Mathematica
Balkanica

New Series Vol. 25, 2011, Fasc. 1-2

Weighted Approximation by a Class of Bernstein-Type Operators

Parvan E. Parvanov

Presented by Bl. Sendov

Direct theorem in terms of the weighted K-functional for the uniform weighted approximation errors of a class of Bernstein-type operators are obtained for functions from $C(w)[0,1]$ with weight of the form $x^{\gamma_0}(1-x)^{\gamma_1}$ for $\gamma_0, \gamma_1 \in [-1, 0]$.

AMS classification: 41A36, 41A10, 41A25, 41A17.

Keywords: Bernstein-type operator, Direct theorem, K-functional.

1. Introduction

The class of Bernstein-type operators discussed in this paper are given for natural n by

$$\tilde{B}_n(f,x) = \sum_{k=0}^{n} b_{n,k}(f)P_{n,k}(x),$$

where $P_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$ and the functionals $b_{n,k}(f)$ satisfy the following conditions

(1.1) $\qquad b_{n,0}(f) = f(0)$ and $b_{n,n}(f) = f(1)$;

(1.2) $\qquad b_{n,k}(f)$ are linear and positive;

(1.3) $\qquad \tilde{B}_n(e_i, x) = e_i(x)$ for i=0 and i=1;

(1.4) $\qquad \tilde{B}_n(e_2, x) = e_2(x) + \alpha(n)x(1-x)$.

Here e_i (for $i = 0, 1, 2$) are the functions $e_i(x) = x^i$.

The functional $b_{n,k}(f)$ for $1 \le k \le n-1$ in the operators $\tilde{B}_n$ takes place of $f\left(\frac{k}{n}\right)$ in the classical Bernstein operators [4].

Denote the weight function by

$$w(x) = w(\gamma_0, \gamma_1; x) = x^{\gamma_0}(1-x)^{\gamma_1} \text{ for } x \in (0,1) \text{ and real } \gamma_0, \gamma_1. \tag{1.5}$$

Our main results will concern the values of the powers γ_0, γ_1 in the range $[-1, 0]$. By $\varphi(x) = x(1-x)$ we denote the other weight which is naturally connected with the second derivatives of operators and the error for the function $e_2(x)$. By $D = \frac{d}{dx}$ we denote the first derivative operator.

Let $C(0,1)$ be the space of all continuous functions bounded on (0,1) and let $C(w)(0,1) = \{f : wf \in C(0,1)\}$. The norm in $C(w)(0,1)$ is given by $\|f\|_{C(w)(0,1)} = \sup_{x \in (0,1)} |w(x)f(x)|$. The cases of (weighted) continuity at the end-points of the domain are denoted by $[0,1]$ on the place of $(0,1)$, namely

$$C(w)[0,1] = \left\{ f \in C(w)(0,1) : \exists \lim_{x \to 0+0} w(x)f(x) \text{ and } \lim_{x \to 1-0} w(x)f(x) \right\},$$

$$C_0(w)[0,1] = \left\{ f \in C(w)[0,1] : \lim_{x \to 0+0} w(x)f(x) = \lim_{x \to 1-0} w(x)f(x) = 0 \right\}.$$

The space of smooth functions considered in the paper is given by

$$W^2(w\varphi)(0,1) = \left\{ g, g' \in AC_{loc}(0,1) : w\varphi D^2 g \in L_\infty(0,1) \right\},$$

where $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,1)$ and $L_\infty(0,1)$ denotes the Lebesgue measurable and essentially bounded in $(0,1)$ functions.

In this paper we estimate the rate of weighted approximation by $\tilde{B}_n$ for functions in $C_0(w)[0,1] + \pi_1$, where π_1 is the set of all algebraical polynomials of degree 1. This space serves as a natural generalization on $C[0,1]$ for the unweighted case because $C[0,1] = C_0[0,1] + \pi_1$.

The weighted approximation error will be compared with the K-functional which for every $f \in C(w)(0,1)$ and $t > 0$ is defined by

$$K_w(f,t) = \inf \left\{ \|w(f-g)\| + t\|w\varphi D^2 g\| : g \in W^2(w\varphi)(0,1) \right\}. \tag{1.6}$$

Our main result is a direct inequality. It is a generalization of the result in [3], which treats the case $w = 1$ and Goodman-Sharma operator [1] and [2].

Theorem 1.1. *Let w be given by (1.5) with $\gamma_0, \gamma_1 \in [-1, 0]$. Then for every $f \in C_0(w)[0,1] + \pi_1$ and every $n \in \mathbb{N}$ we have*

$$\|w(\tilde{B}_n f - f)\| \le 2K_w\left(f, \frac{\alpha(n)}{2}\right)$$

Some remarks:

(1.) Both sides of Theorem 1.1. do not change if f is replaced by $f-q$ for any $q \in \pi_1$. Hence, it is enough to prove Theorem 1.1. for functions $f \in C_0(w)[0,1]$.

(2.) Functions from $C(w)[0,1] \backslash (C_0(w)[0,1]+\pi_1)$ are not considered in Theorem 1.1. because neither $\|w(f-U_n f)\| \to 0$ nor $K_w\left(f, n^{-1}\right) \to 0$ when $n \to \infty$ for such functions.

(3.) We consider $\gamma_0, \gamma_1 \geq -1$ because functions $\tilde{B}_n(f) \in C_0(w)[0,1]$ with $\gamma_0, \gamma_1 = -1$.

(4.) We asume $\lim_{n\to\infty} \alpha(n) = 0$ because of the same reasons as in **(2.)**.

2. Main result

We first prove four lemmas concerning any operator L which is satisfying the following two conditions:

(2.1) $\qquad L$ *is linear and positive operator*;

(2.2) $\qquad L(1,x)=1 \;\;,\;\; L(t,x)=x;$

As a corollary from (2.1) and (2.1) we obtain the following property

(2.3) $\qquad f \leq Lf$ for convex function f.

Lemma 2.1. *For every function* $f \in C_0(w)[0,1]$ *we have* $\|wL(f)\| \leq \|wf\|$, *i.e. the norm of the operator is 1.*

Proof.

Let we mention that function $(w)^{-1}$ is concave and then from (2.3)) we have $(w)^{-1} \geq L\left((w)^{-1}\right)$. The last one, (2.1) and (2.2) give

$$\begin{aligned}\|wL(f)\| &= \|wL\left(wf(w)^{-1}\right)\| \\ &\leq \|wf\|\,\|wL\left((w)^{-1}\right)\| \\ &\leq \|wf\|\;\|w(w)^{-1}\| = \|wf\|.\end{aligned}$$

We define

$$K_y(x) \stackrel{\text{def}}{=} \begin{cases} y(x-1) & 0 \le y \le x \le 1; \\ x(y-1) & 0 \le x \le y \le 1, \end{cases}$$

Lemma 2.2. *For every $f \in W^2(w\varphi)$*

$$L(f,x) - f(x) = \int_0^1 \left(L(K_y, x) - K_y(x)\right) f''(y)dy.$$

The above statement is Lemma 3.1 from [3] .

We define $f_w(x) = x f_0(x) + (1-x) f_1(x)$ where

$$f_0(x) = -\int_x^1 \frac{dy}{y^{1+\gamma_0}(1-y)^{\gamma_1}} \quad \text{and} \quad f_1(x) = -\int_0^x \frac{dy}{y^{\gamma_0}(1-y)^{1+\gamma_1}}.$$

Lemma 2.3. *Let $f \in W^2(w\varphi)$, then we have*

$$\|w(Lf - f)\| \le \|w\varphi f''\| \, \|w(Lf_w - f_w\|.$$

Proof. The function $K_y(x)$ is convex and nonpositive. Then from conditions 2.1 and 2.3 it follows that $L(K_y, x) - K_y(x) \ge 0$.

From Lemma 2.2. we have

$$L(f,x) - f(x) = \int_0^1 \frac{L(K_y, x) - K_y(x)}{\varphi(y)} f''(y)\varphi(y)dy.$$

Taking a norm in the above equality we obtain

$$\|w(Lf - f)\| = \left\| w \int_0^1 \frac{L(K_y) - K_y}{w(y)\varphi(y)} w(y) f''(y)\varphi(y)dy \right\|$$

$$\le \|w\varphi f''\| \max_{x\in[0,1]} \left| w(x) \left(L\left(\int_0^1 \frac{K_y(x)}{w(y)\varphi(y)} dy, x \right) - \int_0^1 \frac{K_y(x)}{w(y)\varphi(y)} dy \right) \right|. \tag{2.4}$$

In the right hand side of the above inequality we have the function

$$\int_0^1 \frac{K_y(x)}{w(y)\varphi(y)}dy = \int_0^x \frac{y(x-1)}{y^{1+\gamma_0}(1-y)^{1+\gamma_1}}dy + \int_x^1 \frac{x(y-1)}{y^{1+\gamma_0}(1-y)^{1+\gamma_1}}dy \tag{2.5}$$

$$= -(1-x)\int_0^x \frac{dy}{y^{\gamma_0}(1-y)^{1+\gamma_1}}dy - x\int_x^1 \frac{dy}{y^{1+\gamma_0}(1-y)^{\gamma_1}}dy$$

$$= xf_0(x) + (1-x)f_1(x)$$

$$= f_w(x).$$

Replacing the result of 2.5 in 2.4 we obtain

$$\begin{aligned} \|w(Lf-f)\| &\le \|w\varphi f''\| \max_{x\in[0,1]} |w(x)\left(L(f_w,x) - f_w(x)\right)| \\ &= \|w\varphi f''\|\, \|w(Lf_w - f_w)\|. \end{aligned}$$

■

Lemma 2.4.

$$\|w(Lf_w - f_w)\| \le \|\varphi^{-1}(\cdot)L\left((t-\cdot)^2,\cdot\right)\|.$$

Proof. From the definition of f_w, 2.1 and 2.2 we have

$$\begin{aligned} 0 &\le L(f_w,x) - f_w(x) \\ &= L\left(tf_1(t) + (1-t)f_0(t), x\right) - L(1-t,x)f_0(x) - L(t,x)f_1(x) \\ &= L\left((1-t)\left(f_0(t) - f_0(x)\right), x\right) + L\left(t\left(f_1(t) - f_1(x)\right), x\right). \end{aligned} \tag{2.6}$$

Expanding for $i = 0, 1$ functions $f_i(x+t-x)$ by Taylor's formula:

$$f_0(t) = f_0(x) - \frac{t-x}{x^{\gamma_0}(1-x)^{1+\gamma_1}} + \int_x^t (t-u)f_0''(u)du\,;$$

$$f_1(t) = f_1(x) + \frac{t-x}{x^{1+\gamma_0}(1-x)^{\gamma_1}} + \int_x^t (t-u)f_1''(u)du$$

and using (from definitions of functions) that $f_0''(u) < 0$ and $f_1''(u) < 0$ we obtain

$$(1-t)\left(f_0(t) - f_0(x)\right) \le -\frac{(1-t)(t-x)}{x^{\gamma_0}(1-x)^{1+\gamma_1}}\,; \tag{2.7}$$

$$t\left(f_1(t) - f_1(x)\right) \le \frac{t(t-x)}{x^{1+\gamma_0}(1-x)^{\gamma_1}}. \tag{2.8}$$

Applying the results of 2.7 and 2.8 in 2.6 we have

$$\begin{aligned}
0 &\le w(x)\left(L(f_w,x)-f_w(x)\right)\\
&\le w(x)L\left(-\frac{(1-t)(t-x)}{x^{\gamma_0}(1-x)^{1+\gamma_1}}+\frac{t(t-x)}{x^{1+\gamma_0}(1-x)^{\gamma_1}},x\right)\\
&=\varphi^{-1}(x)L\left((t-x)^2,x\right).
\end{aligned}$$

Taking a norm in the above inequality we prove Lemma 2.4. ∎

Recapitulating results from above four lemmas we obtain

Theorem 2.1. *(Jackson-type inequality). Let L satisfies conditions 2.1 and 2.2. Then for every function $f \in W^2(w\varphi)$ we have*

$$\|w(Lf-f)\| \le \|w\varphi f''\| \ \|\varphi^{-1}(\cdot)L\left((t-\cdot)^2,\cdot\right)\|.$$

Let we mention that 1.2 and 1.3 are the properties 2.1 and 2.2 for operators $\tilde{B}_n$. From 1.3 and 1.4 it follows that

$$\frac{1}{\varphi(x)}\tilde{B}_n((t-x)^2,x)=\frac{\tilde{B}_n(t^2,x)-x^2}{\varphi(x)}=\alpha(n).$$

Above result and Theorem 2.1. give

Theorem 2.2. *For every function $f \in W^2(w\varphi)$ we have*

$$\|w(\tilde{B}_n f-f)\| \le \alpha(n)\|w\varphi f''\|.$$

Theorem 2.2. we use in the proof of Theorem 1.1.

P r o o f o f T h e o r e m 1.1. Let g is an arbitrary function in $W^2(w\varphi)$. Then

$$\|w(\tilde{B}_n f-f)\| \le \|w(\tilde{B}_n f-\tilde{B}_n g)\|+\|w(\tilde{B}_n g-g)\|+\|w(g-f)\|.$$

From Lemma 2.1. and Theorem 2.2. we get

$$\|w(\tilde{B}_n f - f)\| \le 2\|w(f-g)\| + \alpha(n)\|w\varphi g''\| \le 2\left(\|w(f-g)\| + \frac{\alpha(n)}{2}\|w\varphi g''\|\right).$$

Taking an infimum on all $g \in W^2(w\varphi)$ in the above inequality we prove Theorem 1.1. ■

References

[1] T. N. T. Goodman and A. Sharma, A Bernstein-type Operator on the Simplex. *Mathematica Balkanica (new series)*, **5,2,** 1991 , 129-145.

[2] T. N. T. Goodman and A. Sharma, A Modified Bernstein - Shoenberg Operator. *Proc. of Conference on Constructive Theory of Functions, Varna'87, Publ. House of Bulg. Acad. of Sci., Sofia,* 1987, 166-173.

[3] P. E. Parvanov and B. D. Popov, The Limit Case of Bernstein's Operators with Jacobi-weights. *Mathematica Balkanica (new series)*, **8** , 1994, 165-177.

[4] G. G. Lorentz, *Bernstein Polinonials.* Mathematical Expositions, 8, University of Toronto Press, (1953).

Dept. of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia, Bulgaria
E-mail : pparvan@fmi.uni-sofia.bg

Received 03.03.2010