theory with them. The aim of this paper is to give evidence that two special classes of regular semigroups, completely regular semigroups and Clifford semigroups are close to groups in certain aspects. By definition, a completely regular semigroup is a semigroup S with the property that each element of S is contained in some subgroup of S. It turns out that in completely regular semigroups the $\mathcal{H}$-classes are groups, therefore such semigroups are disjoint union of groups. On the other hand, Clifford semigroups, seen as a subclass of completely regular semigroups, are those completely regular semigroups in which idempotents form a semilattice. In the first part of the paper we try to capture the broader picture of completely regular semigroups and groups at once and for this we consider the category of completely regular semigroups **Crs** and that of groups **Grp** which imbeds in the first via the inclusion functor $\mathcal{G} : \mathbf{Grp} \longrightarrow \mathbf{Crs}$

sending each group to itself. Only the existence of the inclusion functor does not reveal much of how these two categories are related to each other, but as we show in Theorem 3.1, $\mathcal{G}$ has a left adjoint F which means that **Grp** is a reflective subcategory of **Crs**. In the second part of the paper we study the classifying spaces of Clifford monoids. We show in Theorem 4.2 that, similar to groups, the classifying space of a Clifford monoid S is a $K(G,1)$ complex where G turns out to be isomorphic to $F(S)$ and then we use this fact to show in the last Theorem 4.3 that **Grp** is equivalent to the category **Cscm** of the classifying spaces of Clifford monoids and homotopy classes of maps between them. These facts about Clifford monoids make them a good candidate for the study of homology and cohomology which will be the aim of a forthcoming paper.

2. Preliminaries

2.1. Coproducts in Grp and the Freyd adjoint functor theorem

We recall the following universal property for the free product $\underset{e\in E}{*} H_e$ of a family of groups H_e with $e \in E$ since it will be useful in the third section. This can be found in [1].

Proposition 2.1 *If G is any group and $h_e : H_e \longrightarrow G$ $(e \in E)$ are group homomorphisms, then there exists a unique homomorphism $h^* : \underset{e\in E}{*} H_e \longrightarrow G$ such that $h^* \circ \iota_e = h_e$ for all e.*

This can be pictured by the following commutative diagram

$$\begin{tikzcd} H_e \arrow[d, "\iota_e"'] \arrow[dr, "h_e"] & \\ \underset{e\in E}{*} H_e \arrow[r, dashed, "h^*"'] & G \end{tikzcd}$$

It turns out that for every $x_1 \cdot ... \cdot x_k \in \underset{e\in E}{*} H_e$ we have

$$h^*(x_1 \cdot ... \cdot x_k) = h_{e_1}(x_1)...h_{e_k}(x_k)$$

where for all $i = 1, ..., k$, $e_i \in E$ is such that $x_i \in H_{e_i}$, and the multiplication on the right hand side is the multiplication in G.

To conclude this section, we give the Freyd Adjoint Functor Theorem (see [4]).

Theorem 2.1 *Given a small complete category A with small hom-sets, a functor $G : A \to X$ has a left adjoint if and only if it preserves all small limits and satisfies the following*

Solution set condition. For each object $x \in X$ there is a small set I and an I-indexed family of arrows $f_i : x \to Ga_i$ such that every arrow $h : x \to Ga$ can be written as a composite $h = Gt \circ f_i$ for some index i and some $t : a_i \to a$.

2.2. Spaces and maps between them

In this section we give a few basic results from Algebraic Topology about how homotopy groups of spaces built in a rather nice way from smaller spaces, are related to the homotopy groups of the later. All we mention here, can be found in [2].

Van Kampen Theorem . *Let a space X be decomposed as a union of path-connected open subsets A_α each of which contains the basepoint $x_0 \in X$. The inclusions $A_\alpha \xrightarrow{incl.} X$ induce homomorphisms $\pi_1(A_\alpha) \to \pi_1(X)$ and then from Proposition 2.1 this family extends to a homomorphism $\Phi : *_\alpha\pi_1(A_\alpha) \to \pi_1(X)$. If $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \xrightarrow{incl.} A_\alpha$, then it is easy to see that the kernel of Φ contains all the elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$.*

Theorem 2.2 (Van Kampen) *If X is the union of path-connected open subsets A_α each of which contains the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi : *_\alpha\pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup generated by all the elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$ and so Φ induces an isomorphism $\pi_1(X) \cong *_\alpha\pi_1(A_\alpha)/N$.*

As it turns out from the proof of Van Kampen theorem, every $[f] \in \pi_1(X)$ is the image of Φ of a factorization of $[f]$ which is an element $[f_1] \cdot ... \cdot [f_k] \in *_\alpha\pi_1(A_\alpha)$ such that each f_i is a loop in some A_α at x_0 and f is homotopic to $f_1 \circ ... \circ f_k$ in X. If Φ is an isomorphism, then Φ^{-1} maps each $[f]$ to such a factorization.

Cells Attachment. Suppose we attach a collection of 2-cells e_α to a path-connected space X via maps $\phi_\alpha : S^1 \to X$, producing a space Y. If γ_α is a path in X from the basepoint x_0 of X to $\phi_\alpha(s_0)$ where s_0 is a basepoint of S^1, then $\gamma_\alpha\phi_\alpha\overline{\gamma_\alpha}$ is a loop at x_0. It follows that the normal subgroup $N \subset \pi_1(X, x_0)$

generated by homotopy classes of all the loops $\gamma_\alpha \phi_\alpha \overline{\gamma_\alpha}$ for varying α lies in the kernel of the map $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by the inclusion $X \to Y$.

Proposition 2.2 *The inclusion* $X \xrightarrow{incl.} Y$ *induces a surjection* $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ *whose kernel is* N. *Thus* $\pi_1(Y) \cong \pi_1(X)/N$.

As an application of the above one can get this.

Corollary 2.1 *For every group* G *there is a 2-dimensional cell complex* X_G *with* $\pi_1(X_G) \cong G$.

One can take as X_G the 2-skeleton of the classifying space BG of G which is obtained from $\bigvee_{g \in G} S^1_g$ by adding 2-cells along paths $g_1 \circ g_2 \circ \overline{g_1 g_2}$. In other words, we take as a presentation of G its multiplication table. Letting φ be the isomorphism of the corollary in this special case, we have

$$\varphi([g]) = g \tag{1}$$

for every homotopy class $[g]$ with $g \in G$.

3. An adjunction situation

We will show in this section that there is an adjunction between the category of groups **Grp** and the category of completely regular semigroups **Crs**. We show exactly in Theorem 3.1 that **Grp** is a reflective subcategory of **Crs** revealing thus another similarity between groups and completely regular semigroups. Throughout this section, for a given completely regular semigroup S, we denote by E the set of its idempotents.

3.1 Some technical lemmas

Let S be a completely regular semigroup, G a group and $h : S \to G$ a homomorphism and suppose that $S = \cup_{e \in E} H_e$ is the disjoint union of S into $\mathcal{H}$-classes. For every $e \in E$ we denote by h_e the restriction of h in H_e. The following is immediate.

Lemma 3.1 *If* S *is completely regular,* G *is a group and* $h : S \to G$ *is a homomorphism, then for every* $\mathcal{H}$*-class* H_e *of* S, $h_e(H_e)$ *is a subgroup of* G.

Lemma 3.2 *If* S *is completely regular,* G *is a group and* $h : S \to G$ *is a homomorphism, then there is a unique homomorphism* $h^* : \mathop{*}_{e \in E} H_e \longrightarrow G$

such that for every $e \in E$, $h^* \circ \iota_e = h_e$ *and that for every* $x_1 \cdot ... \cdot x_k \in \mathop{*}\limits_{e\in E} H_e$, $h^*(x_1 \cdot ... \cdot x_k) = h(x_1)...h(x_k)$.

P r o o f. From Lemma 3.1, h induces a family of homomorphisms

$$\{h_e : H_e \to G | e \in E\} \tag{2}$$

and therefore, from the universal property of Proposition 2.1 there is a unique homomorphism

$$h^* : \mathop{*}\limits_{e\in E} H_e \longrightarrow G$$

induced from the family (2) which satisfies the property stated in the lemma. ■

Make the following notations. Let S be a completely regular semigroup and $S = \bigcup\limits_{e\in E} H_e$ the corresponding decomposition into $\mathcal{H}$-classes. After having identified the elements of E with a single (unit) element 1, we take the free product $\mathop{*}\limits_{e\in E} H_e$ of the disjoint union of groups $\bigcup\limits_{e\in E} H_e$. Denote by

$$\iota_S : S \longrightarrow \mathop{*}\limits_{e\in E} H_e$$

the map which sends each nonidempotent x to (x) and each idempotent e to 1. Note that if $x \in H_e$ for some $e \in E$, then $\iota_S(x) = \iota_e(x)$.

Lemma 3.3 *Let* $h : S \to G$ *be a homomorphism of a completely regular semigroups* S *into a group* G *and* $h^* : \mathop{*}\limits_{e\in E} H_e \to G$ *the induced homomorphism of Lemma 3.2. The following diagram is commutative.*

$$\begin{tikzcd} S \arrow[r, "\iota_S"] \arrow[d, "h"'] & \mathop{*}\limits_{e\in E} H_e \arrow[d, "h^*"] \\ G \arrow[r, "id_G"'] & G \end{tikzcd}$$

P r o o f. The commutativity follows immediately from the definition of ι_S and from Lemma 3.2. ■

3.2 The adjunction theorem

Theorem 3.1 ***Grp*** *is a reflective subcategory of* ***Crs****.*

P r o o f. To prove the claim, we must show that the inclusion functor

$$\mathcal{G} : \mathbf{Grp} \longrightarrow \mathbf{Crs}$$

defined by

$$\mathcal{G}(G) = G$$

has a left adjoint. To do this, we will make use of the Freyd Adjoint Functor Theorem. Note that **Grp** is small complete, it has small hom-sets and the functor $\mathcal{G}$ obviously preserves all small limits. It then remains to prove that $\mathcal{G}$ satisfies the Solution Set Condition. As a first step to prove this, we construct for every completely regular semigroup S a group $F(S)$ and then show that there is a homomorphism from S to $F(S)$ satisfying the other conditions of Solution Set Condition.

If $\{H_e | e \in E\}$ is the corresponding set of $\mathcal{H}$-classes of S, then we let

$$\sim = \{(\iota_S(x) \cdot \iota_S(y), \iota_S(xy)) | x \in H_e, y \in H_f, e, f \in E \text{ and } e \neq f\} \tag{3}$$

and let $\sim^{\#}$ be the congruence generated by $\sim$. Then denote

$$F(S) = (\mathop{*}_{e\in E} H_e)/N$$

where N is the normal subgroup of $\mathop{*}_{e\in E} H_e$ induced by $\sim^{\#}$. We let

$$\nu_S : \mathop{*}_{e\in E} H_e \longrightarrow F(S)$$

be the quotient map.

For every completely regular semigroup S, every group G and every homomorphism $h : S \to G$ we define

$$F(h) : F(S) \longrightarrow G$$

by

$$F(h)(\nu_S(x_1 \cdot ... \cdot x_k)) = h^*(x_1 \cdot ... \cdot x_k),$$

where h^* is the homomorphism of Lemma 3.2. From this lemma, we can write $F(h)$ as

$$F(h)(\nu_S(x_1 \cdot ... \cdot x_k)) = h(x_1)...h(x_k), \tag{4}$$

where the multiplication on the right hand side is the multiplication in G.

Let us show that $F(h)$ is well defined. For this we must show that $F(h)$ is independent from the choice of the representative of $\nu_S(x_1 \cdot ... \cdot x_k)$. Recall from Proposition 1.5.9, [3] that two congruent elements are obtained from each other by a sequence of elementary $\sim$-transitions. Therefore it is enough to show that if the elements $x_1 \cdot ... \cdot x_k$ and $y_1 \cdot ... \cdot y_s$ are obtained from each other by a single elementary $\sim$-transition, then

$$h^*(x_1 \cdot ... \cdot x_k) = h^*(y_1 \cdot ... \cdot y_s).$$

A transition from $\sim$ takes the element

$$x_1 \cdot ... \cdot x_k = x_1 \cdot ... \cdot x_{i-1} \cdot (\iota_S(x) \cdot \iota_S(y)) \cdot x_{i+1} \cdot ... \cdot x_k$$

to

$$y_1 \cdot ... \cdot y_s = x_1 \cdot ... \cdot x_{i-1} \cdot \iota_S(xy) \cdot x_{i+1} \cdot ... \cdot x_k,$$

or conversely, where $x \in H_e, y \in H_f, e, f \in E$ and $e \neq f$. The fact that h^* and h are homomorphisms together with Lemma 3.3 imply

$$\begin{aligned}
&h^*(x_1 \cdot ... \cdot x_{i-1} \cdot \iota_S(xy) \cdot x_{i+1} \cdot ... \cdot x_k) = \\
&h^*(x_1 \cdot ... \cdot x_{i-1}) h^* \iota_S(xy) h^*(x_{i+1} \cdot ... \cdot x_k) = \\
&h^*(x_1 \cdot ... \cdot x_{i-1}) h(xy) h^*(x_{i+1} \cdot ... \cdot x_k) = \\
&h^*(x_1 \cdot ... \cdot x_{i-1}) h(x) h(y) h^*(x_{i+1} \cdot ... \cdot x_k) = \\
&h^*(x_1 \cdot ... \cdot x_{i-1}) h^* \iota_S(x) h^* \iota_S(y) h^*(x_{i+1} \cdot ... \cdot x_k) = \\
&h^*(x_1 \cdot ... \cdot x_{i-1}) h^*(\iota_S(x) \cdot \iota_S(y)) h^*(x_{i+1} \cdot ... \cdot x_k) = \\
&h^*(x_1 \cdot ... \cdot x_{i-1} \cdot (\iota_S(x) \cdot \iota_S(y)) \cdot x_{i+1} \cdot ... \cdot x_k).
\end{aligned}$$

The definition implies immediately that $F(h)$ is a homomorphism.

Finally, to prove the Solution Set Condition for a given S, we take as the I-indexed family the singleton $F(S)$ and as family of arrows, the map $\nu_S \circ \iota_S$. This map is a homomorphism. Indeed, for $x, y \in S - E$ we have

$$(\nu_S \iota_S)(xy) = \nu_S(x \cdot y) = \nu_S(x) \cdot \nu_S(y) = \nu_S(\iota_S(x)) \cdot \nu_S(\iota_S(y)).$$

If $x = e \in E$ and $y \notin E$, then

$$(\nu_S \iota_S)(ey) = \nu_S(\iota_S(y)) = \nu_S(\iota_S(e)) \cdot \nu_S(\iota_S(y)).$$

The case when $x \notin E$ and $y \in E$ is dealt with similarly and the last case when both $x, y \in E$ is obvious.

To conclude the proof, we see that if G is a group and $h : S \to G$ any homomorphism, then (4) implies that for every generator $(\nu_S(\iota_S(x)) \in F(S)$ we have

$$F(h)(\nu_S(\iota_S(x)) = h(x)$$

and we are done. ■

R e m a r k 3.1 To prove Theorem 3.1 we could have used the result of Lemma 1 of [5] but we decided to follow the longer way for the following reason. On the one hand, as we will see in the next section, $F(S)$ turns out to be isomorphic to the homotopy group of the classifying space BS. On the other hand, restricting to the case of Clifford semigroups (see the next section for the definition), it is not hard to check that $F(S) \cong S/\sigma$ where σ is the minimum group congruence in S. Thus, in the case of the category of Clifford semigroups **CS**, the reflector $F : \mathbf{CS} \to \mathbf{Grp}$ can be realized in three different ways. First, as the functor sending S to $(\underset{e\in E}{*} H_e)/N$ where N is the normal subgroup of $\underset{e\in E}{*} H_e$ induced by $\sim^{\#}$. Second, as the functor sending S to the homotopy group of BS, and thirdly, as the functor sending S to S/σ.

4. Classifying spaces of Clifford monoids

The main result of this section states that the classifying space of a Clifford monoid is homotopy equivalent to the classifying space of the colimit group of the diagram representing the monoid. A similar result to this can be found in [8]. We will take as the definition of Clifford semigroups one of the equivalent statements of the following theorem which can be found in [3].

Theorem 4.1 *Let S be a semigroup with E set of idempotents. Then the following statements are equivalent:*
(1) S is a Clifford semigroup;
(2) S is a semilattice of groups;
(3) S is a strong semilattice of groups;
(4) S is regular, and the idempotents are central;
(5) S is regular, and $\mathcal{D}^S \cap (E \times E) = 1_E$.

The proof of this theorem reveals that every Clifford semigroup is completely regular and that idempotents form a semilattice. It turns out that in Clifford semigroups we have

$$\mathcal{R} = \mathcal{L} = \mathcal{H} = \mathcal{D} = \mathcal{J}. \tag{5}$$

In the sequel we will restrict ourselves to Clifford semigroups having a unit element, namely Clifford monoids. We make the following observation. If

1 is the unit of a Clifford monoid S and $x, y \in S$ are such that $xy = 1$, then $x, y \in H_1$. Indeed, 1 is the greatest element of S according the relation $\leq_{\mathcal{H}}$ since for all $s \in S$ we have $s = s1 = 1s$. Assuming now that $xy = 1$, we get $1 \leq_{\mathcal{R}} x$ and $1 \leq_{\mathcal{L}} y$, therefore from (5) and the fact that 1 is maximal we get $x, y \in H_1$ as claimed. This means that the only divisors of 1 are the elements of H_1.

If S is a Clifford monoid with $\mathcal{H}$-classes $\{H_e | e \in E\}$, then we define the map

$$\varphi : \underset{e\in E}{*} H_e \to S$$

by

$$\varphi(x_1 \cdot ... \cdot x_k) = \begin{cases} x_1 ... x_k & \text{if} \quad x_1 \cdot ... \cdot x_k \neq 1_* \\ 1_S & \text{if} \quad x_1 \cdot ... \cdot x_k = 1_* \end{cases}$$

where 1_S is the unit element of S and 1_* is the unit of $\underset{e\in E}{*} H_e$.

Theorem 4.2 *If S is a Clifford monoid, then its classifying space BS is a $K(G, 1)$ space with $G \cong F(S)$.*

P r o o f. Suppose that S is the semilattice of groups $\{H_e | e \in E\}$. Form the wedge sum $\underset{e\in E}{\vee} BH_e$ where BH_e is the classifying space of H_e and the base point $\bullet$ is obtained by identifying the base points of each BH_e. Note that for every $e \in E$, there is an open neighbourhood U_e of $\bullet$ such that $\bullet$ is a deformation retract of U_e. Such U_e can be obtained by removing a different point than $\bullet$ from a chosen 1-cell $g \in H_e$. It follows that each BH_e is a deformation retract of the open neighbourhood $A_e = BH_e \vee_{f \neq e} U_f$, and that for every $n \geq 2$, $\bigcap_{i=1}^{n} A_{e_i} = \vee_e U_e$. Van-Kampen Theorem (Theorem 2.2) implies that

$$\pi_1(\underset{e\in E}{\vee} BH_e) \cong \underset{e\in E}{*} \pi_1(BH_e) \tag{6}$$

where the isomorphism sends each homotopy class $[x_1 \circ ... \circ x_n]$ with $x_i \in BH_{e_i}$ for $i = 1, ..., n$, to the respective factorization $[x_1] \cdot ... \cdot [x_n]$. On the other hand, extending the isomorphisms of (1) to the free product, we get the isomorphism

$$\underset{e\in E}{*} \pi_1(BH_e) \cong \underset{e\in E}{*} H_e \tag{7}$$

which sends $[x_1] \cdot ... \cdot [x_n]$ to $\iota_S(x_1) \cdot ... \cdot \iota_S(x_n)$. Since cells of dimension higher than 2 do not influence π_1 of the complex, we need to consider only BS^2, the 2-skeleton of BS. Note that BS^2 is obtained from $(\underset{e\in E}{\vee} BH_e)^2$ by adding 2-cells along paths

$$\{x \circ y \circ \overline{xy} | x \in H_e, y \in H_f, e, f \in E \text{ and } e \neq f\},$$

where $\overline{xy}$ is the inverse path of xy. Now Proposition 2.2, (6) and (7) imply that $\pi_1(BS^2) \cong \pi_1(\bigvee_{e\in E} BH_e)/N$ where N is the normal subgroup generated by

$$\{\iota_S(x) \cdot \iota_S(y) \cdot (\iota_S(xy))^{-1} | x \in H_e, y \in H_f, e, f \in E \text{ and } e \neq f\}$$

and therefore is isomorphic to $F(S)$. Note that at this stage we did not use the fact that S is a Clifford monoid. All we needed, was S being a disjoint union of groups.

To prove that BS is a $K(F(S), 1)$ space we consider the diagram

$$E(\underset{e\in E}{*} H_e) \xrightarrow{p} B(\underset{e\in E}{*} H_e) \xrightarrow{\phi} BS$$

where p is the covering space projection for $B(\underset{e\in E}{*} H_e)$ and ϕ is defined recursively by

$$\phi(\star) = \bullet$$

where $\star$ is the 0-cell of $B(\underset{e\in E}{*} H_e)$ and on each r-skeleton with $r \geq 1$, by

$$\phi(\star \xrightarrow{x_1^{(1)}\cdot\ldots\cdot x_{n_1}^{(1)}} \star \qquad \ldots \qquad \star \xrightarrow{x_1^{(r)}\cdot\ldots\cdot x_{n_r}^{(r)}} \star) =$$

$$\bullet \xrightarrow{\varphi(x_1^{(1)}\cdot\ldots\cdot x_{n_1}^{(1)})} \bullet \qquad \ldots \qquad \bullet \xrightarrow{\varphi(x_1^{(r)}\cdot\ldots\cdot x_{n_r}^{(r)})} \bullet\,,$$

where each $x_1^{(j)} \cdot \ldots \cdot x_{n_j}^{(j)}$ is in the reduced form. To see how the induction works, we note first that it is easy to define ϕ by the above formula on the 1-skeleton, then suppose that we have already defined it on the $(r-1)$-skeleton and want to extend it to the r-skeleton. To realize the extension, its enough to see how the restriction of ϕ in the boundary of any r-cell

$$\delta = (\star \xrightarrow{x_1^{(1)}\cdot\ldots\cdot x_{n_1}^{(1)}} \star \qquad \ldots \qquad \star \xrightarrow{x_1^{(r)}\cdot\ldots\cdot x_{n_r}^{(r)}} \star)\,,$$

extends to the whole cell δ. We can apply the extension lemma (Lemma 4.7, [2]) by taking in the role of X the closed cell δ and in the role of A the subcomplex of $B(\underset{e\in E}{*} H_e)$ spanned by the closed boundary cells of δ. The restriction of ϕ on this subcomplex yields a continuous map with image the subcomplex of BS spanned by the closed boundary cells of the r-cell of BS

$$\sigma = \bullet \xrightarrow{\varphi(x_1^{(1)}\cdot\ldots\cdot x_{n_1}^{(1)})} \bullet \qquad \ldots \qquad \bullet \xrightarrow{\varphi(x_1^{(r)}\cdot\ldots\cdot x_{n_r}^{(r)})} \bullet\,.$$

Since $\pi_{r-1}(\sigma) = 0$, then extension lemma applies giving the desired extension over δ which from the cellular approximation theorem (Theorem 4.8,[2]) maps the closed cell δ homeomorphically to the closed cell σ.

Note that ϕ is not surjective in general since if there is $e \in E$ which has no non-idempotent divisors, then $\bullet \xrightarrow{e} \bullet$ is not in the image of ϕ. Also observe that for the interior σ^0 of each cell

$$\sigma = (\bullet \xrightarrow{x_1} \bullet ... \bullet \xrightarrow{x_k} \bullet) \in BS$$

we have that $\phi^{-1}(\sigma^0)$ is either empty if for some $s = 1, ..., k$, $\varphi^{-1}(x_s) = \emptyset$, or otherwise, it is the disjoint union of the interiors δ^0 of cells

$$\delta = (\star \xrightarrow{y_{1i}} \star ... \star \xrightarrow{y_{kj}} \star) \in B(\underset{e \in E}{*} H_e)$$

where $y_{st} \in \varphi^{-1}(x_s)$ for $s = 1, ..., k$, and the restriction of ϕ in each δ^0 is a homeomorphism.

Finally, we want to show that ϕ is a covering map. Take as the open cover of BS the set of open cells of positive dimension together with U_{1_S} where U_{1_S} is the 1-cell corresponding to 1_S "minus" a point $\circ$ different than $\bullet$. Since 1_S has no divisors belonging to classes different from H_1, we see that $\phi^{-1}(U_{1_S}) = U_{1_*}$ where U_{1_*} is obtained from U_{1_*} by removing the point $\phi^{-1}(\circ)$. Recalling now that p is a covering map, one can deduce that the composite $\phi \circ p$ is a covering map too proving the claim. ■

In the sequel we denote by **Cscm** the category with objects Clifford monoids and with morphisms, homotopic maps between them. Theorem 4.2 and Theorem 1B.8, [2] imply the following.

Corollary 4.1 *For every completely regular monoid S, $BS \simeq BF(S)$.*

Lemma 4.1 *Every group homomorphism $\varphi : G \to G'$ induces a map $B(\varphi) : BG \to BG'$ between the respective classifying spaces of G and G' and if $\varphi_1, \varphi_2 : G \to G'$ are different homomorphisms, then $B(\varphi_1)$ and $B(\varphi_2)$ are not homotopic.*

Proof. Since for every group G we have $\pi_1(G) \cong G$, we can utilize Proposition 1B.9, [2]. ■

Define

$$\mathbf{B} : \mathbf{Grp} \to \mathbf{Cscm}$$

on objects by

$$\mathbf{B}(G) = BG$$

for every group G where BG is the classifying space of G, and on morphisms by

$$\mathbf{B}(\varphi) = B(\varphi),$$

with $B(\varphi)$ as defined in Lemma 4.1. We show that this assignment is functorial. For $g \in G$ suppose that we have chosen g' to be homotopic with $\varphi(g)$ and g'' to be homotopic with $\varphi(g')$. In other words, we have that $B(\varphi')B(\varphi)(g) = g''$. On the other hand, since g' is homotopic to $\varphi(g)$, then $\varphi'(g')$ is homotopic to $\varphi'\varphi(g)$. Therefore the restrictions of mappings $B(\varphi'\varphi)$ and $B(\varphi')B(\varphi)$ on the 1-skeleton of BG are homotopic, then as the proof of Proposition 1B.9, [2] shows, they can be extended to homotopic maps $B(\varphi'\varphi)$ and $B(\varphi')B(\varphi)$ over every skeleta and as such they can be regarded as the same morphism in **Cscm**.

The following holds true.

Theorem 4.3 ***Grp*** *is equivalent to* ***Cscm****.*

Proof. To prove the theorem, we must show that B is an equivalence of categories (see Theorem 1, p. 93, [4]). Indeed, B is faithful from Lemma 4.1, it is obviously full and lastly every $BS \in$ **Cscm** is isomorphic to $BF(S)$ where $F(S)$ is the group of Theorem 3.1. ∎

Corollary 4.2 ***Cscm*** *is a reflective subcategory of* ***Crs****.*

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