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On Li's Coefficients for Some Classes of L-Functions

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We study the generalized Li coefficients associated with the class $\mathcal{S}^{\sharp\flat}$ of functions containing the Selberg class and (unconditionally) the class of all automorphic L-functions attached to irreducible unitary cuspidal representations of $GL_N(\mathbb{Q})$ and the class of L-functions attached to the Rankin-Selberg convolution of two unitary cuspidal automorphic representations π and π' of $GL_m(\mathbb{A}_F)$ and $GL_{m'}(\mathbb{A}_F)$. We deduce a full asymptotic expansion of the Archimedean contribution to these coefficients and investigate the contribution of the non-archimedean term. Obtained results are applied to automorphic L-functions. Also, a bound towards a generalized Ramanujan conjecture for the Archimedean Langlands parameters $\mu_\pi(v,j)$ of π is derived.

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1. Introduction

The Riemann hypothesis (RH), formulated by B. Riemannn in 1859 is one of the most important conjectures in mathematics. It states that all non-trivial zeros of the Riemann zeta function are on the critical line Re $s = 1/2$ and has been resisting all attempts to prove it.

Various arithmetical, geometrical and algebraic objects can be described by the so-called global L-functions, which are formally similar to the Riemann zeta-function. They can be associated to elliptic curves, number fields, Maass forms, Dirichlet characters, ... The hypothesis that all non-trivial zeros of global L-functions are on the line Re $s = 1/2$ is nowadays called Generalized Riemann hypothesis (GRH). While the global L-functions are seemingly independent of each other, they have similar analytic properties and they are also assumed to satisfy GRH. This was a motivation for mathematicians to try to understand, or at least classify the class of all objects for which GRH holds true.

The Langlands program is an attempt to understand all global L-functions and to relate them to automorphic forms. Common properties of functions which conjecturally satisfy GRH are appointed. A. Selberg [16] has given a set of precise axioms which are believed to characterize the L-functions for which GRH holds. Elements of the Selberg class are Dirichlet series with an Euler product representation, meromorphic continuation and a functional equation of the right shape. Although the exact nature of the class is conjectural, it is assumed that it is possible to classify its elements and give us an insight into their relationship to automorphic forms and the GRH.

Consequences of RH or GRH are various and important. They include many propositions which are known to be true under these hypotheses and some statements equivalent to the RH or GRH. Statements equivalent to RH give us an opportunity to restate the Riemann hypothesis in a different language, even in an entirely different disciplines, so we gain more possible tools for proving it. There are three main categories of statements equivalent to RH: purely number-theoretical statements, statements closely related to the analytic properties of the zeta function and cross-disciplinary statements.

One of the statements equivalent to RH, closely related to analytic properties of zeta function is the Li positivity criterion, proved by X.-J. Li in 1997 [12], stating that RH is equivalent to the non-negativity of the set of coefficients

$$\lambda_n = \sum_{\rho}{}^{*} \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right). \tag{1}$$

Here, the sum runs over the non-trivial zeros of the Riemann zeta function, counted with multiplicities, and $*$ indicates that the sum is taken in the sense of the limit as $|\mathrm{Im}\rho| \to \infty$.

Recently it was proved that it is actually sufficient to probe the Li coefficients for their large n behavior. Namely, A. Voros [22] has proved that the Riemann hypothesis true is equivalent to the tempered growth of λ_n (as $\frac{1}{2} n \log n$), determined by its archimedean part, while the Riemann hypothesis false is equivalent to the oscillations of λ_n with exponentially growing amplitude.

The Li criterion motivated many numerical calculations concerning RH. The generalized Li coefficients attached to global L functions and the generalized Li criterion as well may serve as a motivation for numerical calculations concerning GRH, possibly producing some new ideas for proving it.

E. Bombieri and J. C. Lagarias [3] have noticed that the Li criterion can be formulated for a general multiset of complex numbers and obtained an arithmetic expression of the Li coefficient λ_n in the form

$$\lambda_n = S_\infty(n) - S_{NA}(n) + 1, \tag{2}$$

where S_∞ denotes a contribution from archimedean places (Γ-factors) and S_{NA} is a contribution of finite (non-archimedean) places.

J. C. Lagarias [11] has defined the generalized Li coefficient attached to an irreducible cuspidal unitary automorphic representation π of $GL_m(\mathbb{Q})$ and proved the generalized Li criterion in this case. He obtained an arithmetic expression for these coefficients completely analogous to (2) and determined the asymptotic behavior of both the Archimedean and the finite part of generalized Li coefficients in this case.

F. C. S. Brown [4] has determined zero-free regions of Dirichlet and Artin L-functions (under the Artin hypothesis) in terms of sizes of the corresponding generalized Li coefficients.

Our main objective is to formulate and prove generalization of the Li criterion for some classes of L functions, to find the arithmetic expression and investigate the asymptotic behavior of generalized Li coefficients. Also, we shall analyze consequences of the obtained results under GRH. Classes which will be treated are the Selberg class and the class of L-functions attached to the Rankin-Selberg convolution of two unitary cuspidal automorphic representations π and π' of $GL_m(\mathbb{A}_F)$ and $GL_{m'}(\mathbb{A}_F)$.

2. Preliminaries

2.1. **The Selberg class of functions.** The Selberg class of functions $\mathcal{S}$, introduced by A. Selberg in [16], is a general class of Dirichlet series F satisfying the following properties:

(i) (Dirichlet series) F posses a Dirichlet series representation

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

that converges absolutely for $\mathrm{Re} s > 1$.

(ii) (Analytic continuation) There exists an integer $m \geq 0$ such that $(s-1)^m F(s)$ is an entire function of finite order. The smallest such number is denoted by m_F and called a polar order of F.

(iii) (Functional equation) The function F satisfies the functional equation $\Phi_F(s) = w\overline{\Phi_F(1-\bar{s})}$, where $\Phi_F(s) = F(s)Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$, with $Q_F > 0$, $r \geq 0$, $\lambda_j > 0$, $|w| = 1$, $\mathrm{Re}\mu_j \geq 0$, $j = 1, \ldots, r$. Though the numbers $\lambda_1, \ldots, \lambda_r$ are not unique, it can be shown (see, e.g. [15]) that the number $d_F = 2\sum_{j=1}^{r} \lambda_j$ is an invariant, called the degree of F. Furthermore, the number $\xi_F = 2\sum_{j=1}^{r} (\mu_j - 1/2)$ is also an invariant (see [15] , p. 43]) called the ξ invariant.

(iv) (Ramanujan conjecture) For every $\epsilon > 0$ $a_F(n) \ll n^{\epsilon}$.

(v) (Euler product)

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_F(n)}{n^s},$$

where $b_F(n) = 0$, for all $n \neq p^m$ with $m \geq 1$ and p prime, and $b_F(n) \ll n^\theta$, for some $\theta < \frac{1}{2}$.

An extended Selberg class $\mathcal{S}^\sharp$ is a class of functions satisfying conditions (i), (ii) and (iii).

It is conjectured that the Selberg class coincides with the class of all automorphic L-functions. Some properties such as Ramanujan conjecture, boundedness of coefficients in the Dirichlet series representation of $\log L(s,\pi)$ and the bound $\mathrm{Re}\mu_j \geq 0$ on the archimedean Langlands parameters have not yet been proved.

In order to apply our results unconditionally to automorphic L-functions attached to irreducible unitary automorphic representations of $GL_N(\mathbb{Q})$, we shall focus on the class $\mathcal{S}^{\sharp\flat}$ of functions satisfying axioms (i), (ii) and the following two axioms:

(iii') (Functional equation) The function F satisfies the functional equation $\Phi_F(s) = w\overline{\Phi_F(1-\bar{s})}$, where $\Phi_F(s) = F(s)Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s+\mu_j)$, with $Q_F > 0$, $r \geq 0$, $\lambda_j > 0$, $|w| = 1$, $\mathrm{Re}\mu_j > -\frac{1}{4}$, $\mathrm{Re}(\lambda_j + 2\mu_j) > 0$, $j = 1,\ldots,r$.

Let us note that (iii') implies that $\mathrm{Re}(\lambda_j + \mu_j) > 0$.

(v') The logarithmic derivative of the function F possesses a Dirichlet series representation

$$\frac{F'}{F}(s) = -\sum_{n=1}^{\infty} \frac{c_F(n)}{n^s},$$

converging absolutely for $\mathrm{Re}s > 1$.

It can be shown that introduced class $\mathcal{S}^{\sharp\flat}$ contains Selberg class.

2.2. **Rankin-Selberg L-functions.** The Rankin-Selberg L-function attached to the product $\pi \times \widetilde{\pi}'$ of two unitary cuspidal automorphic representations of $GL_m(\mathbb{A}_F)$ and $GL_{m'}(\mathbb{A}_F)$ is given, for $\mathrm{Re}s > 1$, by an absolutely convergent Euler product of local factors

$$L\left(s, \pi_f \times \widetilde{\pi}'_f\right) = \prod_{\mathfrak{p}<\infty} L(s, \pi_\mathfrak{p} \times \widetilde{\pi}'_\mathfrak{p}), \tag{3}$$

as proved in [8, Th. 5.3.]. Here, $\widetilde{\pi}$ denotes the contragradient representation of π. For any place v of F, $\widetilde{\pi}_v$ is equivalent to the complex conjugate $\overline{\pi}_v$ [6], hence $L\left(s, \pi_f \times \widetilde{\pi}'_f\right) = \overline{L\left(\bar{s}, \widetilde{\pi}_f \times \pi'_f\right)}$.

Let us put

$$L\left(s, \pi_\infty \times \widetilde{\pi}'_\infty\right) = \prod_{v\in S_\infty} L(s, \pi_v \times \widetilde{\pi}'_v),$$

where S_∞ denotes the set of all infinite places of F. Then, as proved in [7, 8, 9, 17, 18, 19, 20] (see also [5, Th. 9.1. and Th. 9.2.], the complete Rankin-Selberg L-function

$$\Lambda(s, \pi \times \widetilde{\pi}') = L(s, \pi_f \times \widetilde{\pi}'_f) L(s, \pi_\infty \times \widetilde{\pi}'_\infty)$$

extends to a meromorphic function of order 1 on the whole complex plane, bounded (away from its possible poles) in vertical strips. It has simple poles at $s = 1 + it_0$ and $s = it_0$, arising form $L(s, \pi_f \times \widetilde{\pi}'_f)$ if and only if $m = m'$ and $\pi' \cong \pi \otimes |\det|^{it_0}$, for some $t_0 \in \mathbb{R}$. Otherwise, it is a holomorphic function. Finally, $\Lambda(s, \pi \times \widetilde{\pi}')$ satisfies the functional equation

$$\Lambda(s, \pi \times \widetilde{\pi}') = \epsilon\left(\pi \times \widetilde{\pi}'\right) Q\left(\pi \times \widetilde{\pi}'\right)^{\frac{1}{2}-s} \Lambda(1-s, \widetilde{\pi} \times \pi'), \tag{4}$$

where $Q\left(\pi \times \widetilde{\pi}'\right) > 0$ is the arithmetic conductor and $\epsilon\left(\pi \times \widetilde{\pi}'\right)$ is a complex number of modulus 1.

3. Results

The arithmetic formulas for generalized Li coefficients are obtained using explicit formulas, proved in [21] for the class $\mathcal{S}^{\sharp\flat}$ and in [14] for the Rankin-Selberg L-function. Proofs are based on results of Jorgenson and Lang [10] on explicit formulas in the fundamental class of functions and results on expanding their class of test functions to which the explicit formula applies, obtained in papers [1] and [2].

3.1. Generalized Li coefficients and Li criterion.

The generalized Li coefficient attached to $F \in \mathcal{S}^{\sharp\flat}$ can be defined analogously as (1), by

$$\lambda_F(n) = \sum_{\rho \in Z(F)}{}^{*} \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right), \tag{5}$$

where $Z(F)$ denotes set of all nontrivial zeros of the function F.

The *-convergence of the series (5) was proved in [21] using the explicit formula with suitably chosen test function.

Proposition 3.1 ([21] Li coefficient for the class $\mathcal{S}^{\sharp\flat}$)**.** *Let $F \in \mathcal{S}^{\sharp\flat}$ such that $0 \notin Z(F)$. Then, the series (5) is *-convergent for every integer n. Moreover, the series* $\mathrm{Re}\lambda_F(n) = \sum_{\rho \in Z(F)}{}^{*} \mathrm{Re}\left(1 - \left(1 - \frac{1}{\rho}\right)^n\right)$ *converges absolutely for all integers n.*

The Generalized Li criterion for the class $\mathcal{S}^{\sharp\flat}$ was also proved in [21].

Proposition 3.2 ([21] Li criterion for the class $\mathcal{S}^{\sharp\flat}$)**.** *Let $F \in \mathcal{S}^{\sharp\flat}$ such that $0 \notin Z(F)$. Then, all non-trivial zeros of F lie on the line* $\mathrm{Re} s = \frac{1}{2}$ *if and only if* $\mathrm{Re}\lambda_F(n) \geq 0$, *for all $n \in \mathbb{N}$.*

The definition of the generalized Li coefficient attached to the product $\pi \times \widetilde{\pi}'$ is given by

$$\lambda_{\pi,\pi'}(n) = \sum_{\rho \in Z(L)}{}^{*} \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right). \tag{6}$$

The existence of coefficients (6) and the generalized Li criterion in this setting are proved in the following propositions.

Proposition 3.3 ([14] Li coefficient for the Rankin-Selberg L-function)**.** *The generalized Li coefficients $\lambda_{\pi,\pi'}(n)$ are well defined for every integer n.*

Proposition 3.4 ([14] Li criterion for the Rankin-Selberg L-function)**.** *All nontrivial zeros of $L(s, \pi_f \times \widetilde{\pi}'_f)$ lie on the line* $\mathrm{Re} s = \frac{1}{2}$ *if and only if* $\mathrm{Re}\lambda_{\pi,\pi'}(n) \geq 0$, *for all $n \in \mathbb{N}$.*

3.2. **An arithmetic formula for the Li coefficients.** The arithmetic formulas for the generalized Li coefficients are given in the next two theorems.

Theorem 3.5 ([21] Arithmetic formula for the class $\mathcal{S}^{\sharp\flat}$)**.** *Let $F \in \mathcal{S}^{\sharp\flat}$ be a function such that $0 \notin Z(F)$. Then, for all $n \in \mathbb{N}$*

$$\lambda_F(-n) = m_F + n \log Q_F + \sum_{l=1}^{n} \binom{n}{l} \gamma_F(l-1) + \sum_{l=1}^{n} \binom{n}{l} \eta_F(l-1)$$

where

$$\eta_F(0) = \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \mu_j) \quad and \quad \eta_F(l-1) = \sum_{j=1}^{r} (-\lambda_j)^l \sum_{k=0}^{\infty} \frac{1}{(\lambda_j + \mu_j + k)^l},$$

for $l \geq 2$.

Theorem 3.6 ([14] Arithmetic formula for the Rankin-Selberg L-function)**.** *Let π and π' be two automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_F)$ and $GL_{m'}(\mathbb{A}_F)$, respectively. Then, for all $n \in \mathbb{N}$ and $t_0 \in \mathbb{R}\backslash\{0\}$*

$$\begin{aligned} \lambda_{\pi,\pi'}(-n) &= \sum_{j=1}^{n} \binom{n}{j} \gamma_{\pi,\pi'}(j-1) + \delta_{\pi,\pi'}(0) \\ &\quad + \frac{n}{2}(\log Q_{\pi \times \widetilde{\pi}'} - dmm' \log \pi) \end{aligned}$$

$$+\delta_{\pi,\pi'}(t_0) \left(1 - \left(1 + \frac{1}{it_0}\right)^n + 1 - \left(1 - \frac{1}{1 - it_0}\right)^n\right)$$

$$- \quad dmm' + \sum_{l=1}^{dmm'} \left(\frac{\mu_{\pi \times \widetilde{\pi}'}(l)}{1 + \mu_{\pi \times \widetilde{\pi}'}(l)}\right)^n + \sum_{j=1}^{n} \binom{n}{j} \eta_{\pi,\pi'}(j-1) \tag{7}$$

where

$$\eta_{\pi,\pi'}(0) = \frac{1}{2}\sum_{l=1}^{dmm'} \frac{\Gamma'}{\Gamma}\left(\frac{3+\mu_{\pi\times\widetilde{\pi}'}(l)}{2}\right),$$

$$\eta_{\pi,\pi'}(j-1) = \frac{(-1)^j}{2^j}\sum_{l=1}^{dmm'}\sum_{t=0}^{\infty}\frac{1}{\left(t+\frac{3+\mu_{\pi\times\widetilde{\pi}'}(l)}{2}\right)^j}, \quad \text{for } j \geq 2$$

and $\gamma_{\pi,\pi'}(k)$ *are the coefficients in the Laurent (Taylor) series expansion of* $\frac{L'}{L}$ *at* $s = 1$.

3.3. **Asymptotic behavior of the Li coefficients.** In order to investigate the asymptotic behavior of the Li coefficients we will treat the archimedean and non-Archimedean (finite) contribution to the nth Li coefficient separately. First, we shall write the Li coefficient (5) as

$$\lambda_F(-n) = S_\infty(n,F) + S_{NA}(n,F),$$

where

$$S_\infty(n,F) = m_F + n\log Q_F + \sum_{l=1}^{n}\binom{n}{l}\eta_F(l-1)$$

is the Archimedean contribution, while

$$S_{NA}(n,F) = \sum_{j=1}^{n}\binom{n}{l}\gamma_F(l-1)$$

is the finite (non-Archimedean) term.

The next theorem gives us the full asymptotic expansion of the Archimedean contribution to the nth Li coefficient in terms of $n\log n$, n, n^0 and odd negative powers of n.

Theorem 3.7 ([13] Archimedean contribution to the Li coefficient for $\mathcal{S}^{\sharp\flat}$)**.** *Let* $F \in \mathcal{S}^{\sharp\flat}$ *be a function non-vanishing at zero. Then, for an arbitrary* $K \in \mathbb{N}$

$$\begin{aligned} S_\infty(n,F) &= \frac{d_F}{2}n\log n + nC_F + \frac{\xi_F}{2} \\ &+ m_F + \frac{d_F}{4} - \frac{d_F}{2}\sum_{k=1}^{K}\frac{B_{2k}}{2k}n^{1-2k} + O_{F,K}(n^{-2K}), \end{aligned}$$

as $n \to \infty$, *where* $C_F = \log Q_F + \frac{d_F}{2}(\gamma - 1) + \sum_{j=1}^{r}\lambda_j\log\lambda_j$ *and* B_{2k} *are Bernoulli numbers.*

Let

$$\lambda_F(n,T) = \sum_{\substack{\rho \\ |\mathrm{Im}\rho|<T}} \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right).$$

denote an incomplete nth Li coefficient to the height T. The following result gives us the representation of the finite contribution to the nth Li coefficient in terms of the incomplete nth Li coefficient to the height $\sqrt{n}$, up to the error term $O(\sqrt{n}\log n)$.

Theorem 3.8 ([13] Non-Archimedean contribution to Li coefficient for the class $\mathcal{S}^{\sharp\flat}$). *Let $F \in \mathcal{S}^{\sharp\flat}$ be a function non-vanishing at zero. Then,*

$$S_{NA}(n,F) = -\lambda_F(-n, \sqrt{n}) + O(\sqrt{n}\log n).$$

Combining the last two theorems we obtain an asymptotic behavior of the nth Li coefficient attached to the function $F \in \mathcal{S}^{\sharp\flat}$, as $n \to \infty$.

Corollary 3.9 ([13]). *Let $F \in \mathcal{S}^{\sharp\flat}$ be a function non-vanishing at zero. Then, for all $n \in \mathbb{N}$*

$$\lambda_F(-n) = \frac{d_F}{2} n\log n + nC_F - \lambda_F(-n,\sqrt{n}) + O(\sqrt{n}\log n)$$

where $C_F = \log Q_F + \frac{d_F}{2}(\gamma - 1) + \sum\limits_{j=1}^{r} \lambda_j \log \lambda_j$ and γ is the Euler constant.

Since $\mathcal{S}^{\sharp\flat}$ contains all automorphic L-functions, attached to irreducible unitary automorphic representations of $GL_N(\mathbb{Q})$, we immediately obtain the following corollary.

Corollary 3.10 ([13]).

$$\begin{aligned} S_\infty(n,\pi) &= \frac{N}{2} n\log n + n\left(\frac{1}{2}\log Q(\pi) + \frac{N}{2}(\gamma - 1 - \log(2\pi))\right) \\ &+ \frac{1}{2}\sum_{j=1}^{N} \kappa_j(\pi) + \delta(\pi) - \frac{N}{4} - \frac{N}{2}\sum_{k=1}^{K} \frac{B_{2k}}{2k} n^{1-2k} + O_{\pi,K}(n^{-2K}), \end{aligned}$$

as $n \to \infty$, for an arbitrary, fixed $K \in \mathbb{N}$.

The above result improves the result of Lagarias [11], showing that $S_\infty(n,\pi)$ has a full asymptotic expansion in terms of n^{-k} (k-odd). Furthermore, it shows that only the constant term depends on the sum of Archimedean Langlands parameters κ_j and the terms with negative degrees depend singly on N.

An analogous decomposition of the generalized Li coefficient attached to the Rankin-Selberg L-function is somewhat more complicated as well as the results for the asymptotic behavior of the Archimedean term. Coefficient (6) can be written as $\lambda_{\pi,\pi'}(-n) = S_\infty(n,\pi,\pi') + S_{NA}(n,\pi,\pi')$,where

$$S_\infty(n,\pi,\pi') = \delta_{\pi,\pi'}(0) + \frac{n}{2}\left(\log Q_{\pi\times\pi'} - dmm'\log\pi\right)$$
$$-dmm' + \sum_{l=1}^{dmm'}\left(\frac{\mu_{\pi\times\widetilde{\pi}'}(l)}{1+\mu_{\pi\times\widetilde{\pi}'}(l)}\right)^n + \sum_{j=1}^{n}\binom{n}{j}\eta_{\pi,\pi'}(j-1)$$

is an Archimedean contribution and

$$S_{NA}(n,\pi,\pi') = \sum_{j=1}^{n}\binom{n}{j}\gamma_{\pi,\pi'}(j-1)$$
$$+\delta_{\pi,\pi'}(t_0)\left(1-\left(1+\frac{1}{it_0}\right)^n + 1 - \left(1-\frac{1}{1-it_0}\right)^n\right)$$

is a finite term. An incomplete nth Li coefficient to the height T is defined as

$$\lambda_{\pi,\pi'}(n,T) = \sum_{\substack{\rho \\ |\mathrm{Im}\rho|<T}}\left(1-\left(1-\frac{1}{\rho}\right)^n\right).$$

The main results in this case are the following two theorems and a corollary.

Theorem 3.11 ([14] Archimedean contribution - Rankin-Selberg L-function)**.** *Let π and π' be two automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_F)$ and $GL_{m'}(\mathbb{A}_F)$ respectively. Then, for an arbitrary $K\in\mathbb{N}$*

$$S_\infty(n,\pi,\pi') = \delta_{\pi,\pi'}(0) + \frac{n}{2}dmm'\log n + \frac{1}{2}\sum_{l=1}^{dmm'}\mu_{\pi\times\widetilde{\pi}'}(l)$$
$$+\frac{n}{2}\left(\log Q_{\pi\times\pi'} - dmm'\left(\log 2\pi - \gamma + 1\right)\right) + \sum_{l=1}^{dmm'}\left(\frac{\mu_{\pi\times\widetilde{\pi}'}(l)}{1+\mu_{\pi\times\widetilde{\pi}'}(l)}\right)^n$$
$$-\frac{1}{4}dmm' - \frac{dmm'}{2}\sum_{k=1}^{K}\frac{B_{2k}}{2k}n^{1-2k} + O_{\pi,\pi',K}(n^{-2K}),$$

as $n\to\infty$, where B_{2k} are the Bernoulli numbers.

Theorem 3.12 ([14] Non-Archimedean contribution - Rankin-Selberg L-function)**.** *Let π and π' be two automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_F)$ and $GL_{m'}(\mathbb{A}_F)$ respectively. Then,*

$$S_{NA}(n,\pi,\pi') = -\lambda_{\pi,\pi'}(-n,\sqrt{n}) + O(\sqrt{n}\log n).$$

Corollary 3.13 ([14])**.** *Let π and π' be two automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_F)$ and $GL_{m'}(\mathbb{A}_F)$ respectively. Then, for all $n \in \mathbb{N}$*

$$\begin{aligned}\lambda_{\pi,\pi'}(-n) &= \frac{dmm'n}{2}\log n + nC_{\pi,\pi'} - \lambda_{\pi,\pi'}(-n,\sqrt{n}) \\ &+ O(\sqrt{n}\log n) + \sum_{l=1}^{dmm'}\left(\frac{\mu_{\pi\times\widetilde{\pi}'}(l)}{1+\mu_{\pi\times\widetilde{\pi}'}(l)}\right)^n,\end{aligned}$$

where $C_{\pi,\pi'} = \frac{1}{2}\left(\log Q_{\pi\times\pi'} + dmm'\left(\gamma - 1 - \log 2\pi\right)\right)$.

3.4. **Consequences of the Generalized Riemann hypothesis.** The simplification of the corollaries 3.9 and 3.13 can be made in the case when GRH holds true. The results are as follows.

Corollary 3.14. *Let $F \in \mathcal{S}^{\sharp\flat}$ be a function non-vanishing at zero. If the generalized Riemann hypothesis holds for $F(s)$ then*

$$\lambda_F(n,\sqrt{n}) = O(\sqrt{n}\log n)$$

and

$$\lambda_F(-n) = \frac{d_F}{2}n\log n + nC_F + O(\sqrt{n}\log n).$$

Corollary 3.15. *Let π and π' be two automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_F)$ and $GL_{m'}(\mathbb{A}_F)$ respectively. Assume the GRH for $L\left(s,\pi_f\times\widetilde{\pi}'_f\right)$. Then,*

$$\begin{aligned}\lambda_{\pi,\pi'}(-n) = \frac{dmm'}{2}n\log n + \sum_{l=1}^{dmm'}\left(\frac{\mu_{\pi\times\widetilde{\pi}'}(l)}{1+\mu_{\pi\times\widetilde{\pi}'}(l)}\right)^n \\ + \frac{n}{2}\left(\log Q_{\pi\times\widetilde{\pi}'} - dmm'(1+\log 2\pi - \gamma)\right) + O_{\pi,\pi'}\left(\sqrt{n}\log n\right).\end{aligned}$$

An interesting consequence of the GRH for $L\left(s,\pi_f\times\widetilde{\pi}'_f\right)$ is a bound towards the Ramanujan conjecture for the Archimedean Langlands parameters. This result is obtained by comparison of the expressions for the generalized Li coefficients under GRH obtained in two different ways in [14].

Theorem 3.16 ([14])**.** *Let π and π' be two automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_F)$ and $GL_{m'}(\mathbb{A}_F)$ respectively. Under the GRH for $L\left(s,\pi_f\times\widetilde{\pi}'_f\right)$ one has*

$$\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l) \geq -\frac{1}{2},\ \text{for all } l = 1,\ldots,dmm'.$$

Corollary 3.17 ([14]). *Let π be an automorphic unitary cuspidal representation of $GL_m(\mathbb{A}_F)$, unramified at the archimedean place $v \in S_\infty$. Then GRH for the function $L(s, \pi_f \times \widetilde{\pi}_f)$ implies the bound*

$$|\mathrm{Re}\mu_\pi(v,j)| \leq \frac{1}{4}$$

for all $j = 1, \ldots, m$.

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