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Subclasses of Starlike Functions of Complex Order Involving Generalized Hypergeometric Functions

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Making use of the generalized hypergeometric functions, we define a new subclass of uniformly convex functions and a corresponding subclass of starlike functions with negative coefficients and obtain coefficient estimates, extreme points, the radii of close to convexity, starlikeness and convexity and neighbourhood results for the class $TS^{\tau}_{\lambda}(\alpha, \beta, \gamma)$. In particular, we obtain integral means inequalities for the function $f(z)$ belongs to the class $TS^{\tau}_{\lambda}(\alpha, \beta, \gamma)$ in the unit disc.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $\mathcal{U} = \{z : z \in \mathcal{D}, \ |z| < 1\}$. Also denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{1.2}$$

introduced and studied by Silverman [21].For functions $f \in \mathcal{A}$ given by (1.1) and $g(z) \in A$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}. \tag{1.3}$$

For positive real values of $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$ ($\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m; z) &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\ l, m \in N_0 &:= N \cup \{0\}; z \in U) \end{aligned} \tag{1.4}$$

where N denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\dots(a+n-1), & n \in N. \end{cases} \tag{1.5}$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial and others; for example see [3] and [4,23].

Let $H(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m) : \mathcal{A} \to \mathcal{A}$ be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m))(f)](z) &:= z\ {}_lF_m(\alpha_1, \alpha_2, \dots \alpha_l; \beta_1, \beta_2 \dots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \omega_n(\alpha_1; l; m)\ a_n z^n \end{aligned} \tag{1.6}$$

where

$$\omega_n(\alpha_1; l; m) = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \frac{1}{(n-1)!}. \tag{1.7}$$

For notational simplification in our investigation, we write

$$H_m^l[\alpha_1]f(z) = H(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m)f(z).$$

Recently Srivastava and et.al.[23] defined the linear operator $\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}$ as follows:

$$\mathcal{L}_{\lambda,\alpha_1}^{0} f(z) = f(z),$$

$$\mathcal{L}^{1,\alpha_1}_{\lambda,l,m}f(z) = \mathcal{L}^{\alpha_1}_{\lambda,l,m}f(z)$$

$$= (1-\lambda)H^l_m[\alpha_1]f(z) + \lambda(H^l_m[\alpha_1]f(z))'; (\lambda \geq 0), \tag{1.8}$$

$$\mathcal{L}^{2,\alpha_1}_{\lambda,l,m}f(z) = \mathcal{L}^{\alpha_1}_{\lambda,l,m}(\mathcal{L}^{1,\alpha_1}_{\lambda,l,m}f(z)) \tag{1.9}$$

and in general,

$$\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}f(z)$$

$$= \mathcal{L}^{\alpha_1}_{\lambda,l,m}(\mathcal{L}^{\tau-1,\alpha_1}_{\lambda,l,m}f(z)), (l \leq m+1;\ l,m \in N_0 = N \cup \{0\}; z \in U). \tag{1.10}$$

If the function $f(z)$ is given by (1.1), then we see from (1.6), (1.7),(1.8) and(1.10) that

$$\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}f(z) \quad := \quad z + \sum_{n=2}^{\infty} \omega^{\tau}_n(\alpha_1;\lambda;l;m)\ a_n\ z^n \tag{1.11}$$

where

$$\omega^{\tau}_n(\alpha_1;\lambda;l;m)$$

$$= \left(\frac{(\alpha_1)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_m)_{n-1}}\ \frac{[1+\lambda(n-1)]}{(n-1)!}\right)^{\tau}, (n \in N \setminus \{1\}, \tau \in N_0). \tag{1.12}$$

unless otherwise stated. We note that when $\tau = 1$ and $\lambda = 0$ the linear operator $\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}$ would reduce to the familiar Dziok-Srivastava linear operator given in [5], includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [4], Owa [16]and Ruscheweyh [19]. Motivated by Gooodman [7], [8], Rønning [17], [18] introduced and studied the following subclasses of $\mathcal{A}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_p(\alpha,\beta)$ uniformly $\beta-$starlike functions if it satisfies the condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|,\ (-1 < \alpha \leq 1; \beta \geq 0) z \in U \tag{1.13}$$

and is said to be in the class $\mathcal{UCV}(\alpha,\beta)$, uniformly $\beta-$convex functions if it satisfies the condition

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|,\ (-1 < \alpha \leq 1; \beta \geq 0) z \in U, \tag{1.14}$$

Indeed it follows from (1.13) and (1.14) that

$$f \in \mathcal{UCV}(\alpha,\beta) \Leftrightarrow zf' \in \mathcal{S}_p(\alpha,\beta). \tag{1.15}$$

Recently, many papers in the theory of univalent functions have been devoted to mapping and characterization properties of various linear integral or integro-differential operators in the class S (of normalized analytic and univalent functions in the open unit disk U), and in its subclasses (as the classes $\mathcal{S}^*$ of the starlike functions and $\mathcal{K}$ of the convex functions in $\mathcal{U}$) by Kiryakova [11] and Kiryakova et.al. [12], using various linear operators (also see [4,5,14,15,19,23]). further, the interesting geometric properties of the function classes $\mathcal{UCV}$ and $\mathcal{S}_p$ were discussed by Kanas et.al. in [9],[10] and in [14]. Motivated by Altintas et al. [1], Murugusundaramoorthy and Magesh [14] and Murugusundaramoorthy and Srivastava [16], we define a subclass uniformly starlike functions of complex order using the generalized derivative operator $\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}$ and discuss its characterization properties.

For $-1 \leq \alpha < 1$, $\beta \geq 0$ and $\gamma \in \mathcal{C}\backslash\{0\}$ we let $\mathcal{S}_\lambda^\tau(\alpha,\beta,\gamma)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z))'}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z)}-\alpha\right)\right\}$$

$$(1.16)\qquad > \beta\left|1+\frac{1}{\gamma}\left(\frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z))'}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z)}-1\right)\right|,\quad z\in U$$

where $\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z)$ is given by (1.11). We also let $\mathcal{TS}_\lambda^\tau(\alpha,\beta,\gamma) = \mathcal{S}_\lambda^\tau(\alpha,\beta,\gamma)\cap\mathcal{T}$. The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bound, extreme points, radii of close to convexity, starlikeness and convexity for the class $\mathcal{TS}_\lambda^\tau(\alpha,\beta,\gamma)$. Further, we obtain Neighbourhood results and integral means inequalities for aforementioned class.

2 Basic properties

Now we obtain the necessary and sufficient condition for functions functions $f(z)$ in the class $\mathcal{TS}_\lambda^\tau(\alpha,\beta,\gamma)$.

Theorem 2.1. *A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in the class $\mathcal{TS}_\lambda^\tau(\alpha,\beta,\gamma)$,is*

$$(2.1)\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_n^\tau(\alpha_1;\lambda;l;m)a_n \leq (1-\alpha)+|\gamma|(1-\beta),$$

where $-1 \leq \alpha < 1$, $\beta \geq 0$ and $\gamma \in \mathcal{C}\backslash\{0\}$.

Proof. Assume that $f(z) \in \mathcal{TS}_\lambda^\tau(\alpha,\beta,\gamma)$, then

$$\mathrm{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z))'}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z)}-\alpha\right)\right\} > \beta\left|1+\frac{1}{\gamma}\left(\frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z))'}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z)}-1\right)\right|$$

$$\mathrm{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z(1-\alpha)-\sum_{n=2}^{\infty}(n-\alpha)\omega_n^\tau(\alpha_1;\lambda;l;m)a_nz^n}{z-\sum_{n=2}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)a_nz^n}\right)\right\}$$

$$>\beta\left|1-\frac{1}{\gamma}\left(\frac{\sum_{n=2}^{\infty}(n-1)\omega_n^\tau(\alpha_1;\lambda;l;m)a_nz^n}{z-\sum_{n=2}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)a_nz^n}\right)\right|.$$

Letting $z\to 1^-$ along the real axis, we have

$$\left\{1+\frac{1}{|\gamma|}\left(\frac{(1-\alpha)-\sum_{n=2}^{\infty}(n-\alpha)\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n|}{1-\sum_{n=2}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n|}\right)\right\}$$

$$>\beta\left[1-\frac{1}{|\gamma|}\left(\frac{\sum_{n=2}^{\infty}(n-1)\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n|}{1-\sum_{n=2}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n|}\right)\right].$$

The simple computation leads the desired inequality

$$\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_n^\tau(\alpha_1;\lambda;l;m)a_n \le (1-\alpha)+|\gamma|(1-\beta).$$

Conversely, suppose that (2.1) is true for $z\in\mathcal{U}$. Then

$$\mathrm{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z))'}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z)}-\alpha\right)\right\}-\beta\left|1+\frac{1}{\gamma}\left(\frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z))'}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z)}-1\right)\right|>0$$

if

$$1+\frac{1}{|\gamma|}\left(\frac{(1-\alpha)-\sum_{n=2}^{\infty}(n-\alpha)\omega_n^\tau(\alpha_1;\lambda;l;m)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)a_n|z|^{n-1}}\right)$$

$$\beta\left[1-\frac{1}{|\gamma|}\left(\frac{\sum_{n=2}^{\infty}(n-1)\omega_n^\tau(\alpha_1;\lambda;l;m)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)a_n|z|^{n-1}}\right)\right]\ge 0.$$

That is if

$$\sum_{n=2}^{\infty}[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_n^{\tau}(\alpha_1;\lambda;l;m)a_n \leq (1-\alpha)+|\gamma|(1-\beta),$$

which completes the proof.

Corollary 2.1 *Let the function $f(z)$ defined by (1.2) belongs $\mathcal{TS}_{\lambda}^{\tau}(\alpha,\beta,\gamma)$. Then*

$$a_n \leq \frac{[(1-\alpha)+|\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_n^{\tau}(\alpha_1;\lambda;l;m)} \tag{2.2}$$

$n \geq 2$, $-1 \leq \alpha < 1$, $\beta \geq 0$ and $\gamma \in \mathcal{C}\backslash\{0\}$, with equality for

$$f(z) = z - \frac{[(1-\alpha)+|\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_n^{\tau}(\alpha_1;\lambda;l;m)}z^n.$$

In the following theorem we state the extreme points for the functions of the class $\mathcal{TS}_{\lambda}^{\tau}(\alpha,\beta,\gamma)$ without proof.

Theorem 2.2. *(Extreme points) Let $f_1(z) = z$ and $f_n(z) = z - \frac{[(1-\alpha)+|\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_n^{\tau}(\alpha_1;\lambda;l;m)}z^n$ for $n=2,3,4,\ldots$.*

Then $f(z) \in \mathcal{TS}_{\lambda}^{\tau}(\alpha,\beta,\gamma)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=1}^{\infty}\chi_n f_n(z)$, where $\chi_n \geq 0$ and $\sum_{n=1}^{\infty}\chi_n = 1$.

3 Close-to-convexity, starlikeness and convexity

We now obtain the radii of close-to-convexity, starlikeness and convexity results for functions in the class $\mathcal{TS}_{\lambda}^{\tau}(\alpha,\beta,\gamma)$.

Theorem 3.1. *Let $f \in \mathcal{TS}_{\lambda}^{\tau}(\alpha,\beta,\gamma)$. Then f is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, that is Re $\{f'(z)\} > \delta$, where*

$$r_1 = \inf_{n\geq 2}\left[\frac{(1-\delta)}{n}\frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{[(1-\alpha)+|\gamma|(1-\beta)]}\omega_n^{\tau}(\alpha_1;\lambda;l;m)\right]^{\frac{1}{n-1}}.$$

P r o o f. Given $f \in \mathcal{T}$, and f is close-to-convex of order δ, we have

$$|f'(z)-1| < 1-\delta. \tag{3.1}$$

For the left hand side of (3.1) we have

$$|f'(z)-1| \leq \sum_{n=2}^{\infty} na_n|z|^{n-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} a_n|z|^{n-1} < 1.$$

Using the fact, that $f \in TS_{\lambda}^{\tau}(\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{(1-\alpha)+|\gamma|(1-\beta)} \omega_n^{\tau}(\alpha_1;\lambda;l;m)a_n < 1.$$

We can say (3.1) is true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{(1-\alpha)+|\gamma|(1-\beta)} \omega_n^{\tau}(\alpha_1;\lambda;l;m).$$

Or, equivalently,

$$|z| \leq \left[\frac{(1-\delta)[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{n[(1-\alpha)+|\gamma|(1-\beta)]} \omega_n^{\tau}(\alpha_1;\lambda;l;m)\right]^{\frac{1}{n-1}},$$

which completes the proof.

Theorem 3.2. *Let $f \in TS_{\lambda}^{\tau}(\alpha,\beta,\gamma)$. Then*

1. *f is starlike of order $\delta (0 \leq \delta < 1)$ in the disc $|z| < r_2$; that is, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, where*

$$r_2 = \inf_{n\geq 2}\left\{\frac{(1-\delta)}{(n-\delta)} \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{[(1-\alpha)+|\gamma|(1-\beta)]} \omega_n^{\tau}(\alpha_1;\lambda;l;m)\right\}^{\frac{1}{n-1}}.$$

2. *f is convex of order δ $(0 \leq \delta < 1)$ in the unit disc $|z| < r_3$, that is $Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, where*

$$r_3 = \inf_{n\geq 2}\left\{\frac{(1-\delta)}{n(n-\delta)} \frac{[(n+|\gamma|)(1-\beta)-(\alpha-\beta)]}{[(1-\alpha)+|\gamma|(1-\beta)]} \omega_n^{\tau}(\alpha_1;\lambda;l;m)\right\}^{\frac{1}{n-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (2).

Proof. Given $f \in \mathcal{T}$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta. \tag{3.2}$$

For the left hand side of (3.2), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n\ |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n\ |z|^{n-1}}.$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n\ |z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{TS}_{\lambda}^{\tau}(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[(n+|\gamma|)(1-\beta) - (\alpha-\beta)]}{(1-\alpha) + |\gamma|(1-\beta)} \omega_n^{\tau}(\alpha_1; \lambda; l; m) a_n\ < 1.$$

We can say (3.2) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} < \frac{[(n+|\gamma|)(1-\beta) - (\alpha-\beta)]}{(1-\alpha) + |\gamma|(1-\beta)} \omega_n^{\tau}(\alpha_1; \lambda; l; m).$$

Or, equivalently,

$$|z|^{n-1} < \frac{(1-\delta)[(n+|\gamma|)(1-\beta) - (\alpha-\beta)]}{(n-\delta)[(1-\alpha) + |\gamma|(1-\beta)]} \omega_n^{\tau}(\alpha_1; \lambda; l; m)$$

which yields the starlikeness of the family.

Using the fact that f is convex if and only if zf' is starlike, we can prove (2), on lines similar the proof of (1).

4 Integral Means

In order to find the integral means inequality and to verify the Silverman Conjuncture [22], we need the following subordination result, due to Littlewood [13].

Lemma 4.1. *If the functions $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with $g(z) \prec f(z)$, then*

$$(4.1)\quad \int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta, \quad \eta > 0, \quad z = re^{i\theta} \text{ and } 0 < r < 1.$$

Applying Theorem 2.1 with extremal function and Lemma 4.1, we prove the following theorem.

Theorem 4.2. *Let $\eta > 0$. If $f(z) \in \mathcal{TS}_\lambda^\tau(\alpha, \beta, \gamma)$, and $\{\Phi(\alpha, \beta, \gamma, n)\}_{n=2}^\infty$ is non-decreasing sequence, then for $z = re^{i\theta}$ and $0 < r < 1$, we have*

$$(4.2)\quad \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta$$

where

$$f_2(z) = z - \frac{(1-\alpha) + |\gamma|(1-\beta)}{\Phi(\alpha, \beta, \gamma, 2)} z^2$$

and

$$\Phi(\alpha, \beta, \gamma, n) = [(n + |\gamma|)(1-\beta) - (\alpha - \beta)]\omega_n^\tau(\alpha_1; \lambda; l; m).$$

P r o o f. Let $f(z)$ of the form (1.2) and $f_2(z) = z - \frac{(1-\alpha)+|\gamma|(1-\beta)}{\Phi(\alpha,\beta,\gamma,2)} z^2$, then we must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^\infty a_n z^{n-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{(1-\alpha) + |\gamma|(1-\beta)}{\Phi(\alpha, \beta, \gamma, 2)} z\right|^\eta d\theta.$$

By Lemma 4.1, it suffices to show that

$$1 - \sum_{n=2}^\infty a_n z^{n-1} \prec 1 - \frac{(1-\alpha) + |\gamma|(1-\beta)}{\Phi(\alpha, \beta, \gamma, 2)} z$$

Setting

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}=1-\frac{(1-\alpha)+|\gamma|(1-\beta)}{\Phi(\alpha,\beta,\gamma,2)}w(z). \tag{4.3}$$

From (4.3) and (2.1), we obtain

$$\begin{aligned} |w(z)| &= \left|\sum_{n=2}^{\infty}\frac{\Phi(\alpha,\beta,\gamma,n)}{(1-\alpha)+|\gamma|(1-\beta)}a_nz^{n-1}\right| \\ &\le |z|\sum_{n=2}^{\infty}\frac{\Phi(\alpha,\beta,\gamma,n)}{(1-\alpha)+|\gamma|(1-\beta)}a_n \\ &\le |z|<1. \end{aligned}$$

This completes the proof of Theorem 4.2.

5 Inclusion relations involving $N_\delta(e)$.

To study about the inclusion relations involving $N_\delta(e)$ we need the following definitions. Following [1], [6], [20], we define the n,δ neighborhood of function $f(z)\in\mathcal{T}$ by

$$N_\delta(f)=\left\{g\in\mathcal{T}:g(z)=z-\sum_{n=2}^{\infty}b_nz^n \text{ and } \sum_{n=2}^{\infty}n|a_n-b_n|\le\delta\right\}. \tag{5.1}$$

Particulary for the identity function $e(z)=z$, we have

$$N_\delta(e)=\left\{g\in\mathcal{T}:g(z)=z-\sum_{n=2}^{\infty}b_nz^n \text{ and } \sum_{n=2}^{\infty}n|b_n|\le\delta\right\}. \tag{5.2}$$

Theorem 5.1. *Let*

$$\delta=\frac{2[(1-\alpha)+|\gamma|(1-\beta)]}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_2^{\mathcal{T}}(\alpha_1;\lambda;l;m)}. \tag{5.3}$$

Then $\mathcal{T}S_\lambda^{\mathcal{T}}(\alpha,\beta,\gamma)\subset N_\delta(e)$.

Proof. For $f \in \mathcal{T}S_\lambda^\tau(\alpha,\beta,\gamma)$, Theorem 2.1 yields:

$$[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_2^\tau(\alpha_1;\lambda;l;m)\sum_{n=2}^{\infty} a_n \le (1-\alpha)+|\gamma|(1-\beta),$$

so that

$$\sum_{n=2}^{\infty} a_n \le \frac{(1-\alpha)+|\gamma|(1-\beta)}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_2^\tau(\alpha_1;\lambda;l;m)}. \tag{5.4}$$

On the other hand, from (2.1) and (5.4) we have

$$(1-\beta)\omega_2^\tau(\alpha_1;\lambda;l;m)\sum_{n=2}^{\infty} na_n$$

$$\le (1-\alpha)+|\gamma|(1-\beta)+[(\alpha-\beta)-|\gamma|(1-\beta)]\omega_2^\tau(\alpha_1;\lambda;l;m)\sum_{n=2}^{\infty} a_n$$

$$\le (1-\alpha)+|\gamma|(1-\beta)+[(\alpha-\beta)-|\gamma|(1-\beta)]\omega_2^\tau(\alpha_1;\lambda;l;m)$$

$$\times\frac{(1-\alpha)+|\gamma|(1-\beta)}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_2^\tau(\alpha_1;\lambda;l;m)}$$

$$\le \frac{[(1-\alpha)+|\gamma|(1-\beta)]2(1-\beta)]}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]}$$

$$\sum_{n=2}^{\infty} na_n \le \frac{2[(1-\alpha)+|\gamma|(1-\beta)]}{[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_2^\tau(\alpha_1;\lambda;l;m)}. \tag{5.5}$$

Now we determine the neighborhood for each of the class $\mathcal{T}S_\lambda^\tau(\alpha,\beta,\gamma)$ which we define as follows. A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{T}S_\lambda^\tau(\alpha,\beta,\gamma)$ if there exists a function $g \in \mathcal{T}S_\lambda^\tau(\alpha,\beta,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\eta, \ \ (z\in U, \ \ 0\le\eta<1). \tag{5.6}$$

Theorem 5.2. *If* $g \in \mathcal{T}S_\lambda^\tau(\alpha,\beta,\gamma)$ *and*

$$\eta = 1-\frac{\delta[(2+|\gamma|)(1-\beta)-(\alpha-\beta)]\omega_2^\tau(\alpha_1;\lambda;l;m)}{2[((2+|\gamma|)(1-\beta)-(\alpha-\beta))\omega_2^\tau(\alpha_1;\lambda;l;m)-((1-\alpha)+|\gamma|(1-\beta))]}. \tag{5.7}$$

Then $N_\delta(g) \subset \mathcal{T}S_\lambda^\tau(\alpha,\beta,\gamma,\eta)$.

Proof. Suppose that $f \in N_\delta(g)$ then we find from (5.5) that

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \delta$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2}.$$

Next, since $g \in \mathcal{T}S_\lambda^\tau(\alpha, \beta, \gamma)$, we have

$$\sum_{n=2}^{\infty} b_n \leq \frac{2[(1-\alpha) + |\gamma|(1-\beta)]}{[(2+|\gamma|)(1-\beta) - (\alpha - \beta)]\omega_2^\tau(\alpha_1; \lambda; l; m)}.$$

So that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} b_n}$$

$$\leq \frac{\delta}{2} \times \frac{[(2+|\gamma|)(1-\beta) - (\alpha - \beta)]\omega_2^\tau(\alpha_1; \lambda; l; m)}{[((2+|\gamma|)(1-\beta) - (\alpha - \beta))\omega_2^\tau(\alpha_1; \lambda; l; m) - ((1-\alpha) + |\gamma|(1-\beta))]}$$

$$\leq 1 - \eta.$$

provided that η is given precisely by (5.7). Thus by definition, $f \in \mathcal{T}S_\lambda^\tau(\alpha, \beta, \gamma, \eta)$ for η given by (5.7), which completes the proof.

Concluding Remarks: By suitably specializing the various parameters involved in Theorem 2.1 to Theorem 5.2, presented in this paper, we can deduce numerous further corollaries and consequences of each of these results.

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References

[1] O. Altintas, O. Ozkan and H.M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, *Appl. Math. Letters.*, **13**, 2000, 63–67.

[2] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, **135**, 1969, 429–446.

[3] B.C. Carlson, *Special Functions of Applied Mathematics,* Academic Press, New York, 1977.

[4] B.C. Carlson, and Shaffer, S.B. Starlike and prestarlike hypergrometric functions, *SIAM J. Math. Anal.*, **15**, 2002, 737–745.

[5] J. Dziok, and Srivastava, H.M. Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transform and Spec. Funct.*, **14**, 2003, 7–18.

[6] A.W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, **8**, 1957, 598–601.

[7] A.W. Goodman, On uniformly convex functions, *Ann. polon. Math.*, **56**, 1991, 87–92.

[8] A.W. Goodman, On uniformly starlike functions, *J. Math. Anal. & Appl.*, **155**, 1991, 364–370.

[9] S. Kanas, and Srivastava, H.M. Linear operators associated with k-uniformly convex functions, *Integral Transform. Spec. Funct.*,**9** , No.2, 2000, 121–132.

[10] S. Kanas, and Yaguchi, T. Subclasses of k-uniformly convex and starlike functions defined by generalized derivative -II, *Publ. Inst. Math. (Beograd) (N.S.)*, **69**, No.83, 2001, 91–100.

[11] V.S. Kiryakova On two Saigo's fractional integral operators in the class of univalent functions, *Fract. Calc. Appl. Anal.*, **9**, No. 2, 2006, 159–176.

[12] V. S. Kiryakova, M. Saigo, S. Owa, Distortion and characterization theorems for starlike and convex functions related to generalized fractional calculus, In:*New Development of Convolutions (Surikaisekikenkyusyo Kokyuroku)* , **1012**, Kyoto University (1997), 25-46.

[13] J.E. Littlewood, On inequalities in theory of functions, *Proc. London Math. Soc.*, **23**, 1925, 481–519.

[14] G. Murugusundaramoorthy, and Magesh, N. Linear operators associated with a subclass of uniformly $\beta-$convex functions, *Proc. National Conference on Geometry, Analysis, Fluid Mechanics and Computer Applications, Kuvempu University, Shimoga, Karnataka, December 9-11, 2004,* 103–110.

[15] G. Murugusundaramoorthy and Srivastava, H. M Neighborhoods of certain classes of analytic functions of complex order, *J. Inequal. Pure Appl. Math.*, **5**(2), (2004), Art.24, 1–8.

[16] S. Owa, On the distortion theorems - I, *Kyungpook. Math. J.*, **18**, 1978, 53–59.

[17] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118**, 1993, 189–196.

[18] F. Rønning, Integral representations for bounded starlike functions, ,*Annal. Polon. Math.*, **60**, 1995, 289–297.

[19] S. Ruscheweyh, New criteria for Univalent Functions, *Proc. Amer. Math. Soc.*, **49**, 1975, 109–115.

[20] S. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81**, 1981, 521 -527.

[21] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51**, 1975, 109–116.

[22] H. Silverman, Integral means for univalent functions with negative coefficients, *Houston J. Math.*, **23**, 1997, 169–174.

[23] H.M. Srivastava, Li, S.H and Tang, H. Certain classes of k-uniformly close-to-convex functions and other related functionsdefined by using Dziok-Srivastava operator, *Bull. Math.Anal.Appln..*, **13** , 2009, 49–63.

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