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Canonical Objects in Classes of $(n, \mathcal{V})$-Groupoids

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Free algebras are very important in studying classes of algebras, especially varieties of algebras. Any algebra that belongs to a given variety of algebras can be characterized as a homomorphic image of a free algebra of that variety. Describing free algebras is an important task that can be quite complicated, since there is no general method to resolve this problem. The aim of this work is to investigate classes of groupoids, i.e. algebras with one binary operation, that satisfy certain identities or other conditions, and look for free objects in such classes.

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1. **Introduction**

This paper is a review of a part of my doctoral thesis "*Free and injective objects in some classes of n-groupoids*". The thesis was prepared during the last three years at the Institute of Mathematics in the Faculty of Natural Sciences and Mathematics, "Ss. Cyril and Methodius" University, Skopje, Macedonia, and some of its parts were supported by the Macedonian Academy of Sciences and Arts through the project "Algebraic Structures".

We introduce the basic idea of this work. For the notation and basic notions of universal algebra the reader is referred to [12] and [13].

Let X be an arbitrary nonempty set whose elements are called variables and $\mathbf{T}_X = (T, \cdot)$ be the set of all groupoid terms over X in signature $\cdot$. The terms are denoted by $t, u, v, w \dots$. Note that $\mathbf{T}_X$ is an absolutely free groupoid over X, where the operation is defined by $(t, u) \mapsto tu$. The groupoid $\mathbf{T}_X$ is injective, i.e. if $x, y, v, w \in T$, then $xy = vw \Rightarrow x = v, y = w$. The set X is the set of primes in $\mathbf{T}_X$ and generates $\mathbf{T}_X$. (An element a of a groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be *prime* in $\boldsymbol{G}$ if and only if $a \neq xy$, for all $x, y \in G$.) These two properties of $\mathbf{T}_X$ characterize all absolutely free groupoids ([1]; Lemma 1.5): A

groupoid $\boldsymbol{H} = (H, \cdot)$ *is an absolutely free groupoid if and only if it satisfies the following two conditions:* $\boldsymbol{H}$ *is injective and the set of primes in* $\boldsymbol{H}$ *is nonempty and generates* $\boldsymbol{H}$. We refer to this proposition as *Bruck Theorem* for the class of all groupoids.

Let $\mathcal{V}$ be a variety of groupoids, i.e. a class of groupoids defined by a certain set of identities (or, equivalently, a class of groupoids that is hereditary and closed under homomorphic images and direct products). For a given variety $\mathcal{V}$ of groupoids, a free groupoid of a special form, called canonical form, is constructed. Namely, if X is a non-empty set and $\mathbf{T}_X$ is the term groupoid over X, then a $\mathcal{V}$-*canonical groupoid* $\boldsymbol{R} = (R, *)$ *over* X is a groupoid that satisfies the following conditions:

(c_0) $X \subseteq R \subseteq T$ and $t \in R \Rightarrow P(t) \subseteq R$, where $P(t)$ is the set of subterms of the term t defined by:
$t \in X \Rightarrow P(t) = \{t\}$ and $t = t_1t_2 \Rightarrow P(t_1t_2) = \{t_1t_2\} \cup P(t_1) \cup P(t_2)$;

(c_1) $tu \in R \Rightarrow t * u = tu$ and

(c_2) $\boldsymbol{R}$ is a $\mathcal{V}$-free groupoid over X.

Using suitable properties of the obtained $\mathcal{V}$-canonical groupoid, we introduce the notion of $\mathcal{V}$-injective groupoid that is defined separately for each particular variety $\mathcal{V}$ [6]. Then the class of $\mathcal{V}$-free groupoids can be characterized by the class of $\mathcal{V}$-injective groupoids in the following way: *A groupoid* $\boldsymbol{H} = (H, \cdot)$ *is a* $\mathcal{V}$*-free groupoid if and only if* $\boldsymbol{H}$ *is* $\mathcal{V}$*-injective and the set of* $\mathcal{V}$*-prime elements in* $\boldsymbol{H}$ *is non-empty and generates* $\boldsymbol{H}$. (An element $a \in G$ is said to be $\mathcal{V}$-*prime* if and only if any equation of the form $a = bc$ is a consequence of the axioms in $\mathcal{V}$.) We call this property "*Bruck Theorem for the variety* $\mathcal{V}$".

Such characterizations are given for some classes of $(n, \mathcal{V})$-groupoids.

2. $(n, \mathcal{V})$-Groupoids

Let $\mathcal{V}$ be a variety of groupoids. A groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be $(n, \mathcal{V})$-*groupoid* if and only if any subgroupoid generated by n elements of $\boldsymbol{G}$ belongs to the variety $\mathcal{V}$. The class of $(n, \mathcal{V})$-groupoids is denoted by $(n, \mathcal{V})$. If $n = 1$, then $(1, \mathcal{V})$-groupoids are called *power* $\mathcal{V}$*-groupoids.* In that case, the variety $\mathcal{V}$ is a subclass of the class $(1, \mathcal{V})$, and more generally $\mathcal{V}$ is a subclass of the class $(n, \mathcal{V})$. For any positive integers n, k, the class $(n + k, \mathcal{V})$ is a subclass of the class $(n, \mathcal{V})$. We give a description of canonical objects in the classes of power-commutative groupoids, power left and right idempotent groupoids, power-slim groupoids and biassociative groupoids. Also, a characterization by injective objects for some of this classes is given.

Throughout the paper we will use the concept of groupoid power and some of its properties stated in [4]. By $\mathbf{E} = (E, \cdot)$ we will denote the term groupoid over the set $\{e\}$. The elements of E are called *groupoid powers* and will be denoted by $f, g, h, \ldots$. For any groupoid $\boldsymbol{G} = (G, \cdot)$, each element $f \in E$ induces a transformation $f^{\boldsymbol{G}} : G \to G$, called an *interpretation* of f in $\boldsymbol{G}$, defined by:

$$e^{\boldsymbol{G}}(x) = x, \;\; (gh)^{\boldsymbol{G}}(x) = g^{\boldsymbol{G}}(x)h^{\boldsymbol{G}}(x)$$

for any $g, h \in E$ and $x \in G$. We will write $f(x)$ instead of $f^{\boldsymbol{G}}(x)$ when $\boldsymbol{G}$ is understood.

In the sequel we will present without proofs some of the main results of this part of the thesis.

The class of commutative groupoids, i.e. groupoids that satisfy the identity $xy \approx yx$, is a variety of groupoids, here denoted by $\mathcal{C}om$. We investigate a class of groupoids larger than $\mathcal{C}om$, called ***the class of power-commutative groupoids***. It will be denoted by $\mathcal{P}_c$.

If $\boldsymbol{G}$ is a groupoid, then any subgroupoid of $\boldsymbol{G}$ generated by an element $a \in G$ (denoted by $\langle\, a \,\rangle$) is called *cyclic subgroupoid* of $\boldsymbol{G}$ with a generator a [2]. The cyclic subgroupoids are characterized in [5]: if $a \in G$, then $\langle\, a \,\rangle = \{f(a) : f \in E\}$. A groupoid $\boldsymbol{G}$ is said to be *power-commutative* if and only if every cyclic subgroupoid of $\boldsymbol{G}$ is commutative. Clearly, every commutative groupoid is power-commutative. The set of all 2×2 matrices under the multiplication is a nontrivial example of a power-commutative groupoid. Moreover, all semigroups are power-commutative groupoids. Directly from the definition we obtain that $\boldsymbol{G} \in \mathcal{P}_c$ if and only if $\boldsymbol{G}$ is a union of commutative cyclic subgroupoids of $\boldsymbol{G}$. This result enables to obtain an axiom system for $\mathcal{P}_c$, i.e. the class of power-commutative groupoids $\mathcal{P}_c$ is a variety of groupoids defined by the system of identities $\{f(x)g(x) \approx g(x)f(x) : f, g \in E\}$.

In order to give a description of free objects in the variety $\mathcal{P}_c$, we will introduce an ordering of terms. Namely, let X be a linearly ordered set and let that relation be denoted by $\leq$. An extension of the relation $\leq$ from X to T is defined as follows.

Let $t, u \in T$. (0) If $t, u \in X$, then $t \leq u$ in X implies that $t \leq u$ in T; (1) If $|\,t\,| < |\,u\,|$, then $t < u$, where $|\,t\,|$ is the length of the term t defined by $|\,t\,| = 1$, if $t \in X$, $|\,uv\,| = |\,u\,| + |\,v\,|$, if $t = uv$; (2) If $|\,t\,| = |\,u\,| \geq 2$ and $t \neq u$, where $t = t_1t_2$, $u = u_1u_2$, then $t < u \;\Leftrightarrow [\, t_1 < u_1 \;\vee\; (t_1 = u_1 \;\;\wedge\;\; t_2 < u_2)\,]$. *The relation $\leq$ is a linear ordering in T.*

A term $t \in T$ is said to be *order-regular* if and only if

$$t \in X \;\vee\; (t = t_1t_2 \in T \setminus X \;\;\wedge\;\; t_1 \leq t_2).$$

Specially, a groupoid power $f \in E$ is order-regular if and only if

$$f = e \quad \vee \quad (f = f_1 f_2 \quad \wedge \quad f_1 \leq f_2).$$

We will use canonical commutative groupoids constructed as follows. Define a subset T_c of T by

$$T_c = \{t \in T : \text{every subterm of } t \text{ is order-regular}\}$$

and an operation $\odot$ on T_c by

$$t, u \in T_c \;\Rightarrow\; t \odot u = \begin{cases} tu, & \text{if } t \leq u \\ ut, & \text{if } u < t. \end{cases} \tag{2.1}$$

Then $\mathbf{T}_c = (T_c, \odot)$ is a canonical commutative groupoid over X.

Specially, $\mathbf{E}_c = (E_c, \odot)$ is a canonical commutative groupoid over $\{e\}$, where $E_c = \{f \in E$: every subterm of f is order-regular$\}$ and $\odot$ is defined by (2.1).

A term t is said to be *primitive* in $\mathbf{T}_X$ if and only if $t \neq f(u)$ for any $u \in T$ and any $f \in E \setminus \{e\}$; and t is said to be *potent* (or *nonprimitive*) in $\mathbf{T}_X$ if and only if $t = f(u)$ for some $u \in T$ and $f \in E \setminus \{e\}$. The following proposition is true ([3]): *For any potent term t there is a unique primitive term u and a unique groupoid power $f \in E \setminus \{e\}$ such that $t = f(u)$*. In that case we say that: *u is the base of t, f is the power of t* and denote $\underline{t}$, $t^{\sim}$, respectively.

Define the carrier of a free groupoid in $\mathcal{P}_c$ by

$$R = \{t \in T : u \in P(t) \;\Rightarrow\; u^{\sim} \in E_c\}, \tag{2.2}$$

and an operation $*$ on R by

$$t, u \in R \;\Rightarrow\; t * u = \begin{cases} tu, & \text{if } tu \in R \\ ut, & \text{if } \underline{t} = \underline{u} \text{ and } u^{\sim} < t^{\sim}. \end{cases} \tag{2.3}$$

One can obtain that $\boldsymbol{R} = (R, *)$ *defined by* (2.2) *and* (2.3) *is a free power-commutative groupoid over X in canonical form.* We will use the properties of the canonical groupoid $\boldsymbol{R} = (R, *)$ in $\mathcal{P}_c$ related to the elements of $\boldsymbol{R}$ that are not prime, to define a subclass of the class $\mathcal{P}_c$ that is larger than the class of $\mathcal{P}_c$-free groupoids, called the class of $\mathcal{P}_c$-injective groupoids. The class of $\mathcal{P}_c$-injective groupoids will be successfully defined if the following two conditions are satisfied. Firstly, the class of $\mathcal{P}_c$-injective groupoids should unable the characterization of $\mathcal{P}_c$-free groupoids: any $\mathcal{P}_c$-injective groupoid $\boldsymbol{H}$ whose set of primes is nonempty and generates $\boldsymbol{H}$, to be $\mathcal{P}_c$-free. Secondly, the class of $\mathcal{P}_c$-free groupoids has to be a proper subclass of the class of $\mathcal{P}_c$-injective groupoids. This is done for the class of power-commutative groupoids, i.e. the *Bruck Theorem for $\mathcal{P}_c$ holds* and *the class of $\mathcal{P}_c$-free groupoids is a proper subclass of the class of $\mathcal{P}_c$-injective groupoids.*

In [7] a variety $\mathcal{U}$ of left and right idempotent groupoids, i.e. $\mathcal{U} = Var(x^2y^2 \approx xy)$, is investigated. We investigate a larger class, called ***the class of power left and right idempotent groupoids***, that will be denoted by $\mathcal{P}_\mathcal{U}$. A groupoid $\boldsymbol{G} = (G, \cdot)$ is *power left and right idempotent* if and only if every cyclic subgroupoid of $\boldsymbol{G}$ is left and right idempotent, i.e. belongs to $\mathcal{U}$. The elements of any groupoid in the class $\mathcal{P}_\mathcal{U}$ have almost trivial powers, i.e. *if $f(x)$ is a power of x, then either $f(x) = x$ or $f(x) = x^2$, for any nontrivial groupoid power f*. As a consequence we obtain that *the class $\mathcal{P}_\mathcal{U}$ is a variety of groupoids defined by the identities $x^2 \approx x^2x \approx xx^2 \approx x^2x^2$*. For details the reader is referred to [3].

Define the carrier of the desired $\mathcal{P}_\mathcal{U}$-canonical groupoid $\boldsymbol{R}$ by

$$R = \{t \in T : (\forall u \in P(t))\, |\, u^{\sim}\, | \leq 2\}, \tag{2.4}$$

and an opperation $*$ on R by

$$t, u \in R \Rightarrow t * u = \begin{cases} tu, & \text{if } tu \in R \\ v^2, & \text{if } \underline{t} = \underline{u} = v,\ |\, t^{\sim}\, | + |\, u^{\sim}\, | \geq 3. \end{cases} \tag{2.5}$$

One can show that *the groupoid $\boldsymbol{R} = (R, *)$ defined by* (2.4) *and* (2.5) *is a free power left and right idempotent groupoid over X in canonical form.*

We use the properties of the obtained $P_\mathcal{U}$-canonical groupoid $(R, *)$ that are related to the elements in $(R, *)$ that are not prime. Namely, *if $t \in R$, then $t * t$ is an idempotent element in $\boldsymbol{R}$; t is idempotent in $\boldsymbol{R}$ if and only if t is a square in $\boldsymbol{R}$ and if t is idempotent in $\boldsymbol{R}$, then there is a unique nonidempotent $u \in R$, i.e. $u \neq u * u$, such that $t = u * u$.* Also, *for every $t \in R \setminus X$ there is a unique pair $(u, v) \in R \times R$ such that $t = uv = u * v$.* We say that (u, v) is the *pair of divisors* of t in $\boldsymbol{R}$. If $u = v$, then u is a *divisor* of t.

Define a $P_\mathcal{U}$-injective groupoid in the following way. A groupoid $\boldsymbol{H} = (H, \cdot)$ is said to be *$\mathcal{P}_\mathcal{U}$-injective* if and only if the following conditions are satisfied:

(0) $\boldsymbol{H} \in \mathcal{P}_\mathcal{U}$

(1) If $a \in H$ is idempotent, then there is a unique nonidempotent $c \in H$, such that $a = c^2$ and the equality $a = xy$ holds if and only if $\{x, y\} \subseteq \{c, c^2\}$. (In that case c is the *divisor* of a or c is the *base* of a.)

(2) If $a \in H$ is nonidempotent and nonprime in $\boldsymbol{H}$, then there is a unique pair $(c, d) \in H \times H$, such that $a = cd$ and $\underline{c} \neq \underline{d}$.

(Note thet c, d can be both idempotents; one idempotent and the other nonidempotent; both nonidempotents.)

It is proved in [3] that the *Bruck Theorem for* $\mathcal{P}_{\mathcal{U}}$ *holds*, that *neither of the classes* $\mathcal{P}_{\mathcal{U}}$*-free and* $\mathcal{P}_{\mathcal{U}}$*-injective groupoids is hereditary* and that *the class of* $\mathcal{P}_{\mathcal{U}}$*-free groupoids is a proper subclass of the class of* $\mathcal{P}_{\mathcal{U}}$*-injective groupoids.*

An Evans' result ([8]) is used to show that the word problem is solvable for the variety $\mathcal{P}_{\mathcal{U}}$. Note that if a partial groupoid A is strongly embeddable into a power left and right idempotent groupoid, then it satisfies the following condition:

(j_0) if $a \in A$ is such that a^2 is defined, then a^2a, aa^2 and a^2a^2 are also defined and $a^2a = aa^2 = a^2a^2 = a^2$.

For a partial groupoid A satisfying (j_0) we define a groupoid $(G, \circ)$ as follows:

(j_1) if xy is defined in A, then $x \circ y = xy$

(j_2) if x^2 is not defined in A, then $x \circ x = x$

(j_3) if xy is not defined in A and $x \neq y$, then $x \circ y = c$, where c is a fixed element in A.

It is shown that *if A is a partial groupoid satisfying* (j_0)*, then* $(G, \circ)$ *defined above by* (j_1) – (j_3) *is a power left and right idempotent groupoid.* As a special case of the Evans' Theorem we obtain the following theorem: *if every partial* $\mathcal{P}_{\mathcal{U}}$*-groupoid is embeddable into a* $\mathcal{P}_{\mathcal{U}}$*-groupoid, then the word problem is solvable for the variety* $\mathcal{P}_{\mathcal{U}}$. As a corollary, we have that the *word problem for the variety* $\mathcal{P}_{\mathcal{U}}$ *is solvable.*

The variety of groupoids that satisfy the identity $x(yz) \approx xz$ is called the variety of *slim groupoids*. We investigate ***the class of power-slim groupoids***, i.e. the class of groupoids such that every cyclic subgroupoid satisfies the identity $x(yz) \approx xz$. Our purpose is to construct free objects in that class. First we will give a description of the free slim groupoids (slightly different then the given description in [11]). The variety of slim groupoids will be denoted by $\mathcal{V}_s$. Define a subset F_s of T by

$$F_s = \{t \in T : \ (\forall u, v, w \in T)\ u(vw) \notin P(t)\} \tag{2.6}$$

and an operation $*$ on F_s by

$$t, u \in F_s \Rightarrow t * u = \begin{cases} tu, & \text{if } u \in X \\ tu_2, & \text{if } u = u_1u_2 \ \wedge \ u_2 \in X. \end{cases} \tag{2.7}$$

The groupoid $\boldsymbol{F}_s = (F_s, *)$ *defined by* (2.6) *and* (2.7) *is a canonical slim groupoid over* X. Specially, if $X = \{e\}$, then the canonical slim groupid over $\{e\}$ is denoted by $\mathbf{E}_s = (E_s, *)$, where $E_s = \{f \in E : (\forall g, h, j \in E)\ g(hj) \notin P(f)\}$, i.e. $E_s = \{e^n : \ n \geq 1\}$, and $f, g \in E_s \ \Rightarrow \ f * g = fe$.

A groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be a *power-slim groupoid* if and only if every cyclic subgroupoid of $\boldsymbol{G}$ is a slim groupoid. The class of such groupoids will be denoted by $\mathcal{P}_s$. Using the characterization of cyclic groupoids, we obtain that *$\mathcal{P}_s$ is a variety of groupoids defined by the set of identities* $\{f(x)\,(\,g(x)h(x)\,) \approx f(x)h(x) : f, g, h \in E\}$.

Define a subset R of T by

$$R = \{t \in T : \ (\forall u \in P(t))\ u^{\sim} \in E_s\} \tag{2.8}$$

and an operation $*$ on R by

$$t, u \in R \Rightarrow \ t * u = \begin{cases} tu, \text{ if } \ (tu)^{\sim} \in E_s \\ t\,\underline{u}, \text{ if } \ \underline{t} = \underline{u} \ \wedge \ |\,u^{\sim}\,| \geq 2. \end{cases} \tag{2.9}$$

One can show that *the groupoid* $\boldsymbol{R} = (R, *)$ *defined by* (2.8) *and* (2.9) *is a canonical power-slim groupoid over* X. *The groupoid* $\boldsymbol{R} = (R, *)$ *is right cancellative and it is not left cancellative.*

In the paper [9] ***the variety of biassociative groupoids***, denoted by $\mathcal{B}ass$ is considered. A groupoid $\boldsymbol{G}$ is said to be *biassociative* if and only if every subgroupoid generated by at most two elements of $\boldsymbol{G}$ is a subsemigroup. Free objects are constructed using a chain of partial biassociative groupoids that satisfy certain properties. The obtained free objects are not canonical. In [10] the obtained free objects have canonical form.

Let $\boldsymbol{G} = (G, \cdot)$ be a groupoid and $a, b \in G$. We denote by $\langle\, a, b \,\rangle$ the subgroupoid of $\boldsymbol{G}$ generated by a, b and by $\langle\, a \,\rangle$ the subgroupoid generated by a. Clearly, $\langle\, a \,\rangle \subseteq \langle\, a, b \,\rangle$ and if $b \in \langle\, a \,\rangle$, then $\langle\, a, b \,\rangle = \langle\, a \,\rangle$; specially, $\langle\, a, a \,\rangle = \langle\, a \,\rangle$. The subgroupoids $\langle\, a, b \,\rangle$ and $\langle\, b, a \,\rangle$ are equal.

Let $a_1, a_2, \ldots, a_n$ be a finite sequence of elements in a groupoid $\boldsymbol{G}$. We denote by $a_1a_2 \ldots a_n$ the product of the sequence $a_1, a_2, \ldots, a_n$ in $\boldsymbol{G}$ defined as follows:

i) if $n = 3$, then $a_1a_2a_3 \stackrel{\text{df}}{=} a_1(a_2a_3)$ and

ii) if $n \geq 3$, then $a_1a_2 \ldots a_n \stackrel{\text{df}}{=} a_1(a_2 \ldots a_n)$.

We call $a_1a_2 \ldots a_n$ the *main product* of the sequence $a_1, a_2, \ldots, a_n$. If $n = 1$ and $n = 2$, then a_1 and a_1a_2 will also be called the main products of the sequences a_1 and a_1, a_2 respectively.

Let $t, u \in T$ and $\langle\, t, u \,\rangle$ be the subgroupoid of $\mathbf{T}_X$ generated by t, u. Each element x of $\langle\, t, u \,\rangle$ is a product of a finite sequence of elements $x_1, \ldots, x_n$ $(n \geq 1)$, where each x_i is either t or u, i.e. $\{x_1, x_2, \ldots, x_n\} \subseteq \{t, u\}$. Any such product is constructed by the two generators t, u and therefore we call it a *binary product* or shortly *biproduct*. Thus, if a term $x \in T$ is an element of $\langle\, t, u \,\rangle$, then we say that x *has a representation as a biproduct* (or shortly, *x is a biproduct*) *with the generating pair* $\{t,u\}$ and denote it by $x_{\langle t,u \rangle}$. (In this case we also say that x is the *carrier* of the biproduct $x_{\langle t,u \rangle}$.) *If* $t,u,x \in T$, *where* $x \in \langle t,u \rangle$, $t \notin \langle u \rangle$ *and* $u \notin \langle t \rangle$, *then* x *has a unique representation as a biproduct with the generating pair* $\{t,u\}$.

A biproduct $x_{\langle t,u \rangle}$ of a term x is said to be *maximal* in $\mathbf{T}_X$ if and only if for any biproduct $x_{\langle \alpha,\beta \rangle}$ of x, the hierarchy $\chi_{\langle \alpha,\beta \rangle}(x)$ does not exceed the hierarchy $\chi_{\langle t,u \rangle}(x)$, i.e. $\chi_{\langle \alpha,\beta \rangle}(x) \leq \chi_{\langle t,u \rangle}(x)$. (For details the reader is referred to [10].)

Let $x = x_1x_2\dots x_m$ be the main product of $x_1, x_2, \dots, x_m$ in $\mathbf{T}_X$.

If $\{x_1, x_2, \dots, x_m\} \subseteq \{t,u\}$, for some terms t,u of T, then we call $x_1x_2\dots x_m$ the *main biproduct* of x in $\mathbf{T}_X$ with the generating pair $\{t,u\}$ and denote it by $x_{t,u}$. (If $u = t$, i.e. the generating "pair" is $\{t,t\}$, we write x_t instead of $x_{t,t}$.)

If $x = x_1x_2\dots x_m$ *and* $x = x'_1x'_2\dots x'_n$ *are main biproducts of* x *in* $\mathbf{T}_X$ *with the same generating pair* $\{t,u\}$, *then* $m = n$ *and* $x_i = x'_i$, *for* $i = 1,2,\dots,m$. *Specially, any maximal biproduct of* $x \in \mathbf{T}_X$, *that is a main biproduct, is uniquely determined.*

We define the desired groupoid $\boldsymbol{R} = (R,*)$ by:

(2.10)
$$R = \{x \in T : \text{every biproduct of any subterm of } x \text{ is a main biproduct}\}$$

and an operation $*$ on R as follows.

Let $x,y \in R$, $x = x_1x_2\dots x_m$, $y = y_1y_2\dots y_n$ be maximal biproducts and put

$$x * y = \begin{cases} xy, \text{ if } xy \in R \\ x_1x_2\dots x_my_1y_2\dots y_n, \text{ if } xy \notin R. \end{cases} \tag{2.11}$$

The groupoid $\boldsymbol{R} = (R,*)$, *defined by* (2.10) *and* (2.11) *is a canonical biassociative groupoid over* X.

The problem of power $\mathcal{V}$-groupoids can be expanded to power $\mathcal{V}$-ternary groupoids or power $\mathcal{V}$-n-ary groupoids. For instance, we can investigate power-commutative ternary groupoids and power-semicommutative ternary groupoids, since a canoninical description of free objects in the varieties of commutative ternary groupoids and semicommutative ternary groupoids are obtained in the thesis.

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