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Integral Representations of the Logarithmic Derivative of the Selberg Zeta Function

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We point out the importance of the integral representations of the logarithmic derivative of the Selberg zeta function valid up to the critical line, i.e. in the region that includes the right half of the critical strip, where the Euler product definition of the Selberg zeta function does not hold. Most recent applications to the behavior of the Selberg zeta functions associated to a degenerating sequence of finite volume, hyperbolic manifolds of dimension 2 and 3 are surveyed. The research problem consists in extending this kind of integral representations to the setting of the locally symmetric spaces of rank 1.

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1. Introduction

The Selberg trace formula, introduced by A. Selberg in 1956, describes the spectrum of the hyperbolic Laplacian in terms of geometric data involving the lengths of geodesics on the Riemann surface. It is formally similar to the explicit formulas relating the zeros of the Riemann zeta function to prime numbers. The Selberg zeta zeros correspond to eigenvalues of the Laplacian, and the primes corresponding to primitive geodesics. Motivated by this analogy, Selberg [14] introduced the Selberg zeta function of a Riemann surface, whose analytic properties are encoded by the Selberg trace formula. The Selberg zeta function has properties similar to the Riemann zeta function. It is defined by the infinite product similar to the classical Euler product, but taken over closed geodesics rather than primes. It also satisfies a functional equation relating values at s with values at $1-s$. Selberg proved that the Selberg zeta functions satisfy the analogue of the Riemann hypothesis, i.e. all zeros of the Selberg zeta function in the case of the compact Riemann surface lie on the critical line $\mathrm{Re}(s) = 1/2$.

2. The Selberg zeta function

Let $H = \{z = x + iy : y > 0\}$ denote the upper half-plane equipped with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. Möbius transformations $z \to (az+b)/(cz+d)$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ form the group PSL(2,$\mathbb{R}$) that acts on H by homeomorphisms which preserve the hyperbolic distance. Discrete subgroups of PSL(2,$\mathbb{R}$) are called Fuchsian groups.

Let X be a non-compact, hyperbolic Riemann surface of a finite volume with $n_1 \geq 1$ cusps. Then, X can be identified with $\Gamma\backslash H$, where $\Gamma \subseteq$PSL$(2,\mathbb{R})$ is a Fuchsian group of the first kind containing n_1 inequivalent parabolic elements. Let $\Im$ denote the fundamental domain of X and $|\Im|$ its volume. We put

$$\bar{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\}$$

and denote by v the multiplier system of a weight $m \in \mathbb{R}$ on $\bar{\Gamma}$. For an r dimensional unitary representation ψ of Γ, the function W defined by $W(T) = \psi(t)v(t)$, $T \in \Gamma$ is a unitary $r \times r$ multiplier system of the weight m. This kind of a system has been introduced in order to develop L^2 spectral theory of operators

$$\Delta_m = y^2(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}) - imy\frac{\partial}{\partial x}$$

on X, since $-\Delta_m$ is essentially a self-adjoint operator on the space D_m of all twice continuously differentiable functions $f : H \to V$ (V is an r dimensional vector space over $\mathbb{C}$) such that f and $\Delta_m f$ are square integrable on $\Im$ and satisfy the equality

$$f(Sz) = \frac{(cz+d)^m}{|cz+d|^m} W(S) f(z)$$

for all $z \in H$ and $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}$. The operator$-\Delta_m$ has the unique self-adjoint extension $-\tilde{\Delta}_m$ to the space $\tilde{D}_m$. Let $\{T_j\}$, $j = 1, 2, ..., n_1$ denote all parabolic classes of the group Γ. By m_j we shall denote the multiplicity of 1 as an eigenvalue of the matrix $W(T_j)$, and $n_1^* = \sum_{j=1}^{n_1} m_j$ will be the degree of singularity of W. The operator $-\tilde{\Delta}_m$ has both a discrete and a continuous spectrum in the case $n_1^* \geq 1$ and only a discrete spectrum in the case $n_1^* = 0$. The discrete spectrum will be denoted as $\{\lambda_n\}_{n \geq 0}$ ($0 \leq \lambda_0 < \lambda_1 < ..., \lambda_n \to \infty$). One of the properties of the continuous spectrum is that, for a fixed $j \in \{1, 2, ..., n_1\}$ and $1 \leq h \leq r$ it is possible to choose the orthonormal vector columns f_{h_j} so that $W(T_j^{-1}) f_{h_j} = e^{2\pi i \alpha_{h_j}} f_{h_j}$, where $0 \leq \alpha_{h_j} < 1$ and $\alpha_{h_j} = 0$ iff $1 \leq h \leq m_j$ (see [9]).

The group Γ contains inequivalent hyperbolic, elliptic and parabolic classes. Following [9], we denote the set of inequivalent hyperbolic resp. elliptic classes by $\{P\}$ resp. $\{R\}$ whereas the set of inequivalent, primitive hyperbolic classes is denoted by $\{P_0\}$. The Selberg zeta function associated to the pair (Γ, W) is defined as the Euler product

$$Z_{\Gamma,W}(s) = \prod_{\{P_0\}_\Gamma} \prod_{k=0}^{\infty} \det(I_r - W(P_0)N(P_0)^{-s-k}) \tag{1}$$

converging absolutely for $\mathrm{Re}(s) > 1$, where $N(P)$ denotes the norm of the class $\{P\}$. All elements of an elliptic class are conjugate in $\mathrm{SL}(2,\mathbb{R})$ to a rotation for some angle $\theta \in (0,\pi)$. The order of the primitive element R_0 associated to R is denoted by $M_R/2$.

3. **The Selberg trace formula**

The most important tool in analysis of the spectrum of the operator $-\tilde{\Delta}_m$ is the Selberg trace formula [[9], p. 412, Th. 6.3.]. M. Avdispahić and L. Smajlović [5] have proved that the Selberg trace formula (written in a different form) holds true for a larger class of test functions. They have proved the following theorem.

Theorem 3.1. *Let* $n_1^* \geq 1$ and *suppose that* $h(r)$ *satisfies the conditions*

S1. $h(r) = h(-r)$;

S2. $h(r)$ *is analytic in the strip* $|\mathrm{Im}(r)| < \frac{1}{2} + \delta$, *for some* $\delta > 0$;

S3'. *For* $j = 0, 1, 2$, $h^{(j)}(r) = O\left(\left(1 + |r|^2\right)^{-1}\right)$ *in the strip* $|\mathrm{Im}(r)| < \frac{1}{2} + \delta$, with $0 < \delta < 1/4$. *Then, the formula*

$$\sum_{n=0}^{K} h(r_n) + \int_0^{\infty} h(t) dR(t) - \frac{|\mathfrak{F}|}{2\pi} \int_0^{\infty} t h(t) r(t)\, dt + \frac{n_1^*}{2\pi} \int_0^{\infty} h(t) H(t)\, dt =$$

$$\frac{2}{\sqrt{2\pi}} \sum_{\substack{\{P_0\}_\Gamma \\ Tr P_0 > 2}} \sum_{m=1}^{\infty} \frac{Tr\left(W(P_0)^m\right) \log N(P_0)}{N(P_0)^{m/2} - N(P_0)^{-m/2}} \widehat{h}\left(m \log N(P_0)\right) + \tag{2}$$

$$\sum_{\substack{\{R\}_\Gamma \\ 0<\theta(R)<\pi}} \frac{Tr\left(W(R)\right) i e^{i(m-1)\theta}}{2\sqrt{2\pi} M_R \sin\theta} \int_{-\infty}^{\infty} \frac{e^u - e^{2i\theta}}{\cosh u - \cos 2\theta} \widehat{h}(u)\, e^{\frac{m-1}{2}u} du +$$

$$\frac{1}{\sqrt{2\pi}}\sum_{\alpha_{h_j}\neq 0}\left(\frac{1}{2}-\alpha_{h_j}\right)\int\limits_{-\infty}^{\infty}\frac{e^u-1}{\cosh u-1}\widehat{h}\left(u\right)e^{\frac{m-1}{2}u}du+$$

$$\frac{1}{2}h\left(0\right)Tr\left(I_r-\Phi\left(\frac{1}{2}\right)\right)+\frac{2n_1^*}{\sqrt{2\pi}}\int\limits_0^{\infty}\frac{2\widehat{h}\left(u\right)}{\sinh\frac{u}{2}}\left(1-\cosh\frac{mu}{2}\right)du$$

holds true

Here, we put

$$r\left(t\right)=\frac{\sinh 2\pi t}{\cosh 2\pi t+\cos\pi m}-1,$$

$$H(t)=\frac{\Gamma'}{\Gamma}\left(1+it\right)+\frac{\Gamma'}{\Gamma}\left(1-it\right)-2\log t,$$

$$R\left(t\right)=N\left[0\leq r_n\leq t\right]-\frac{1}{4\pi}\int\limits_{-t}^{t}\frac{\phi'}{\phi}\left(\frac{1}{2}+iu\right)du-r\frac{|\mathfrak{F}|}{4\pi}t^2+\frac{n_1^*}{\pi}t\log t$$

$$-\frac{t}{\pi}\left(n_1^*-n_1^*\log 2-\sum_{\alpha_{h_j}\neq 0}\log\left|1-e^{2\pi i\alpha_{h_j}}\right|\right),$$

where $N\left[0\leq r_n\leq t\right]$ counts the non-negative solutions of the equations $\frac{1}{4}+r_n^2=\lambda_n$. For $\lambda_n<\frac{1}{4}$, $n=0,...,K$, we put $r_n=-i\sqrt{\frac{1}{4}-\lambda_n}$, Φ denotes the hyperbolic scattering matrix and $\phi=\det\Phi$. (More about the hyperbolic scattering determinant can be found in [3]. Finally,

$$\hat{h}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h|_{\mathbb{R}}(x)\cdot e^{-itx}dx$$

denotes the Fourier transform of the function h considered as a function on $\mathbb{R}$.

4. **Integral expressions for the logarithmic derivative of the Selberg zeta function**

We shall denote by

$$D_{\Gamma,W}(s)=\frac{Z'_{\Gamma,W}(s)}{Z_{\Gamma,W}(s)}$$

the logarithmic derivative of the Selberg zeta function $Z_{\Gamma,W}(s)$. From (1) it is easily seen that, for $\mathrm{Re}(s)>1$ one has

$$D_{\Gamma,W}(s)=\sum_{\{P\},TrP>2}\frac{\Lambda(P)}{N(P)^s}Tr(W(P)),$$

where $\Lambda(P) = \dfrac{\log N(P_0)}{1 - N(P)^{-1}}$, for a primitive element P_0 such that $P = P_0^n$ for some $n \in \mathbb{N}$. Another representation of the logarithmic derivative $D_{\Gamma,W}(s)$ (with $m = 0$, hence $W = \psi$) given by absolutely and uniformly convergent series in any half plane $\mathrm{Re}(s) > 1 + \varepsilon$ ($\varepsilon > 0$) was obtained by Wakayama in [15].

In many applications it is important to have a representation of the logarithmic derivative $D_{\Gamma,W}(s)$ inside the critical strip, i.e. in the half-plane Re $(s) > 1/2$. M. Avdispahić and L. Smajlović have used Theorem 3.1. in order to obtain new integral representations of the logarithmic derivative of the Selberg zeta function valid up to the critical line. Namely, the functions

$$h_s(r) = \frac{1}{(s-\frac{1}{2})^2 + r^2} \quad \text{and} \quad h_{\alpha,y}(t) = \frac{2\alpha \exp y\alpha}{\alpha^2 + t^2} \cos yt$$

where $\mathrm{Re}(s) > \frac{1}{2}$ and $y > 0$ satisfy assumptions of Theorem 3.1. and do not satisfy assumptions of the classical trace formula [[9], p. 412, Th. 6.3.]. Inserting those functions into the display (2) they have proved the following theorem.

Theorem 4.1. [5] a) *For* $\mathrm{Re}(s) > 1/2$

$$\frac{D_{\Gamma,W}(s)}{2s-1} = \sum_{n=0}^{K} \frac{1}{(s-1/2)^2 + r_n^2} + \int_0^\infty \frac{dR(t)}{\left(s-\frac{1}{2}\right)^2 + t^2} - \frac{|\mathfrak{F}|}{2\pi} \int_0^\infty \frac{t \cdot r(t)\, dt}{\left(s-\frac{1}{2}\right)^2 + t^2} +$$

$$\frac{n_1^*}{2\pi} \int_0^\infty \frac{H(t)\, dt}{\left(s-\frac{1}{2}\right)^2 + t^2} - \frac{1}{2} \sum_{\alpha_{h_j} \neq 0} \left(\frac{1}{2} - \alpha_{h_j}\right) \int_0^\infty \frac{\sin \pi m \cdot dt}{\left(\left(s-\frac{1}{2}\right)^2 + t^2\right)(\cosh 2\pi t + \cos \pi m)}$$

$$(3) \quad - \sum_{\substack{\{R\}_\Gamma \\ 0<\theta(R)<\pi}} \frac{Tr(W(R))}{2M_R \sin\theta} \int_0^\infty \frac{1}{\left(s-\frac{1}{2}\right)^2 + t^2} \frac{\cosh 2(\pi-\theta)t + e^{i\pi m} \cosh 2\theta t}{\cosh 2\pi t + \cos \pi m} dt -$$

$$\frac{n_1^*}{2(2s-1)} \left(\frac{\Gamma'}{\Gamma}\left(s + \frac{m}{2}\right) + \frac{\Gamma'}{\Gamma}\left(s - \frac{m}{2}\right) - 2\frac{\Gamma'}{\Gamma}(s)\right) - \frac{1}{(2s-1)^2} Tr\left(I - \Phi\left(\frac{1}{2}\right)\right)$$

b) *For* $\mathrm{Re}\alpha > 0$, $y > 0$ *and* $x = e^y$ *one has*

$$(4) \quad D_{\Gamma,W}\left(\frac{1}{2} + \alpha\right) = \frac{1}{1+x^{2\alpha}} \sum_{N(P)<x} \frac{TrW(P)\Lambda(P)}{N(P)^{\alpha+\frac{1}{2}}} \left(x^{2\alpha} - N(P)^{2\alpha}\right)$$

$$+ \frac{4\alpha x^\alpha}{1+x^{2\alpha}} \left(\sum_{n=0}^{K} \frac{\cos yr_n}{\alpha^2 + r_n^2} + \int_0^\infty \frac{\cos yt dR(t)}{\alpha^2 + t^2} - \frac{|\mathfrak{F}|}{2\pi} \int_0^\infty \frac{t \cdot r(t) \cos yt}{\alpha^2 + t^2} dt\right.$$

$$+\frac{n_1^*}{2\pi}\int_0^\infty \frac{\cos ytH(t)\,dt}{\alpha^2+t^2}-\sum_{\alpha_{h_j}\neq 0}\left(\frac{1}{2}-\alpha_{h_j}\right)\int_0^\infty \frac{\cos yt\sin\pi mdt}{(\alpha^2+t^2)(\cosh 2\pi t+\cos\pi m)}dt$$

$$\left.-\sum_{\substack{\{R\}_\Gamma\\ 0<\theta(R)<\pi}}\frac{Tr\,(W\,(R))}{2M_R\sin\theta}\int_0^\infty \frac{\cos yt}{\alpha^2+t^2}\frac{\cosh 2\,(\pi-\theta)\,t+e^{i\pi m}\cosh 2\theta t}{\cosh 2\pi t+\cos\pi m}dt\right)$$

$$+n_1^*\int_0^\infty \frac{e^{-\alpha t}\left(\cosh\frac{mt}{2}-1\right)}{2\sinh\frac{t}{2}}dt+\frac{n_1^*}{1+x^{2\alpha}}\int_0^y \frac{e^{\alpha t}-e^{(2y-t)\alpha}}{2\sinh\frac{t}{2}}\left(\cosh\frac{mt}{2}-1\right)dt.$$

Theorem 4.1. may also be derived through interpretation of the Selberg trace formula as an explicit formula in the Jorgenson-Lang fundamental class of functions, as in [2].

5. **Some applications**

In order to illustrate the importance of the representation of the logarithmic derivative of the Selberg zeta function that is valid in the right half of the critical strip, we shall state here the main result of the paper [1] and indicate how the representation (3) of the logarithmic derivative of the Selberg zeta function and its three dimensional analogue are used in the proof of this result.

The authors have considered a sequence $\{M_k\}$ of low dimensional, finite volume hyperbolic manifolds whose limiting manifold is M_∞. By $Z_{M_k}(s)$ we shall denote the Selberg zeta function associated to the manifold M_k, while the $\mathbf{Z}_k(s)$ denotes the product of local factors in the Euler product expansion of $Z_{M_k}(s)$ corresponding to pinching geodesics of M_k. The problem of the behavior of the function $Z_{M_k}(s)/\mathbf{Z}_k(s)$ through degeneration (i.e. as $k\to\infty$) was firstly posed by Wolpert in [16]. In the case of dimension $d=2$, the convergence of $Z_{M_k}(s)/\mathbf{Z}_k(s)$ to $Z_{M_\infty}(s)$ in the half plane $\mathrm{Re}(s)>1/2$ was proved by M. Schulze [13] by means of analysis of certain operators. In [1], the unified approach in both dimensions $d=2$ or $d=3$ is provided and the following theorem is proved

Theorem 5.1. *Let $\{M_k\}$ be a degenerating sequence of low dimensional, finite volume hyperbolic manifolds whose limiting manifold is M_∞. Let $\{Z_{M_k}(s)\}$ be the associated sequence of Selberg zeta functions, normalized to have convergent Euler product in the half-plane Re $s>(d-1)$ and functional equations which* *relate values at s to values at $(d-1)-s$. Then, for all Re $s>(d-1)/2$ we have*

$$Z_{M_k}(s)/\mathbf{Z}_k(s)=Z_{M_\infty}(s)+o(1), \quad as \quad k\to\infty.$$

The statement of the above theorem was proved in [12] and [6] (in dimensions $d=2$ and $d=3$ respectively) but in the region that does not include the right half of the critical strip. (Actually, the right half of the critical strip is, due to the functional equation, the maximal region where the statement of Theorem 5.1. could hold true).

It was Theorem 4.1. a) that enabled the authors of [1], to use investigations conducted in [12], [10] and [6] about the behavior of heat kernels through degeneration, and prove that the convergence through degeneration of the Selberg zeta functions holds in the half-plane $\mathrm{Re}(s)>(d-1)/2$.

The representation (4) is itself of a number-theoretical interest, since it allows one to consider a limiting process as $y\to\infty$ on the right-hand side of (4), the left hand side being independent of y. This observation was used in [4] in order to evaluate analogues of Euler-Stieltjes constants for the Selberg zeta functions and determine the upper and lower bounds for some special values of $D_{\Gamma,Id}$ that improve the bounds obtained in [11].

6. **A possible generalization**

Let G be a connected semisimple Lie group with finite center, K its maximal compact subgroup. By H we shall denote the symmetric space G/K, assumed to be of a rank one. Let Γ be a discrete, torsion-free subgroup of G such that $\Gamma\backslash G$ is compact, T finite dimensional unitary representation of Γ and χ its character. By the work of R. Gangolli [7], it is possible to define zeta function $Z_\Gamma(s,\chi)$ attached to the data (G,K,Γ,χ), as an Euler product similar to (1). Actually, Gangolli has first proved the trace formula in this setting and derived the representation of logarithmic derivative of the function $Z_\Gamma(s,\chi)$ and the functional equation relating values of $Z_\Gamma(s,\chi)$ at s with the values at $2\rho_0-s$, for a constant $\rho_0>0$, whose dependance on the data (G,K,Γ,χ) is fully explained in [7]. This yields to a natural notion of the critical strip $0\leq\mathrm{Re}(s)\leq2\rho_0$ and the critical line $\mathrm{Re}(s)=\rho_0$. A similar zeta function can also be defined in the case of a non-compact space $\Gamma\backslash G$.

The main goal of our future research is to use the trace formula proved in [7] and [8] and try to obtain the analogue of Theorem 3.1. in the setting of rank one symmetric spaces. A further goal is to obtain the representation of the logarithmic derivative of the function $Z_\Gamma(s,\chi)$ (analogous to Theorem 4.1.)

valid in the half plane $\mathrm{Re}(s) > \rho_0$ (i.e. up to the critical line) in both compact and non-compact setting.

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