

Splines in Numerical Integration

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We gave a short review of several results which are related to the role of splines (cardinal, centered or interpolating) in numerical integration. Results deal with the problem of approximate computation of the integrals with spline as a weight function, but also with the problem of approximate computation of the integrals without weight function. Besides, we presented an algorithm for calculation of the coefficients of the polynomials which correspond to the cardinal B-spline of arbitrary order and described five methods for calculation of the moments in the case when cardinal B-spline of order $m, m \in \mathbb{N}$, is a weight function.

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1. Introduction

In the approximation theory, cardinal B-splines have a very important place (in different methods for solving initial and boundary value problems, spline interpolation, multiresolution approximation...). The main topic of the author's PhD thesis is the use of the cardinal B-splines in numerical integration. The obtained results can be grouped in two ways, depending on the type of integral under consideration or the order of the spline applied.

The first grouping refers to the results where the problem of approximate computation of the integral

$$(1.1) \quad \int_0^m \varphi_m(x) f(x) dx$$

(integrals with cardinal B-spline of order m as a weight function) is considered and those where the classical problem of numerical analysis, i.e. the problem of approximate computation of the integral

$$(1.2) \quad \int_a^b f(x) dx$$

is considered.

In organizing this paper (as well as in the PhD thesis) we opt for the second possibility. In the next section, we deal with splines of arbitrary order, while the final one presents the results related to the splines of a low order.

Let us recall here the definition of the cardinal B-spline and list its basic properties.

Definition 1. *Cardinal B-spline of the first order, denoted by $\varphi_1(\cdot)$, is the characteristic function of the interval $[0, 1)$, i.e.*

$$\varphi_1(x) = \begin{cases} 1, & x \in [0, 1) \\ 0, & \text{otherwise} \end{cases}.$$

Cardinal B-spline of order $m, m \in \mathbb{N}$, denoted by $\varphi_m(\cdot)$, is defined as a convolution

$$\begin{aligned} \varphi_m(x) &= (\varphi_{m-1} * \varphi_1)(x) = \int_{\mathbb{R}} \varphi_{m-1}(x-t)\varphi_1(t)dt \\ &= \int_0^1 \varphi_{m-1}(x-t)dt. \end{aligned}$$

Theorem 1. *Cardinal B-spline of order $m, m \in \mathbb{N}$, has the following properties*

- (1) $\text{supp}\varphi_m(\cdot) = [0, m]$;
- (2) $\varphi_m(\cdot) \in C^{m-2}[0, m]$;
- (3) *at each interval $[k, k+1], 0 \leq k \leq m-1$, cardinal B-spline of order m is a polynomial of degree equal to $m-1$;*
- (4)

$$(1.3) \quad (\forall t \in [0, m]) \varphi_m(t) = \frac{t}{m-1} \varphi_{m-1}(t) + \frac{m-t}{m-1} \varphi_{m-1}(t-1), m \geq 2;$$

(5)

$$(1.4) \quad (\forall t \in [0, m]) \varphi'_m(t) = \varphi_{m-1}(t) - \varphi_{m-1}(t-1), m \geq 2;$$

- (6) *Cardinal B-spline is symmetric at the interval $[0, m]$, i.e.*
 $(\forall t \in [0, m]) \varphi_m(t) = \varphi_m(m-t);$
- (7)

$$(1.5) \quad (\forall a \in \mathbb{R}) \sum_{i \in \mathbb{Z}} \varphi_m(i-a) = 1;$$

- (8) *For any m time differentiable function $g(\cdot)$ holds*

$$\int_{\mathbb{R}} \varphi_m(x) g^{(m)}(x) dx = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} g(k);$$

(9) *The cardinal B-spline is a solution of the so-called dilatation equation*

$$(\forall t \in \mathbb{R}) \varphi_m(t) = \frac{1}{2^{m-1}} \sum_{k=0}^m \binom{m}{k} \varphi_m(2t - k).$$

The proof of the theorem, and more details about cardinal B-splines one can find in [1] or [2].

2. Splines of arbitrary order

2.1. Calculation of the coefficients of the cardinal B-spline. The first result presented in this paper is an algorithm for calculation the coefficients of the polynomials which correspond to the cardinal B-spline of order m . By using equalities (1.3) and (1.4) it is easy to obtain differential equation

$$(m - x)\varphi'_m(x) + (m - 1)\varphi_m(x) = m\varphi_{m-1}(x).$$

Evidently, the solution of this equation is a cardinal B-spline of order m . Furthermore, let $x \in [k, k + 1]$ and let $\varphi_m(x) = \sum_{i=0}^{m-1} a_i^{(m,k)} x^i$. By using the previous differential equation and the symmetry of the cardinal B-spline one can prove that the following recurrence relations hold:

$$\begin{aligned} a_{m-1}^{(m,0)} &= \frac{1}{m-1} a_{m-2}^{(m-1,0)}, \\ a_i^{(m,0)} &= 0, m-2 \geq i \geq 0, \\ a_{m-1}^{(m,k)} &= \frac{1}{(m-1)(m-k)} \left[m a_{m-2}^{(m-1,k)} - k(m-1) a_{m-1}^{(m,k-1)} - a_{m-2}^{(m,k-1)} \right], \\ a_i^{(m,k)} &= \frac{m}{i+1-m} \left[(i+1) a_{i+1}^{(m,k)} - a_i^{(m-1,k)} \right], \\ &\quad m-2 \geq i \geq 0, \\ a_{m-1}^{(m,m-k-1)} &= (-1)^{m-1} a_{m-1}^{(m,k)}, \\ a_j^{(m,m-k-1)} &= \frac{(-1)^j}{j!} \sum_{i=0}^{m-j-1} (i+1)(i+2) \dots (i+j) \cdot a_{i+j}^{(m,k)} m^i, \\ &\quad 0 \leq j \leq m-2. \end{aligned}$$

Those relations enable us to construct an algorithm for calculating the coefficients of the cardinal B-spline of order m .

2.2. Calculation of the moments of the cardinal B-spline. Calculating moments of the given weight function has essential significance in construction of the orthogonal polynomials and quadrature rules, as well as in other fields of the approximation theory. On the other hand, a very frequent weight function is exactly the cardinal B-spline (finite elements method, multiresolution approximation,...). In the paragraph below, five methods (without proofs) for calculation of moments

$$\mathbb{M}_{n,m} = \int_0^m \varphi_m(t) t^n dt, n \in \mathbb{N}_0.$$

are provided.

(1)

$$\mathbb{M}_{n,m} = \frac{1}{2^m(2^n - 1)} \sum_{k=1}^m \sum_{l=0}^{n-1} \binom{m}{k} \binom{n}{l} k^{n-l} \mathbb{M}_{l,m};$$

(2)

$$\begin{aligned} \mathbb{M}_{n,m} &= \sum_{k=0}^n \binom{n}{k} \mathbb{M}_{k,m-1} - \frac{1}{m-1} \sum_{k=0}^{n-1} \binom{n}{k} \mathbb{M}_{k+1,m-1} \\ &= 1 + \sum_{k=1}^n \left[\binom{n}{k} - \frac{1}{m-1} \binom{n}{k-1} \right] \mathbb{M}_{k,m-1}; \end{aligned}$$

(3)

$$\mathbb{M}_{n,m} = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} \mathbb{M}_{k,m-1};$$

(4)

$$M_{n,m} = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \frac{k^{m+n}}{(m+n)(m+n-1) \dots (n+1)}.$$

The last method is, of course, the most important result of this paragraph. This method generalizes equality (1.5) and we will formulate it as a theorem.

Theorem 2. *For any $a \in \mathbb{R}$ and every $m \in \mathbb{N}, m \geq 2$, the following equality holds*

$$\mathbb{M}_{n,m} = \sum_{i \in \mathbb{Z}} \varphi_m(i-a) (i-a)^n, 0 \leq n \leq m-1.$$

In the case when a is an integer and m is odd, one can easily check that the previous equality also holds for $n = m$. Those results were recently published [8].

2.3. One point quadrature rule with cardinal B-spline. In this paragraph we are presenting probably the most important result of our research which was published in [7]. The basic formula is a classical one point quadrature rule

$$(2.6) \quad \int_a^b f(x)dx \approx (b-a)f(X),$$

with choosen point $X = (1-\lambda)a + \lambda b$, for some $\lambda \in [0, 1]$.

Let $x_1 < x_2 < \dots < x_{p-1}$ be an arbitrary points from the interval $(0, 1)$ and let the partition

$$\begin{array}{ccccccccc} 0, & x_1, & x_2, & \dots, & x_{p-1}, & & & & \\ 1, & x_1 + 1, & x_2 + 1, & \dots, & x_{p-1} + 1, & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ m-1, & x_1 + m-1, & x_2 + m-1, & \dots, & x_{p-1} + m-1, & & & & \\ m, & & & & & & & & \end{array}$$

of the interval $[0, m]$ be given. We call this kind of mesh “quaziuniform”. Furthermore, let the points $X_k + i, 0 \leq i \leq m-1$, where $X_k = (1-\lambda_k)x_k + \lambda_k x_{k+1}$, for some $\lambda_k \in [0, 1], 0 \leq k \leq p-1$, at each interval of the given partition, be chosen (naturally, $x_0 = 0$ and $x_p = 1$).

At each interval of partition, we will use the formula (2.6) with chosen points $X_k + i, 0 \leq i \leq m-1, 0 \leq k \leq p-1$, to approximately compute the integral (1.1). After summation, we will obtain

$$(2.7) \quad \int_0^m \varphi_m(x)f(x)dx \approx \sum_{j=0}^{p-1} (x_{j+1} - x_j) \sum_{i=0}^{m-1} \varphi_m(X_j + i)f(X_j + i).$$

Theorem 3. *If $f(x) = x^n, 0 \leq n \leq m-1$, then the quadrature rule (2.7) is exact, for each $m \in \mathbb{N}$.*

In accordance with the previous theorem formula (2.7) has, conditionally speaking, algebraic degree of exactness equal to $m-1$. In some special cases this degree of exactness can be improved up to m .

2.4. Numerical integration using cardinal B-splines. One of the classical problems in the mathematical analysis is, of course, the calculation of the integral (1.2). Since this integral cannot, in general, be calculated exactly, there is a large number of methods for its approximate computation. We are presenting here a new method which is obtained by using cardinal B-splines. The main idea is to approximate the function $f(\cdot)$ by its projection on the cardinal B-spline space

$$\hat{f}(x) = \sum_{k \in \mathbb{Z}} c_k \varphi_m^{(j,k)}(x),$$

where $\varphi_m^{(j,k)}(x) = 2^{\frac{j}{2}}\varphi_m(2^jx - k)$. The coefficients of the approximation are solution of the corresponding system of linear equations. This approach leaves $m - 2$ coefficients free, so additional conditions can be chosen depending on the concrete situation. One possibility is to choose some integer nodes to have multiplicity two. Finally, one obtains

$$\begin{aligned} \int_0^m f(x)dx &\approx \int_0^m \widehat{f}(x)dx = 2^{-\frac{j}{2}} \left(\sum_{k=-m+1}^{-1} c_k (1 - \mu_{0,m}^{-k}) \right. \\ &\quad \left. + \sum_{k=0}^{(2^j-1)m} c_k + \sum_{k=(2^j-1)m+1}^{2^jm-1} c_k \mu_{0,m}^{2^jm-k} \right), \end{aligned}$$

where

$$\mu_{n,m}^x = \int_0^m \varphi_m(t)t^n dt$$

are the so-called shortened spline moments. Those moments, for example, can be calculated recursively

$$\begin{aligned} \mu_{n,1}^x &= \int_0^x \varphi_1(t)t^n dt = \begin{cases} 0, & x < 0 \\ \frac{x^{n+1}}{n+1}, & x \in [0, 1] \\ \frac{1}{n+1}, & x > 1 \end{cases}, \\ \mu_{n,m}^x &= \int_0^x \varphi_m(t)t^n dt \\ &= \frac{x^{n+1}\varphi_m(x)}{n+1} - \frac{1}{n+1} \int_0^x \varphi_{m-1}(t)t^{n+1} dt \\ &\quad + \frac{1}{n+1} \int_0^{x-1} \varphi_{m-1}(t)(t+1)^{n+1} dt \\ &= \frac{1}{n+1} \left(x^{n+1}\varphi_m(x) - \mu_{n+1,m-1}^x + \sum_{k=0}^{n+1} \binom{n+1}{k} \mu_{k,m-1}^{x-1} \right). \end{aligned}$$

This result was inspired by the paper [5].

3. Splines of given (low) order

3.1. Some modifications of the trapezoidal rule. This section begins with the result recently published in [6]. The main idea is based on the fact that the construction of the quadratic interpolating spline leaves one parameter free. By an appropriate choice of the free parameter we obtain reproduction of some well known quadrature rules, but also a wide class of new quadrature rules.

Except the reproduction of the Simpsons and Ermits quadrature rule, we obtain the following formulas:

$$\begin{aligned}
\int_a^b f(x)dx &\approx \frac{h}{2} \left(f_0 + 2 \sum_{k=1}^{2m-1} f_k + f_{2m} \right) - \frac{h^3}{6} \sum_{k=1}^m f''_{2k-1}; \\
&\approx \frac{h}{2} (f_0 + f_1) - \frac{h^3 f''(X)}{12} + Q_S[f, x_1, b, 2m]; \\
&\approx \frac{h}{2} \left(f_0 + 2 \sum_{k=1}^{2m} f_k + f_{2m+1} \right) - \frac{h^3}{6} \left(\frac{f''(X)}{2} + \sum_{k=1}^m f''_{2k} \right); \\
&\approx \frac{h}{2} (f_0 + f_1) - \frac{h^2}{12} (f'_1 - f'_0) + Q_S[f, x_1, b, 2m]; \\
&\approx Q_T[f, x_0, x_1] - \frac{h^3}{12} \left[f''(X) + f'''(X) \left(x_0 + \frac{h}{2} - X \right) \right] \\
&\quad + Q_S[f, x_1, b, 2m].
\end{aligned}$$

In the second and the third formulas X is any point from the interval $[x_0, x_1]$ such that $f''(X) = \lambda f''_0 + (1 - \lambda) f''_1$, where $\lambda \in [0, 1]$, while in the last formula X is any point from the interval $[x_0, x_1]$. Furthermore, Q_T and Q_S denote a classical trapezoidal and the Simpsons rule respectively. The accuracy of the obtained formulas is $O(h^4)$.

3.2. A certain class of quadratures with specific weight function. The last result was inspired by the papers [3] and [4], and it is about the so-called quadrature formulas of “practical type”. Hence, we are considering the quadrature formula

$$\int_0^m \varphi_m(x) f(x) dx \approx \sum_{i=1}^5 A_i f(x_i).$$

This formula is of “practical type” if the following conditions hold:

- coefficients A_i , $1 \leq i \leq 5$, are symmetric, i.e. ($A_1 = A_5$ and $A_2 = A_4$);
- nodes x_i , $1 \leq i \leq 5$, are symmetric, rational numbers from the interval $[0, m]$.

We proved that in the case when the weight function is cardinal B-spline of order two (hat function), i.e. in the case when the weight function is a cardinal B-spline of order four (cubic B-spline), the maximal algebraic degree of exactness of such type of quadratures is equal to five.

References

- [1] C. Chui, *An Introduction to Wavelets*, Academic Press, inc., Boston San Diego New York London Sydney Tokyo Toronto, 1992.
- [2] C. Chui, *Wavelets: A mathematical tool for signal analysis*, Society for Industrial and Applied Mathematics, Philadelphia, 1997.
- [3] E. Constantinescu, A certain class of quadratures, *Acta Univ. Apulensis Math. Inform.*, **7**, 2004, 119-116.
- [4] E. Constantinescu, A certain class of quadratures, *Gen. Math.*, **10**, No. 1-2, 2002, 75-84.
- [5] H. Hashish, S. H. Behiry, N. A. El-Shamy, Numerical Integration Using Wavelets, *Appl. Math. Comput.*, 2009
- [6] Z. Udovičić, Some modifications of the trapezoidal rule, *Sarajevo J. Math.*, **2**(15), 2006, 237-245.
- [7] Z. Udovičić, One point quadrature rule with cardinal B-spline, *J. Math. Anal. Appl.*, **360**, 2009, 432-438.
- [8] Z. Udovičić, Calculation of the moments of the cardinal B-spline, *Sarajevo J. Math.*, accepted.
- [9] Z. Udovičić, A certain class of quadratures with hat function as a weight function, *Acta Univ. Apulensis Math. Inform.*, accepted.

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