Weighted Approximation by a Class of Bernstein-Type Operators

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Direct theorem in terms of the weighted K-functional for the uniform weighted approximation errors of a class of Bernstein-type operators are obtained for functions from $C(w)[0, 1]$ with weight of the form $x^{\gamma_0}(1 - x)^{\gamma_1}$ for $\gamma_0, \gamma_1 \in [-1, 0]$.


Keywords: Bernstein-type operator, Direct theorem, K-functional.

1. Introduction

The class of Bernstein-type operators discussed in this paper are given for natural $n$ by

$$\tilde{B}_n(f, x) = \sum_{k=0}^{n} b_{n,k}(f) P_{n,k}(x),$$

where $P_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}$ and the functionals $b_{n,k}(f)$ satisfy the following conditions

(1.1) $b_{n,0}(f) = f(0)$ and $b_{n,n}(f) = f(1)$;

(1.2) $b_{n,k}(f)$ are linear and positive;

(1.3) $\tilde{B}_n(e_i, x) = e_i(x)$ for $i=0$ and $i=1$;

(1.4) $\tilde{B}_n(e_2, x) = e_2(x) + \alpha(n)x(1 - x)$. 
Here $e_i$ (for $i = 0, 1, 2$) are the functions $e_i(x) = x^i$.

The functional $b_{n,k}(f)$ for $1 \leq k \leq n - 1$ in the operators $\tilde{B}_n$ takes place of $f \left( \frac{k}{n} \right)$ in the classical Bernstein operators [4].

Denote the weight function by

(1.5) \quad w(x) = w(\gamma_0, \gamma_1; x) = x^{\gamma_0}(1 - x)^{\gamma_1} \quad \text{for} \quad x \in (0, 1) \quad \text{and} \quad \text{real} \quad \gamma_0, \gamma_1.

Our main results will concern the values of the powers $\gamma_0, \gamma_1$ in the range $[-1, 0]$. By $\varphi(x) = x(1 - x)$ we denote the other weight which is naturally connected with the second derivatives of operators and the error for the function $e_2(x)$. By $D = \frac{d}{dx}$ we denote the first derivative operator.

Let $C(0, 1)$ be the space of all continuous functions bounded on $(0,1)$ and let $C(w)(0,1) = \{f : w f \in C(0,1)\}$. The norm in $C(w)(0,1)$ is given by $\| f \|_{C(w)(0,1)} = \sup_{x \in (0,1)} |w(x)f(x)|$. The cases of (weighted) continuity at the end-points of the domain are denoted by $[0,1]$ on the place of $(0,1)$, namely

$$C(w)[0,1] = \left\{ f \in C(w)(0,1) : \exists \lim_{x \to 0+0} w(x)f(x) \text{ and } \lim_{x \to 1-0} w(x)f(x) \right\},$$

$$C_0(w)[0,1] = \left\{ f \in C(w)(0,1) : \lim_{x \to 0+0} w(x)f(x) = \lim_{x \to 1-0} w(x)f(x) = 0 \right\}.$$  

The space of smooth functions considered in the paper is given by

$$W^2(w\varphi)(0,1) = \{ g, g' \in AC_{loc}(0,1) : w\varphi D^2g \in L_\infty(0,1) \},$$  

where $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,1)$ and $L_\infty(0,1)$ denotes the Lebesgue measurable and essentially bounded in $(0,1)$ functions.

In this paper we estimate the rate of weighted approximation by $\tilde{B}_n$ for functions in $C_0(w)[0,1] + \pi_1$, where $\pi_1$ is the set of all algebraical polynomials of degree 1. This space serves as a natural generalization on $C[0,1]$ for the unweighted case because $C[0,1] = C_0[0,1] + \pi_1$.

The weighted approximation error will be compared with the K-functional which for every $f \in C(w)(0,1)$ and $t > 0$ is defined by

(1.6) \quad K_w(f,t) = \inf \{ \| w(f - g) \| + t\| w\varphi D^2g \| : g \in W^2(w\varphi)(0,1) \}.

Our main result is a direct inequality. It is a generalization of the result in [3], which treats the case $w = 1$ and Goodman-Sharma operator [1] and [2].

**Theorem 1.1.** Let $w$ be given by (1.5) with $\gamma_0, \gamma_1 \in [-1,0]$. Then for every $f \in C_0(w)[0,1] + \pi_1$ and every $n \in \mathbb{N}$ we have

$$\| w(\tilde{B}_n f - f) \| \leq 2K_w \left( f, \frac{\alpha(n)}{2} \right)$$
Some remarks:
(1.) Both sides of Theorem 1.1. do not change if $f$ is replaced by $f - q$ for any $q \in \pi_1$. Hence, it is enough to prove Theorem 1.1. for functions $f \in C_0(w)[0,1]$.
(2.) Functions from $C(w)[0,1]\backslash(C_0(w)[0,1]+\pi_1)$ are not considered in Theorem 1.1. because neither $\|w(f - U_nf)\| \to 0$ nor $K_w(f, n^{-1}) \to 0$ when $n \to \infty$ for such functions.
(3.) We consider $\gamma_0, \gamma_1 \geq -1$ because functions $\tilde{B}_n(f) \in C_0(w)[0,1]$ with $\gamma_0, \gamma_1 = -1$.
(4.) We assume $\lim_{n \to \infty} \alpha(n) = 0$ because of the same reasons as in (2.).

2. Main result

We first prove four lemmas concerning any operator $L$ which is satisfying the following two conditions:

(2.1) \hspace{1cm} L \text{ is linear and positive operator;}
(2.2) \hspace{1cm} L(1, x) = 1, \ L(t, x) = x;

As a corollary from (2.1) and (2.1) we obtain the following property

(2.3) \hspace{1cm} f \leq Lf \text{ for convex function } f.

Lemma 2.1. For every function $f \in C_0(w)[0,1]$ we have $\|wL(f)\| \leq \|wf\|$, i.e. the norm of the operator is 1.

Proof.
Let we mention that function $(w)^{-1}$ is concave and then from (2.3)) we have $(w)^{-1} \geq L((w)^{-1})$. The last one, (2.1) and (2.2) give

\[
\|wL(f)\| = \|wL(wf(w)^{-1})\| \\
\leq \|wf\| \|wL((w)^{-1})\| \\
\leq \|wf\| \|w(w)^{-1}\| = \|wf\|.
\]
We define
\[ K_y(x) \overset{\text{def}}{=} \begin{cases} 
  y(x - 1) & 0 \leq y \leq x \leq 1; \\
  x(y - 1) & 0 \leq x \leq y \leq 1,
\end{cases} \]

**Lemma 2.2.** For every \( f \in W^2(w \varphi) \)
\[ L(f, x) - f(x) = \int_0^1 (L(K_y, x) - K_y(x)) f''(y) dy. \]

The above statement is Lemma 3.1 from [3].

We define \( f_w(x) = x f_0(x) + (1 - x) f_1(x) \) where
\[ f_0(x) = - \int_1^x \frac{dy}{y^{1+\gamma_0}(1 - y)^{\gamma_1}} \quad \text{and} \quad f_1(x) = - \int_0^x \frac{dy}{y^{\gamma_0}(1 - y)^{1+\gamma_1}}. \]

**Lemma 2.3.** Let \( f \in W^2(w \varphi) \), then we have
\[ \|w(Lf - f)\| \leq \|w \varphi f''\| \|w(Lf_w - f_w)\|. \]

**Proof.** The function \( K_y(x) \) is convex and nonpositive. Then from conditions 2.1 and 2.3 it follows that \( L(K_y, x) - K_y(x) \geq 0 \).

From Lemma 2.2. we have
\[ L(f, x) - f(x) = \int_0^1 \frac{L(K_y, x) - K_y(x)}{\varphi(y)} f''(y) \varphi(y) dy. \]

Taking a norm in the above equality we obtain
\[ \|w(Lf - f)\| = \left\| \int_0^1 \frac{L(K_y) - K_y}{w(y) \varphi(y)} w(y) f''(y) \varphi(y) dy \right\| \]
\[ \leq \|w \varphi f''\| \max_{x \in [0, 1]} w(x) \left( L \left( \int_0^1 \frac{K_y(x)}{w(y) \varphi(y)} dy, x \right) - \int_0^1 \frac{K_y(x)}{w(y) \varphi(y)} dy \right) \quad (2.4) \]
In the right hand side of the above inequality we have the function

\begin{align*}
(2.5) \quad \int_0^1 \frac{K_y(x)}{w(y)\varphi(y)} dy & = \int_0^x \frac{y(x-1)}{y^{1+\gamma_0}(1-y)^{1+\gamma_1}} dy + \int_x^1 \frac{x(y-1)}{y^{1+\gamma_0}(1-y)^{1+\gamma_1}} dy \\
& = -(1-x) \int_0^x \frac{dy}{y^{\gamma_0}(1-y)^{1+\gamma_1}} dy - x \int_x^1 \frac{dy}{y^{1+\gamma_0}(1-y)^{\gamma_1}} dy \\
& = xf_0(x) + (1-x)f_1(x) \\
& = f_w(x).
\end{align*}

Replacing the result of 2.5 in 2.4 we obtain

\[\|w(Lf-f)\| \leq \|w\varphi f''\| \max_{x \in [0,1]} |w(x)(L(f_w,x) - f_w(x))|\]

\[= \|w\varphi f''\| \|w(Lf_w - f_w)\|.
\]

**Lemma 2.4.**

\[\|w(Lf_w - f_w)\| \leq \|\varphi^{-1}(\cdot)L((t - \cdot)^2, \cdot)\|.
\]

**Proof.** From the definition of \(f_w, 2.1\) and 2.2 we have

\begin{align*}
(2.6) \quad 0 & \leq L(f_w,x) - f_w(x) \\
& = L(tf_1(t) + (1-t)f_0(t), x) - L(1-t,x)f_0(x) - L(t,x)f_1(x) \\
& = L((1-t)(f_0(t) - f_0(x)), x) + L(t(f_1(t) - f_1(x)), x).
\end{align*}

Expanding for \(i = 0, 1\) functions \(f_i(x + t - x)\) by Taylor's formula:

\[f_0(t) = f_0(x) - \frac{t-x}{x^{\gamma_0}(1-x)^{1+\gamma_1}} + \int_x^t (t-u)f_0''(u)du;\]

\[f_1(t) = f_1(x) + \frac{t-x}{x^{1+\gamma_0}(1-x)^{\gamma_1}} + \int_x^t (t-u)f_1''(u)du
\]

and using (from definitions of functions) that \(f_0''(u) < 0\) and \(f_1''(u) < 0\) we obtain

\begin{align*}
(2.7) \quad (1-t)(f_0(t) - f_0(x)) & \leq -\frac{(1-t)(t-x)}{x^{\gamma_0}(1-x)^{1+\gamma_1}}; \\
(2.8) \quad t(f_1(t) - f_1(x)) & \leq \frac{t(t-x)}{x^{1+\gamma_0}(1-x)^{\gamma_1}}.
\end{align*}
Applying the results of 2.7 and 2.8 in 2.6 we have

\[
0 \leq w(x) (L(f_w, x) - f_w(x)) \\
\leq w(x)L \left( -\frac{(1-t)(t-x)}{x^{\gamma_0}(1-x)^{1+\gamma}} + \frac{t(t-x)}{x^{1+\gamma_0}(1-x)^{\gamma}} \right) \\
= \varphi^{-1}(x)L \left((t-x)^2, x\right).
\]

Taking a norm in the above inequality we prove Lemma 2.4. ■

Recapitulating results from above four lemmas we obtain

**Theorem 2.1.** (Jackson-type inequality). Let \( L \) satisfies conditions 2.1 and 2.2. Then for every function \( f \in W^2(w\varphi) \) we have

\[
\|w(Lf - f)\| \leq \|w\varphi f''\| \|\varphi^{-1}(\cdot)L \left((t - \cdot)^2, \cdot\right)\|.
\]

Let we mention that 1.2 and 1.3 are the properties 2.1 and 2.2 for operators \( \tilde{B}_n \). From 1.3 and 1.4 it follows that

\[
\frac{1}{\varphi(x)}\tilde{B}_n((t-x)^2, x) = \frac{\tilde{B}_n(t^2, x) - x^2}{\varphi(x)} = \alpha(n).
\]

Above result and Theorem 2.1. give

**Theorem 2.2.** For every function \( f \in W^2(w\varphi) \) we have

\[
\|w(\tilde{B}_nf - f)\| \leq \alpha(n)\|w\varphi f''\|.
\]

Theorem 2.2. we use in the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let \( g \) is an arbitrary function in \( W^2(w\varphi) \). Then

\[
\|w(\tilde{B}_nf - f)\| \leq \|w(\tilde{B}_nf - \tilde{B}_ng)\| + \|w(\tilde{B}_ng - g)\| + \|w(g - f)\|.
\]

From Lemma 2.1. and Theorem 2.2. we get
\[ \|w(\tilde{B}_n f - f)\| \leq 2\|w(f - g)\| + \alpha(n)\|w\varphi''\| \leq 2 \left( \|w(f - g)\| + \frac{\alpha(n)}{2}\|w\varphi''\| \right). \]

Taking an infimum on all \(g \in W^2(w\varphi)\) in the above inequality we prove Theorem 1.1.

References


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