Mathematica
Balkanica

New Series Vol. 25, 2011, Fasc. 1-2

Long Time Approximations for Solutions of Evolution Equations from Quasimodes: Perturbation Problems [1]

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Considering first and second order evolution problems associated with self-adjoint compact operators on Hilbert spaces, we provide estimates for the time t in which functions of the type $e^{\pm\mu t}u$ and $e^{i\sqrt{\mu}t}u$ or $e^{\pm\sqrt{\mu}t}u$ approach their solutions $\mathbf{u}(t)$ when the initial data deal with *quasimodes* (u,μ) of the operators under consideration; μ is a positive number. We establish a general framework which involves the case where operators, spaces and quasimodes depend on a small parameter of perturbation ε, $\varepsilon > 0$. We apply the results to operators arising in a spectral boundary homogenization problem, and we highlight time-dependent solutions concentrating their support asymptotically (as $\varepsilon \to 0$) along lines for long times; ε measures the periodicity of the structure.

AMS Subj. Classification: 47A10, 47D06, 35B25, 35B27

Key Words: spectral analysis, evolution equations, asymptotic analysis, boundary homogenization.

1 Introduction

Let $A : H \longrightarrow H$ be a linear, self-adjoint, non-negative, and compact operator on a separable Hilbert space H; a *quasimode with remainder* $r > 0$ for the operator A is a pair $(u, \mu) \in H \times \mathbb{R}$, with $\|u\|_H = 1$ and $\mu > 0$, such that $\|Au - \mu u\|_H \leq r$. The closeness of a quasimode in the space $H \times \mathbb{R}$ to the eigenelements of the operator A is provided by Lemma 2.1: for "small"

[1] Partially supported by the Spanish MICINN: MTM2009-12628

r, u approaches a certain linear combination of eigenfunctions associated with the eigenvalues of A in "small" intervals containing $[\mu - r, \mu + r]$. Throughout the text, if no confusion arises, we shall refer to u as the *quasimode*, which is associated with the *almost eigenfrequency* μ and the *reminder* r.

The value of quasimodes in describing asymptotics for low and high frequency vibrations in certain singularly perturbed spectral problems, which depend on a small positive parameter ε, has been highlighted recently in many papers: see Lobo&Pérez [8], Lobo&Pérez [10], Pérez [15], and Sanchez-Hubert [16] in this respect. For these problems, the spaces and the operators under consideration depend on the parameter of perturbation ε, and also the quasimode function u, the almost eigenfrequency μ and the reminder r may depend on this parameter; namely $(u,\ \mu,\ r) \equiv (u^\varepsilon,\ \mu^\varepsilon,\ r^\varepsilon)$, $\varepsilon \to 0$. Obtaining approaches to eigenfunctions individually becomes difficult and constructing quasimodes $(u^\varepsilon,\ \mu^\varepsilon)$ for spectral problems in certain spaces allows us to construct *standing waves* of the type $e^{\mathrm{i}\sqrt{\mu^\varepsilon}\,t}u^\varepsilon$ which approach for long times the solutions $\mathbf{u}^\varepsilon(t)$ of the associated wave equations when the initial data are related to $(u^\varepsilon,\ \mu^\varepsilon)$.

In this connection, certain approaches and estimates for vibrating systems have been obtained in Pérez [15], and Lobo&Pérez [10]: this involves considering the associated wave equations, and hence, dealing with unbounded operators on $\mathcal{H}$ in the general setting of spectral problems for sesquilinear, continuous, symmetric and coercive forms on $\mathcal{V}$; $\mathcal{V}$ and $\mathcal{H}$ being two separable Hilbert spaces, $\mathcal{V} \subset \mathcal{H}$ with a dense and compact imbedding.

However, not all spectral problems can be set in the above-mentioned framework. In this paper, we provide a new abstract framework for the case of first and second order evolution problems associated with bounded operators on Hilbert spaces, and more specifically, non-negative, self-adjoint and compact operators A. The results are different from those in previous papers: different second order evolution problems are considered and reduced to systems of equations of first order which do not involve, in general, semigroups of contractions as is the case in Lobo&Pérez [10] and Pérez [15] (cf. (2.5), (2.7), (2.17) and Remark 4. 7.)). See Dautray&Lions [4], Engel&Nagel [5], Kato [6], and Sanchez-Hubert&Sanchez-Palencia [16] for a general theory and for examples of applications.

By analogy with wave equations, throughout the paper, we refer to products of functions $u(x)\tau(t)$ as *standing waves* and we state their connection with solutions of certain evolution problems; more precisely, their relation with solutions of these problems when the initial data are related to quasimodes. Often in applications, approaches to eigenfunctions are provided by quasimodes which can be constructed explicitly (cf. Arnold [1], Babich&Buldyrev [2], and Pérez

[13], for instance). If so, from the quasimodes, it can be useful to construct certain functions which depend on the spatial and time variables separately and approach solutions of the evolution problems. Note that these functions are not solutions of the homogeneous differential equation, but we prove that they behave as *elementary solutions* for long times, and during this period of time (which we determine) they simultaneously approach elementary solutions (which are unknown) and unknown solutions which cannot be obtained by separation of variables. This is one of the aims of this paper (see Section 2). We show that the period of time depends on the problem, the numbers μ and r arising in the definition of the quasimodes, the above mentioned closeness of the quasimodes to the eigenelements of A, and new parameters related with the norm or the accretivity of the operators under consideration (cf. Remark 2.3, and Remarks 3.1, 4.2 and 4.3. when there is also dependence on the perturbation parameter ε). We state the results for ε-dependent evolution problems and apply them to a specific problem (see Sections 3–4).

The structure of the paper is as follows: The statements of the general results are given in Sections 2.1 and 2.2 for second order evolution problems; their proofs are in Section 2.3. In Section 2.1 (Section 2.2, respectively) we consider problems with the elementary solutions which are trigonometric functions (exponential functions, respectively) of the time variable. Section 2.4 contains the results for first order evolution problems. All these results allow a wide range of applications when considering problems depending on a perturbation parameter, and, for the sake of completeness, in Section 3 we summarize the abstract framework and the results which can be applied to singularly perturbed spectral problems (see Theorems 3.1–3.4 and Remarks 3.1 - 3.2.

In Section 4, we apply the results of Section 3 to a family of compact operators arising in a homogenization problem which is related to models in Geophysics (cf. for instance Dascalu&Ionescu [3] and Pérez [14]). We introduce the spectral boundary homogenization problem (4.4) and the associated ε-dependent operators (4.6). Problem (4.4) is a spectral problem posed in a domain of $\mathbb{R}^2$ with strongly alternating boundary conditions of the Stkelov type on a part of the boundary Σ; the perturbation parameter ε measures the periodicity of the structure on Σ. In Section 4.1 we gather the results on the quasimodes constructed in Pérez [14]. In Section 4.2 we provide the standing waves approaching the solutions of the evolution problem (4.17), which concentrate asymptotically their support along Σ. Near this part of the boundary, the standing waves are also strongly oscillating functions (cf. Remarks 4.1 and 4.6). Finally, in this section, we derive the bounds for the discrepancies between solutions and standing waves, and bounds for the time in which the standing waves provide true approaches, that is, the time in which they behave as elementary solutions (see Remarks 4.2– 4.4.). All these bounds are well determined in terms of ε.

We observe that other evolution problems and standing waves associated with (4.5), different from those in this paper, have been considered in Lobo&Pérez [10]. As a matter of fact, the formulation in Lobo&Pérez [10] is within the unbounded operators on fractional Sobolev spaces, and obtaining estimates uses results which are extensions of those in Pérez [15]. This is not possible in the framework of the present paper, and the boundary homogenization problem serves as a sample for applying the wide variety of results in Sections 2 and 3, and to show the value of these results. In this respect, see Remarks 4.7– 4.10.

It should be emphasized that although some bounds for the discrepancies between approximations might be obtained from the theory of semigroups, these bounds may be unknown (cf. Remark 2.2). In addition, our technique is well focused for applications of the results in singularly perturbed spectral problems which have been addressed recently in many papers (cf. Lobo&Pérez [8], Pérez [15], and Sanchez-Hubert&Sanchez-Palencia [16] for further references). There is a lack of bibliography in the literature of applied mathematics on the associated time-dependent perturbed problems here considered. Also the technique makes it possible to highlight resonance phenomena for non-homogeneous differential equations when the non-homogeneous terms involve quasimodes and almost eigenfrequencies. In fact, for problems arising in spectral perturbation theory, the quasimodes often allow us to detect time-dependent solutions concentrating their support asymptotically near points, lines or on certain regions of the space under consideration for long times (cf. also Pérez [15] and Lobo&Pérez [10] in this connection). In this paper, we stress the concentration of supports along a line (see Remark 4.6).

2 General setting of the problems

The results in Section 9 of Visik&Lusternik [17] (cf. Lemmas 12 and 13) and in Section 2 of Lazutkin [7] (cf. Theorem 1) establish the closeness in the space $H \times \mathbb{R}$ of the eigenelements of the operator A defined on H to a given quasimode of A (see the definition in Section 1). For the case of a linear, nonnegative, self-adjoint and compact operator, the results in Visik&Lusternik [17] and Lazutkin [7] can be stated as follows:

Lemma 2.1 *Let $A : H \longrightarrow H$ be a linear, non-negative, self-adjoint, and compact operator on a separable Hilbert space H. Let (u, μ) be a given a quasimode for the operator A of remainder r. Then, in each interval $[\mu - r^*, \mu + r^*]$ containing $[\mu - r, \mu + r]$, under the assumption that the spectrum of A is discrete in both intervals, there are eigenvalues of A, $\{\lambda_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)} \subset [\mu - r^*, \mu + r^*]$ for some index $i(r^*)$ and some natural number $I(r^*) \geq 1$. In addition, there is $u^* \in H$,*

$$\|u^*\|_H = 1, \quad u^* \in \left[u_{i(r^*)+1}, u_{i(r^*)+2}, \cdots, u_{i(r^*)+I(r^*)}\right], \quad u^* = \sum_{k=1}^{I(r^*)} \alpha_k u_{i(r^*)+k}, \tag{2.1}$$

satisfying

$$\|u - u^*\|_H = \left\| u - \sum_{k=1}^{I(r^*)} \alpha_k u_{i(r^*)+k} \right\|_H \leq \frac{2r}{r^*}. \tag{2.2}$$

Here $\{u_{i(r^*)+k}\}_{k=1}^{I(r^*)}$ are the eigenfunctions of A associated with $\{\lambda_{i(r^*)+k}\}_{k=1}^{I(r^*)}$, which we assume orthonormal in H.

In order to introduce a general framework for first and second order evolution problems associated with the operator arising in Lemma 2.1, throughout the section we consider A a linear, compact, self-adjoint and non-negative operator on the separable Hilbert space H.

Let us consider the bounded operator $\mathcal{A}$ defined on $\mathbf{H} = H \times H$ by

$$\begin{pmatrix} 0 & , & \pm I \\ A & , & 0 \end{pmatrix} \tag{2.3}$$

depending on the sign accompanying the identity operator I on H. Then, for any $\bar{\varphi} \in \mathbf{H}$, the evolution problem

$$\left\{ \begin{array}{rcl} \dfrac{d\bar{\mathbf{u}}}{dt} + \mathcal{A}\bar{\mathbf{u}} & = & 0 \\ \bar{\mathbf{u}}(0) & = & \bar{\varphi} \end{array} \right. \tag{2.4}$$

has a unique solution $\bar{\mathbf{u}}(t)$, $\bar{\mathbf{u}} \in C^1([0,T];\mathbf{H})$ for any positive T, which satisfies

$$\|\bar{\mathbf{u}}(t)\|_{\mathbf{H}} \leq e^{\omega t}\|\bar{\varphi}\|_{\mathbf{H}}, \quad \forall t \geq 0, \tag{2.5}$$

for a certain $\omega \geq 0$, $\omega \leq \|\mathcal{A}\|_{\mathcal{L}(\mathbf{H})}$ (cf. Section IX in [6], for instance).

In Sections 2.1 and 2.2 we introduce different second order evolution problems associated with A: see (2.6) and (2.16). The results in Section 2.1

(Section 2.2, respectively) show the connection between quasimodes (u, μ) of the operator A and standing waves of the type $e^{\mathrm{i}\sqrt{\mu}\,t}u$ ($e^{\pm\sqrt{\mu}\,t}u$, respectively) approaching solutions of (2.6) ((2.16), respectively). We emphasize that here we not deal with sesquilinear, continuous and coercive forms, and statements and results are different from those in Lobo&Pérez [10] and Pérez [15], as has already been outlined in Section 1. Section 2.3 contains the proofs of these results, while the results for first order evolution problems associated with A (see (2.39)) are in Section 2.4.

2.1 Approximations from time-dependent trigonometric functions

Following the notations above, for A a compact, self-adjoint and non-negative operator on the Hilbert space H, let us consider the operator $\mathcal{A}$ defined on $\mathbf{H} = H \times H$ by (2.3) with the negative sign accompanying I.

Let us introduce the second order evolution problem

$$(2.6) \qquad \begin{cases} \dfrac{d^2\mathbf{u}}{dt^2} + A\mathbf{u} &= 0 \\ \mathbf{u}(0) &= \varphi \\ \dfrac{d\mathbf{u}}{dt}(0) &= \psi \end{cases}$$

for initial data $(\varphi, \psi) \in H \times H$.

It suffices to consider the change $\mathbf{u_1}(t) = \mathbf{u}(t)$, $\mathbf{u_2}(t) = \dfrac{d\mathbf{u}}{dt}(t)$ in (2.6) to write

$$\frac{d\mathbf{u_1}}{dt}(t) = \mathbf{u_2}(t), \quad \frac{d\mathbf{u_2}}{dt}(t) = -A\mathbf{u_1}(t)$$

and to reduce problem (2.6) to (2.4) for $\bar{\varphi} = (\varphi, \psi) \in H \times H$. The uniqueness of solution $\mathbf{u}(t)$ of (2.6), $\mathbf{u} \in C^2([0,T]; H)$ for any positive T, holds from the uniqueness of solution of (2.4), while inequality (2.5) is improved in the following lemma.

Lemma 2.2 *For $\bar{\varphi} = (\varphi, \psi) \in H \times H$ the solution $\mathbf{u}(t)$ of (2.6) satisfies*

$$(2.7) \quad \|\mathbf{u}(t)\|_H^2 + \left\|\frac{d\mathbf{u}}{dt}(t)\right\|_H^2 \leq 2\left(1 + \|A\|_{\mathcal{L}(H)} + t^2\right)\left(\|\varphi\|_H^2 + \|\psi\|_H^2\right), \quad \forall t \geq 0.$$

Now, considering $\lambda_i > 0$ an eigenvalue of the operator A, it is self-evident that, for a given $\varphi = \alpha u_i$ or $\psi = \beta u_i$, with α, β any constants, u_i any eigenfunction of A associated with the eigenvalue λ_i, the elementary solution of (2.6) is the standing wave

$$\mathbf{u}(t) = \left(\alpha \cos(\sqrt{\lambda_i}\, t) + \beta \frac{\sin(\sqrt{\lambda_i}\, t)}{\sqrt{\lambda_i}} \right) u_i \,.$$

Similarly, for $\varphi = \sum_{k=1}^{I(r^*)} a_k u_{i(r^*)+k}$ and $\psi = \sum_{k=1}^{I(r^*)} b_k u_{i(r^*)+k}$, with a_k, b_k constants, the solution of (2.6) is given by the sum of standing waves

$$\mathbf{u}(t) = \sum_{k=1}^{I(r^*)} \left(a_k \cos(\sqrt{\lambda_{i(r^*)+k}}\, t) + b_k \frac{\sin(\sqrt{\lambda_{i(r^*)+k}}\, t)}{\sqrt{\lambda_{i(r^*)+k}}} \right) u_{i(r^*)+k} \,. \tag{2.8}$$

Also, for solutions of problem (2.6)

$$\mathbf{u}(t) = \sum_{k=1}^{I(r^*)} a_k \left(\cos(\sqrt{\lambda_{i(r^*)+k}}\, t) + \sin(\sqrt{\lambda_{i(r^*)+k}}\, t) \right) u_{i(r^*)+k} \tag{2.9}$$

associated with complex solutions of the form $\mathbf{u}(t) = \sum_{k=1}^{I(r^*)} a_k e^{i\sqrt{\lambda_{i(r^*)+k}}\, t} u_{i(r^*)+k}$, the corresponding initial data are

$$\varphi = \sum_{k=1}^{I(r^*)} a_k u_{i(r^*)+k} \quad \text{and} \quad \psi = \sum_{k=1}^{I(r^*)} a_k \sqrt{\lambda_{i(r^*)+k}}\, u_{i(r^*)+k} \,.$$

In the case where (u, λ) is a quasimode of the operator A with remainder r, and $(\varphi, \psi) = (0, u)$ or $(\varphi, \psi) = (u, 0)$, the following results provide the relation between the solution of (2.6) and the standing waves $(\sqrt{\lambda})^{-1} \sin(\sqrt{\lambda}\, t) u$ or $\cos(\sqrt{\lambda}\, t) u$ respectively.

Theorem 2.1 *Let (u, λ) be a quasimode with remainder r of the operator A arising in (2.6). Let $\{\lambda_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ and $\{u_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ be the eigenvalues and the associated eigenfunctions of the operator A satisfying (2.1)-(2.2). Let us assume that $r^* > r$ and $\lambda - r^* > 0$. Then, for $\varphi = 0$, $\psi = u$, the solution $\mathbf{u}(t)$ of (2.6) satisfies*

$$\left\| \mathbf{u}(t) - \sum_{k=1}^{I(r^*)} \alpha_k \frac{\sin(\sqrt{\lambda_{i(r^*)+k}}\, t)}{\sqrt{\lambda_{i(r^*)+k}}} u_{i(r^*)+k} \right\|_H^2 +$$

$$\left\| \frac{d\mathbf{u}}{dt}(t) - \sum_{k=1}^{I(r^*)} \alpha_k \cos(\sqrt{\lambda_{i(r^*)+k}}\, t) u_{i(r^*)+k} \right\|_H^2 \leq$$

$$2\left(1 + \|A\|_{\mathcal{L}(H)} + t^2\right)\left(\frac{2r}{r^*}\right)^2, \quad \forall t \geq 0. \tag{2.10}$$

In addition, for any $t > 0$, we have

$$\left\| \frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}} u - \mathbf{u}(t) \right\|_H \leq \widetilde{C} \max\left(\frac{2r}{r^*}(C_A + t), \frac{2r}{r^*\sqrt{\lambda}}, \frac{1}{\sqrt{\lambda}}(\sqrt{r^*}\, t + \frac{\sqrt{r^*}}{\sqrt{\lambda} - \sqrt{r^*}}) \right) \tag{2.11}$$

and

$$\left\| \cos(\sqrt{\lambda}\, t) u - \frac{d\mathbf{u}}{dt}(t) \right\|_H \leq \widetilde{C} \max\left(\frac{2r}{r^*}(C_A + t), \frac{2r}{r^*}, \sqrt{r^*}\, t \right), \tag{2.12}$$

where $\widetilde{C}$ and C_A are constants independent of λ, t, r and r^*, $C_A = (1 + \|A\|_{\mathcal{L}(H)})^{1/2}$. Also, the bounds (2.11) and (2.12) hold for the discrepancy

$$\sum_{k=1}^{I(r^*)} \alpha_k \frac{\sin(\sqrt{\lambda_{i(r^*)+k}}\, t)}{\sqrt{\lambda_{i(r^*)+k}}} u_{i(r^*)+k} - \frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}} u$$

and its time derivative.

Theorem 2.2 *Let us consider the assumptions in Theorem 2.1. Then, for $\varphi = u$, $\psi = 0$, the solution $\mathbf{u}(t)$ of (2.6) satisfies*

$$\left\| \mathbf{u}(t) - \sum_{k=1}^{I(r^*)} \alpha_k \cos(\sqrt{\lambda_{i(r^*)+k}}\, t) u_{i(r^*)+k} \right\|_H^2 +$$

$$\left\| \frac{d\mathbf{u}}{dt}(t) + \sum_{k=1}^{I(r^*)} \alpha_k \sqrt{\lambda_{i(r^*)+k}} \sin(\sqrt{\lambda_{i(r^*)+k}}\, t) u_{i(r^*)+k} \right\|_H^2 \leq$$

$$2\left(1 + \|A\|_{\mathcal{L}(H)} + t^2\right)\left(\frac{2r}{r^*}\right)^2, \quad \forall t \geq 0. \tag{2.13}$$

In addition, for any $t > 0$, we have

$$\left\| \cos(\sqrt{\lambda}\, t) u - \mathbf{u}(t) \right\|_H \leq \widetilde{C} \max\left(\frac{2r}{r^*}(C_A + t), \frac{2r}{r^*}, \sqrt{r^*}\, t \right) \tag{2.14}$$

and

$$\left\|\sqrt{\lambda}\sin(\sqrt{\lambda}\,t)u+\frac{d\mathbf{u}}{dt}(t)\right\|_H \leq \widetilde{C}\max\left(\frac{2r}{r^*}(C_A+t),\ \frac{2r}{r^*}\sqrt{\lambda},\ (\sqrt{\lambda}\sqrt{r^*}\,t+\sqrt{r^*}\,)\right),$$

(2.15)

where $\widetilde{C}$ *and* C_A *are constants independent of* λ, t, r *and* r^*, $C_A = (1+\|A\|_{\mathcal{L}(H)})^{1/2}$. *Also, the bounds (2.14) and (2.15) hold for the discrepancy*

$$\sum_{k=1}^{I(r^*)}\alpha_k\cos(\sqrt{\lambda_{i(r^*)+k}}\,t)u_{i(r^*)+k}-\cos(\sqrt{\lambda}\,t)u$$

and its time derivative.

Under the hypotheses in Theorems 2.1 and 2.2, giving the initial data $\varphi = u$, $\psi = \sqrt{\lambda}u$ leads us to obtain approximations for solutions of (2.6) via the function $\cos(\sqrt{\lambda}t)u + \sin(\sqrt{\lambda}t)u$ (see (2.9) and bounds in Section 2.2). The same initial data for a different problem are considered in Theorem 2.3. Also, extensions of the results in Theorems 2.1 and 2.2 for more general initial data can be obtained combining the results in both theorems (see (2.8)): namely, for $\varphi = u$ and $\psi = v$, where (u, λ_1) and (v, λ_2) are quasimodes of the operator A, approaches to solutions of (2.6) are given by sums of standing waves of different frequencies, namely $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$.

2.2 Approximations from time-dependent exponential functions

In this section, we show similar results to those in Section 2.1 for standing waves of the type $e^{\pm\sqrt{\lambda}\,t}u$; we follow the structure of Section 2.1.

For A a compact, self-adjoint and non-negative operator on the Hilbert space H, let us consider the operator $\mathcal{A}$ defined on $\mathbf{H} = H \times H$ by (2.3) with the positive sign accompanying I. Let us introduce the second order evolution problem

$$\left\{\begin{array}{rcl}\dfrac{d^2\mathbf{u}}{dt^2}-A\mathbf{u} & = & 0\\ \mathbf{u}(0) & = & \varphi\\ \dfrac{d\mathbf{u}}{dt}(0) & = & \psi\end{array}\right. \tag{2.16}$$

for initial data $(\varphi, \psi) \in H \times H$.

Performing the change $\mathbf{u_1}(t) = \mathbf{u}(t)$, $\mathbf{u_2}(t) = -\dfrac{d\mathbf{u}}{dt}(t)$ in (2.16), we can write

$$\frac{d\mathbf{u_1}}{dt}(t) = -\mathbf{u_2}(t),\quad \frac{d\mathbf{u_2}}{dt}(t) = -A\mathbf{u_1}(t)$$

and reduce problem (2.16) to (2.4) for $\bar{\varphi} = (\varphi, -\psi) \in H \times H$. Therefore, the uniqueness of solution $\mathbf{u}(t)$ of (2.16), $\mathbf{u} \in C^2([0,T]; H)$ for any positive T, $\mathbf{u}$ satisfying the inequality

$$\|\mathbf{u}(t)\|_H^2 + \left\|\frac{d\mathbf{u}}{dt}(t)\right\|_H^2 \leq 4e^{2\omega t}\left(\|\varphi\|_H^2 + \|\psi\|_H^2\right), \quad \forall t \geq 0, \tag{2.17}$$

for a certain constant $\omega > 0$, holds from the uniqueness of solution of (2.4) and from (2.5) (cf. Lemma 2.2 and Remark 2.1 to compare).

Considering $\lambda_i > 0$ an eigenvalue of the operator A associated with the eigenfunction u_i, it is self-evident that the solution of (2.16), for the initial data $\varphi = u_i$, $\psi = \pm\sqrt{\lambda_i}\, u_i$, is given by the standing wave:

$$\mathbf{u}(t) = e^{\pm\sqrt{\lambda_i}\, t} u_i, \quad \forall t > 0.$$

Similar considerations to those in (2.7)–(2.9) can be outlined by replacing sines and cosines by exponential functions.

In the case where (u, λ) is a quasimode of the operator A with remainder r, and $(\varphi, \psi) = (u, \pm\sqrt{\lambda}\, u) \in H \times H$, the following theorem provides the relation between the solution of (2.16) and the standing wave $\mathbf{u}(t) = e^{\pm\sqrt{\lambda}\, t} u$.

Theorem 2.3 *Let (u, λ) be a quasimode with remainder r of the operator A arising in (2.16). Let $\{\lambda_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ and $\{u_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ be the eigenvalues and the associated eigenfunctions of the operator A satisfying (2.1)-(2.2). Let us assume that $r^* > r$ and $\lambda - r^* > 0$. Then, for $\varphi = u$, $\psi = \pm\sqrt{\lambda}\, u$, the solution $\mathbf{u}(t)$ of (2.16) satisfies*

$$\left\|\mathbf{u}(t) - \sum_{k=1}^{I(r^*)} \alpha_k e^{\pm\sqrt{\lambda_{i(r^*)+k}}\, t} u_{i(r^*)+k}\right\|_H^2 +$$

$$\left\|\frac{d\mathbf{u}}{dt}(t) \mp \sum_{k=1}^{I(r^*)} \alpha_k \sqrt{\lambda_{i(r^*)+k}}\, e^{\pm\sqrt{\lambda_{i(r^*)+k}}\, t} u_{i(r^*)+k}\right\|_H^2 \leq$$

$$4e^{2\omega t}\left(\left(\frac{2r}{r^*}\right)^2 (1+\lambda) + r^*\right), \quad \forall t \geq 0. \tag{2.18}$$

Here, each one of the signs $\pm$ implies a chosen sign for the initial data.

In addition:

i). For $(\varphi, \psi) = (u, -\sqrt{\lambda} u)$, and for any $t > 0$, we have the estimates

$$\left\|e^{-\sqrt{\lambda}\, t} u - \mathbf{u}(t)\right\|_H \leq \widetilde{C} \max\left(\left(\frac{2r}{r^*}(1+\sqrt{\lambda}) + \sqrt{r^*}\right) e^{\omega t}, \sqrt{r^*}\, t\right) \tag{2.19}$$

and

$$\left\| \sqrt{\lambda} e^{-\sqrt{\lambda} t} u + \frac{d\mathbf{u}}{dt}(t) \right\|_H \leq$$

$$\widetilde{C} \max \left((\frac{2r}{r^*}(1+\sqrt{\lambda}) + \sqrt{r^*})\, e^{\omega t},\ \frac{2r}{r^*}\sqrt{\lambda},\ (\sqrt{\lambda}+\sqrt{r^*})\sqrt{r^*}\, t + \sqrt{r^*} \right) \tag{2.20}$$

which also hold for the norms in H of the functions

$$\sum_{k=1}^{I(r^*)} \alpha_k e^{-\sqrt{\lambda_{i(r^*)+k}}\, t} u_{i(r^*)+k} - e^{-\sqrt{\lambda} t} u$$

and

$$\sum_{k=1}^{I(r^*)} \alpha_k \sqrt{\lambda_{i(r^*)+k}} e^{-\sqrt{\lambda_{i(r^*)+k}}\, t} u_{i(r^*)+k} - \sqrt{\lambda} e^{-\sqrt{\lambda} t} u .$$

ii). For $(\varphi, \psi) = (u, \sqrt{\lambda} u)$, and for any $t > 0$, we have the estimates

$$\left\| e^{\sqrt{\lambda} t} u - \mathbf{u}(t) \right\|_H \leq \widetilde{C} \max \left((\frac{2r}{r^*}(1+\sqrt{\lambda}) + \sqrt{r^*}) e^{\omega t},\ \frac{2r}{r^*} e^{\sqrt{\lambda} t},\ \sqrt{r^*}\, t e^{(\sqrt{\lambda}+\sqrt{r^*})\, t} \right) \tag{2.21}$$

and

$$\left\| \sqrt{\lambda} e^{\sqrt{\lambda} t} u - \frac{d\mathbf{u}}{dt}(t) \right\|_H \leq \widetilde{C} \max \Big((\frac{2r}{r^*}(1+\sqrt{\lambda}) + \sqrt{r^*})\, e^{\omega t},\ \frac{2r}{r^*} \sqrt{\lambda}\, e^{\sqrt{\lambda} t},$$

$$(\sqrt{\lambda}+\sqrt{r^*})\sqrt{r^*}\, t e^{(\sqrt{\lambda}+\sqrt{r^*})\, t} + \sqrt{r^*} e^{\sqrt{\lambda} t} \Big) \tag{2.22}$$

which also hold for the norms in H of the functions

$$\sum_{k=1}^{I(r^*)} \alpha_k e^{\sqrt{\lambda_{i(r^*)+k}}\, t} u_{i(r^*)+k} - e^{\sqrt{\lambda} t} u$$

and

$$\sum_{k=1}^{I(r^*)} \alpha_k \sqrt{\lambda_{i(r^*)+k}} e^{\sqrt{\lambda_{i(r^*)+k}}\, t} u_{i(r^*)+k} - \sqrt{\lambda} e^{\sqrt{\lambda} t} u .$$

Here, $\widetilde{C}$ and ω are constants independent of λ, r, r^* and t, with ω appearing in (2.5).

R e m a r k 2. 1. Note that the bounds (2.11)–(2.12) ((2.14)– (2.15), respectively) establish the range of t where the *standing wave* $(\sqrt{\lambda})^{-1}\sin(\sqrt{\lambda}\,t)u$ ($\cos(\sqrt{\lambda}\,t)u$, respectively) approaches the solution $\mathbf{u}(t)$ of problem (2.6) for the initial data $\varphi = 0$ and $\psi = u$ ($\varphi = u$ and $\psi = 0$ respectively), (u, λ) being a given quasimode of the operator A. This range of t is valid for the approach between the solutions of (2.16), $\sum_{k=1}^{I(r^*)} \alpha_k(\sqrt{\lambda_{i(r^*)+k}})^{-1}\sin(\sqrt{\lambda_{i(r^*)+k}}\,t)u_{i(r^*)+k}$, and $(\sqrt{\lambda})^{-1}\sin(\sqrt{\lambda}\,t)u$ ($\sum_{k=1}^{I(r^*)} \alpha_k\cos(\sqrt{\lambda_{i(r^*)+k}}\,t)u_{i(r^*)+k}$ and $\cos(\sqrt{\lambda}\,t)u$, respectively), and also between these solutions and $\mathbf{u}(t)$

The same holds for bounds (2.19)–(2.22), standing waves $e^{\pm\sqrt{\lambda}\,t}u$, solutions of (2.16), $\sum_{k=1}^{I(r^*)} \alpha_k e^{\pm\sqrt{\lambda_{i(r^*)+k}}\,t}u_{i(r^*)+k}$, and initial data $\varphi = u$, $\psi = \pm\sqrt{\lambda}\,u$. We also note that all the results in Theorems 2.1– 2.3 hold in the case where we know that there is only one eigenvalue $\lambda_{i(r^*)}$ in the interval $[\lambda - r^*, \lambda + r^*]$. ■

2.3 On the proofs

Estimates (2.10), (2.13) and (2.18) in Theorems 2.1, 2.2 and 2.3, respectively, are a natural consequence of the estimates for the energy (2.7) and (2.17), and of the definition of quasimode (2.1)-(2.2). We show (2.7) at the end of the section. In order to obtain the rest of the estimates in Theorems 2.1–2.3, below we introduce Propositions 2.1–2.4.

Proposition 2.1 *Let (u, λ) be a quasimode with remainder r of the operator A arising in (2.6) (respectively, (2.16)) . Let $\{\lambda_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ and $\{u_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ be the eigenvalues and the associated eigenfunctions of the operator A satisfying (2.1)-(2.2). Let us assume that $r^* > r$ and $\lambda - r^* > 0$. For given initial data φ, ψ related with u, let us assume that the solution $\mathbf{u}(t)$ of (2.6) (respectively, (2.16)) satisfies*

$$\left\| \mathbf{u}(t) - \sum_{k=1}^{I(r^*)} \alpha_k f_{i(r^*)+k}(t)u_{i(r^*)+k} \right\|_H + \left\| \frac{d\mathbf{u}}{dt}(t) - \sum_{k=1}^{I(r^*)} \alpha_k f'_{i(r^*)+k}(t)u_{i(r^*)+k} \right\|_H \leq$$

$$F(\lambda, r, r^*, t), \quad \forall t > 0, \tag{2.23}$$

for certain functions $f(t)$ and $\{f_{i(r^*)+k}(t)\}_{k=1,2,\cdots,I(r^*)}$, f depending on t and λ, $f_{i(r^*)+k}$ depending on t and $\lambda_{i(r^*)+k}$, and such that f, $f_{i(r^*)+k} \in C^1([0,\infty))$, $k = 1, 2, \cdots, I(r^*)$, and for the function $F(\lambda, r, r^*, t)$ defined by

$$F(\lambda, r, r^*, t) = \frac{2r}{r^*}(2(1 + \|A\|_{\mathcal{L}(H)} + t^2))^{1/2}$$

$$\left(F(\lambda, r, r^*, t) = e^{\omega t}(\frac{2r}{r^*}(1+\sqrt{\lambda})+\sqrt{r^*}), \ \textit{respect.}\right).$$

where ω *is the constant appearing in (2.17). Then,*

$$\|f(t)u - \mathbf{u}(t)\|_H \leq F(\lambda, r, r^*, t) + |f(t)|\frac{2r}{r^*} + \max_{1\leq k\leq I(r^*)} |f_{i(r^*)+k}(t) - f(t)| \tag{2.24}$$

and

$$\left\|f'(t)u - \frac{d\mathbf{u}}{dt}(t)\right\|_H \leq F(\lambda, r, r^*, t) + |f'(t)|\frac{2r}{r^*} + \max_{1\leq k\leq I(r^*)} |f'_{i(r^*)+k}(t) - f'(t)| \tag{2.25}$$

hold for any $t > 0$.

P r o o f. The proof of (2.24) and (2.25) hold the same steps and therefore we only show estimate (2.24).

Consider

$$\|f(t)u - \mathbf{u}(t)\|_H \leq \left\|f(t)u - \sum_{k=1}^{I(r^*)} \alpha_k f_{i(r^*)+k}(t) u_{i(r^*)+k}\right\|_H + \left\|\sum_{k=1}^{I(r^*)} \alpha_k f_{i(r^*)+k}(t) u_{i(r^*)+k} - \mathbf{u}(t)\right\|_H .$$

Then, on account of (2.23), the last term on the right hand side of the above inequality is bounded by $F(\lambda, r, r^*, t)$. Let us consider the first term of the inequality; we can write

$$\left\|f(t)u - \sum_{k=1}^{I(r^*)} \alpha_k f_{i(r^*)+k}(t) u_{i(r^*)+k}\right\|_H \leq \left\|f(t)u - \sum_{k=1}^{I(r^*)} \alpha_k f(t) u_{i(r^*)+k}\right\|_H + \left\|\sum_{k=1}^{I(r^*)} \alpha_k f(t) u_{i(r^*)+k} - \sum_{k=1}^{I(r^*)} \alpha_k f_{i(r^*)+k}(t) u_{i(r^*)+k}\right\|_H \leq$$

$$|f(t)| \left\|u - \sum_{k=1}^{I(r^*)} \alpha_k u_{i(r^*)+k}\right\|_H + \max_{1\leq k\leq I(r^*)} \left|f(t) - f_{i(r^*)+k}(t)\right| \left\|\sum_{k=1}^{I(r^*)} \alpha_k u_{i(r^*)+k}\right\|_H$$

which, on account of the definition of quasimode (2.1)-(2.2), is bounded by

$$|f(t)|\frac{2r}{r^*} + \max_{1\leq k\leq I(r^*)} |f_{i(r^*)+k}(t) - f(t)| .$$

Therefore, (2.24) holds and also (2.25) does by replacing time-dependent functions by their derivatives. Thus, the proposition is proved. ∎

As a consequence of Proposition 2.1, considering the functions $f(t)$ and $\{f_{i(r^*)+k}(t)\}_{k=1}^{I(r^*)}$ to be either the trigonometric or exponential functions appearing in the statements of Theorems 2.1, 2.2 and 2.3, we obtain the bounds in Propositions 2.2, 2.3 and 2.4, repectively.

Proposition 2.2 *Under the hypotheses of Theorem 2.1, for any $t > 0$, we have*

$$\left\|\frac{\sin(\sqrt{\lambda}\,t)}{\sqrt{\lambda}}u - \mathbf{u}(t)\right\|_H \leq \widetilde{C}\max\left(\frac{2r}{r^*}(C_A+t),\ \frac{2r}{r^*\sqrt{\lambda}},\right.$$

$$\left.\frac{1}{\sqrt{\lambda}}\left(\max_{1\leq k\leq I(r^*)}|\sin(\sqrt{\lambda_{i(r^*)+k}}\,t) - \sin(\sqrt{\lambda}\,t)| + \frac{\sqrt{r^*}}{\sqrt{\lambda}-\sqrt{r^*}}\right)\right) \tag{2.26}$$

and

$$\left\|\cos(\sqrt{\lambda}\,t)u - \frac{d\mathbf{u}}{dt}(t)\right\|_H \leq$$

$$\widetilde{C}\max\left(\frac{2r}{r^*}(C_A+t),\frac{2r}{r^*},\max_{1\leq k\leq I(r^*)}|\cos(\sqrt{\lambda_{i(r^*)+k}}\,t) - \cos(\sqrt{\lambda}\,t)|\right). \tag{2.27}$$

The bounds appearing in (2.26)–(2.27) depend on the relation between λ, r, r^*, $\{\lambda_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ and t, and $\widetilde{C}$ and C_A are are constants independent of these parameters, with $C_A = (1+\|A\|_{\mathcal{L}(H)})^{1/2}$.

Proposition 2.3 *Under the hypotheses of Theorem 2.2, for any $t > 0$, we have*

$$\left\|\cos(\sqrt{\lambda}\,t)u - \mathbf{u}(t)\right\|_H \leq$$

$$\widetilde{C}\max\left(\frac{2r}{r^*}(C_A+t),\frac{2r}{r^*},\max_{1\leq k\leq I(r^*)}|\cos(\sqrt{\lambda_{i(r^*)+k}}\,t) - \cos(\sqrt{\lambda}\,t)|\right) \tag{2.28}$$

and

$$\left\|\sqrt{\lambda}\sin(\sqrt{\lambda}\,t)u + \frac{d\mathbf{u}}{dt}(t)\right\|_H \leq \widetilde{C}\max\left(\frac{2r}{r^*}(C_A+t),\ \frac{2r}{r^*}\sqrt{\lambda},\right.$$

$$\left.(\sqrt{\lambda}\max_{1\leq k\leq I(r^*)}|\sin(\sqrt{\lambda_{i(r^*)+k}}\,t) - \sin(\sqrt{\lambda}\,t)| + \sqrt{r^*})\right), \tag{2.29}$$

where the bounds and constants in (2.28)–(2.29) satisfy the relations stated in Proposition 2.2.

Proposition 2.4 *Under the hypotheses of Theorem 2.3, for any $t > 0$, we have*

$$\left\| e^{\pm\sqrt{\lambda}t}u - \mathbf{u}(t) \right\|_H \leq \widetilde{C} \max \left((\frac{2r}{r^*}(1+\sqrt{\lambda}) + \sqrt{r^*}\,) e^{\omega t}, \right.$$

$$\left. \frac{2r}{r^*} e^{\pm\sqrt{\lambda}t}, \max_{1\leq k\leq I(r^*)} |e^{\pm\sqrt{\lambda_{i(r^*)+k}}\,t} - e^{\pm\sqrt{\lambda}t}| \right) \tag{2.30}$$

and

$$\left\| \sqrt{\lambda} e^{\pm\sqrt{\lambda}t}u \mp \frac{d\mathbf{u}}{dt}(t) \right\|_H \leq \widetilde{C} \max \left((\frac{2r}{r^*}(1+\sqrt{\lambda}) + \sqrt{r^*}\,) e^{\omega t}, \frac{2r}{r^*}\sqrt{\lambda}\, e^{\pm\sqrt{\lambda}t}, \right.$$

$$\left. (\sqrt{\lambda}+\sqrt{r^*}) \max_{1\leq k\leq I(r^*)} |e^{\pm\sqrt{\lambda_{i(r^*)+k}}\,t} - e^{\pm\sqrt{\lambda}t}| + \sqrt{r^*} e^{\pm\sqrt{\lambda}t} \right). \tag{2.31}$$

The bounds and constants appearing in (2.30)–(2.31) depend on the relation between λ, r, r^*, $\{\lambda_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ and t, and $\widetilde{C}$ and ω are constants independent of these parameters, with ω appearing in (2.17).

Now, the estimates in Theorems 2.1, 2.2 and 2.3 hold considering the estimates in Propositions 2.2, 2.3 and 2.4 respectively, and the Taylor series error of the sines, cosines and exponential functions in a neighborhood of $\sqrt{\lambda}t$. Below we gather some estimates for errors.

Namely, under the hypotheses for the quasimode in Theorems 2.1 and 2.2 we have:

$$\max_{1\leq k\leq I(r^*)} |\sin(\sqrt{\lambda_{i(r^*)+k}}\,t) - \sin(\sqrt{\lambda}\,t)| \leq \sqrt{r^*}\,t$$

and

$$\max_{1\leq k\leq I(r^*)} |\cos(\sqrt{\lambda_{i(r^*)+k}}\,t) - \cos(\sqrt{\lambda}\,t)| \leq \sqrt{r^*}\,t\,.$$

In the same way, under the hypotheses for the quasimode in Theorem 2.3, and depending on the sign of the argument of the exponential, we have:

$$\max_{1\leq k\leq I(r^*)} |e^{-\sqrt{\lambda_{i(r^*)+k}}\,t} - e^{-\sqrt{\lambda}t}| \leq \sqrt{r^*}\,t$$

and

$$\max_{1\leq k\leq I(r^*)} |e^{\sqrt{\lambda_{i(r^*)+k}}\,t} - e^{\sqrt{\lambda}t}| \leq \sqrt{r^*}\,t\,e^{(\sqrt{\lambda}+\sqrt{r^*})\,t}\,.$$

Consequently, once that we prove Lemma 2.2, Theorems 2.1–2.3 are also proved.

Proof of Lemma 2.2. Let $\mathbf{u}(t)$ be the solution (2.6) for $(\varphi,\psi) \in H \times H$. Let us consider $\{\lambda_i\}_{i\geq 1}$ the set of positive eigenvalues of A, and let us assume that $\{u_i\}_{i\geq 1}$ are the corresponding eigenfunctions which along with the basis $\{v_j\}_{j\geq 1}$ of $Ker(A)$ form an orthonormal basis of H. Let us consider the expansion of the initial data

$$\varphi = \sum_{i\geq 1} \alpha_i u_i + \sum_{j\geq 1} \tilde{\alpha}_j v_j \quad \text{and} \quad \psi = \sum_{i\geq 1} \beta_i u_i + \sum_{j\geq 1} \tilde{\beta}_j v_j$$

with $\alpha_i =< \varphi, u_i >_H$, $\tilde{\alpha}_j =< \varphi, v_j >_H$, $\beta_i =< \psi, u_i >_H$, $\tilde{\beta}_j =< \psi, v_j >_H$. Considering the solution of the associated system (2.4), $\bar{\mathbf{u}}(t) = e^{-\mathcal{A}t}\bar{\varphi}$, and computing the different powers of $\mathcal{A}$, it can be verified that this solution has the components

$$\mathbf{u}(t) = \sum_{i\geq 1} \left(\alpha_i \cos(\sqrt{\lambda_i}\, t) + \beta_i \frac{\sin(\sqrt{\lambda_i}\, t)}{\sqrt{\lambda_i}} \right) u_i + \sum_{j\geq 1} (\tilde{\alpha}_j + t\tilde{\beta}_j) v_j \tag{2.32}$$

and

$$\frac{d\mathbf{u}}{dt}(t) = \sum_{i\geq 1} \left(-\sqrt{\lambda_i}\alpha_i \sin(\sqrt{\lambda_i}\, t) + \beta_i \cos(\sqrt{\lambda_i}\, t) \right) u_i + \sum_{j\geq 1} \tilde{\beta}_j v_j \,. \tag{2.33}$$

Then, on account that $|\cos(t)| \leq 1$ and $|\sin(t)| \leq t$, $\forall t \geq 0$, we can write

$$\|\mathbf{u}(t)\|_H^2 \leq 2 \sum_{i\geq 1} \left(\alpha_i^2 (\cos(\sqrt{\lambda_i}\, t))^2 + \beta_i^2 \left(\frac{\sin(\sqrt{\lambda_i}\, t)}{\sqrt{\lambda_i}} \right)^2 \right) + 2 \sum_{j\geq 1} (\tilde{\alpha}_j^2 + t^2 \tilde{\beta}_j^2),$$

and consequently,

$$\|\mathbf{u}(t)\|_H^2 \leq 2 \left(\|\varphi\|_H^2 + t^2 \|\psi\|_H^2 \right). \tag{2.34}$$

Similarly, we have

$$\left\| \frac{d\mathbf{u}}{dt}(t) \right\|_H^2 \leq 2 \left(\|A\|_{\mathcal{L}(H)} \|\varphi\|_H^2 + \|\psi\|_H^2 \right). \tag{2.35}$$

Gathering (2.34) and (2.35) we can write

$$\|\mathbf{u}(t)\|_H^2 + \left\| \frac{d\mathbf{u}}{dt}(t) \right\|_H^2 \leq 2(1 + \|A\|_{\mathcal{L}(H)}) \|\varphi\|_H^2 + 2(1+t^2)\|\psi\|_H^2 \,, \quad \forall t \geq 0 \,, \tag{2.36}$$

which proves the lemma. ∎

R e m a r k 2. 2. Following the notations in Lemma 2.2 the solution of (2.16) reads

$$\mathbf{u}(t) = \sum_{i\geq 1} \left(\alpha_i \cosh(\sqrt{\lambda_i}\, t) + \frac{\beta_i}{\sqrt{\lambda_i}} \sinh(\sqrt{\lambda_i}\, t) \right) u_i + \sum_{j\geq 1} (\tilde{\alpha}_j + t\tilde{\beta}_j) v_j$$

which makes it difficult to remove the exponential function in (2.17) for general data (φ, ψ). ∎

2.4 On the first order evolution equation

The kinds of results in Sections 2.1 and 2.2, and proofs in Section 2.3, hold in the case where we consider a first order evolution problem, with the suitable modifications.

Let us consider a non-negative, compact and self-adjoint operator A acting on the separable Hilbert space H. Let $\omega^{\pm} > 0$ and $\xi_0^{\pm} > \omega^{\pm}$ be the constants, depending on the sign accompanying the operator A, such that

$$(\pm Au, u)_H + \omega^{\pm} \|u\|_H^2 \geq 0, \quad \forall u \in H, \tag{2.37}$$

and

$$\mathrm{R}(\pm A + \xi_0^{\pm} I) = H. \tag{2.38}$$

Here and hereafter throughout the section, each one of the signs $\pm$ implies a chosen sign $\pm$ accompanying A. Considering that A is non-negative, and the Fredholm alternative, it suffices to take $\omega^+ = 0$ and $\omega^- = \|A\|_{\mathcal{L}(H)}$ to get (2.37) while $\xi_0^+ > 0$ and $\xi_0^- > \|A\|_{\mathcal{L}(H)}$ provide (2.38).

Taking into account (2.37), (2.38) and the Lumer-Phillips Theorem (cf. Section III.8 in [16], for instance), we can assert that for any $\varphi \in H$ the evolution problem

$$\left\{ \begin{array}{rcl} \dfrac{d\mathbf{u}^{\pm}}{dt} \pm A\mathbf{u}^{\pm} & = & 0 \\ \mathbf{u}^{\pm}(0) & = & \varphi \end{array} \right. \tag{2.39}$$

has a unique solution $\mathbf{u}^{\pm} \in C^1([0,T]; H)$ for any positive T, which satisfies

$$\|\mathbf{u}^{\pm}(t)\|_H \leq e^{\omega^{\pm} t} \|\varphi\|_H, \quad \forall t \geq 0, \tag{2.40}$$

with $\omega^+ = 0$. Then, we state the following theorem.

Theorem 2.4 *Let (u, λ) be a quasimode with remainder r of the operator A arising in (2.39). Let $\{\lambda_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ and $\{u_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ be the eigenvalues and the associated eigenfunctions of the operator A satisfying (2.1)-(2.2). Let us assume that $r^* > r$ and $\lambda - r^* > 0$. Then, for $\varphi = u$, and for any $t > 0$, the solution $\mathbf{u}^{\pm}(t)$ of (2.39) satisfies*

$$\left\| e^{-\lambda t} u - \mathbf{u}^{+}(t) \right\|_{H} \leq \widetilde{C} \max\left(\frac{2r}{r^*}, r^* t \right) \tag{2.41}$$

for the positive sign accompanying the operator, while

$$\left\| e^{\lambda t} u - \mathbf{u}^{-}(t) \right\|_{H} \leq \widetilde{C} \max\left(\frac{2r}{r^*} e^{\omega^- t}, \frac{2r}{r^*} e^{\lambda t}, r^* t\, e^{(\lambda + r^*) t} \right) \tag{2.42}$$

for the negative sign. Again $\widetilde{C}$ and $\omega^{\pm}$ are constants independent of λ, r, r^* and t, with $\omega^{\pm}$ appearing in (2.40).

The same bounds hold for

$$\left\| \sum_{k=1}^{I(r^*)} \alpha_k e^{\pm \lambda_{i(r^*)+k} t} u_{i(r^*)+k} - e^{\pm \lambda t} u \right\|_{H}$$

depending on the sign $\pm$, while

$$\left\| \sum_{k=1}^{I(r^*)} \alpha_k e^{\pm \lambda_{i(r^*)+k} t} u_{i(r^*)+k} - \mathbf{u}^{\pm}(t) \right\|_{H} \leq \frac{2r}{r^*} e^{\omega^{\pm} t}, \qquad \textit{with } \omega^+ = 0.$$

Proof. For each one of the signs $\pm$, the last inequality in the statement of the theorem holds from (2.40) and (2.1)-(2.2). Inequalities (2.41) and (2.42) are obtained from (2.24) for $F(\lambda, r, r^*, t) = 2r(r^*)^{-1} e^{\omega^{\pm} t}$, $f(t) \equiv e^{\pm \lambda t}$ and $f_{i(r^*)+k}(t) \equiv e^{\pm \lambda_{i(r^*)+k} t}$, after considering the Taylor series error of the exponential function in a neighborhood of $(\pm \lambda t)$. ∎

Remark 2. 3. As is well-known, the solution of (2.4) (cf. also (2.6) and (2.16) and (2.39)) and, more precisely, of the non-homogeneous equation

$$\begin{cases} \dfrac{d\bar{\mathbf{u}}}{dt} + \mathcal{A}\bar{\mathbf{u}} &= \bar{f}(t) \\ \bar{\mathbf{u}}(0) &= \bar{\varphi} \end{cases}$$

for given $\bar{\varphi} \in \mathbf{H}$, $T > 0$, and $\bar{f} \in C^1([0, T], \mathbf{H})$, is provided by

$$\bar{\mathbf{u}}(t) = e^{-\mathcal{A} t} \bar{\varphi} + \int_0^t e^{-\mathcal{A}(t-s)} \bar{f}(s)\, ds, \quad \forall t \in [0, T]. \tag{2.43}$$

This formula, also allows us to obtain bounds for the discrepancies between the solutions $\bar{\mathbf{u}}(t)$ of (2.4) when the initial data are linear combinations of eigenfunctions (or quasimodes) and the standing waves constructed from the quasimode (u, λ) in Theorems 2.1–2.4. Nevertheless, these bounds involve $d(\lambda, \sigma(A))^{-1}$ which, in general, is unknown and very large, above all when we deal with the very small frequencies of A or with singularly perturbed spectral problems. Further manipulation of the discrepancies is likely to involve using the technique in Section 2.3.

Indeed, considering formula (2.43) for $\mathcal{A} \equiv \pm A$ and choosing the appropriate data $\bar{\varphi} \equiv \varphi$ and $\bar{f}(t) \equiv f(t)$, it is simple to verify the assertion above for problem (2.39) and for the discrepancies considered in Theorem 2.4, while all the bounds that we obtain in Theorem 2.4 (cf. (2.41) and (2.42)) are well determined in terms of well-known parameters. ■

3 General setting for ε-dependent problems

In this section, for the sake of completeness, we state the general framework and results for ε-dependent evolution problems when ε is a small positive parameter.

For each fixed $\varepsilon > 0$, we assume that A^ε is a linear, compact, self-adjoint and non-negative operator on a separable Hilbert space H^ε. Depending on the sign $\pm$, we consider the problem

$$(3.1) \qquad \begin{cases} \dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} \pm A^\varepsilon \mathbf{u}^\varepsilon &= 0 \\ \mathbf{u}^\varepsilon(0) &= \varphi^\varepsilon \\ \dfrac{d\mathbf{u}^\varepsilon}{dt}(0) &= \psi^\varepsilon \end{cases}$$

for given initial data $(\varphi^\varepsilon, \psi^\varepsilon) \in H^\varepsilon \times H^\varepsilon$ related to quasimodes $(u^\varepsilon, \lambda^\varepsilon)$ of the operators, A^ε, and we prove the results in Theorems 3.1-3.3. In order to do this, throughout the section we consider the following hypotheses for the family of operators $\{A^\varepsilon\}_\varepsilon$ on H^ε, ε a small parameter $\varepsilon \in (0, 1)$.

Let $\omega \equiv \omega_\varepsilon$ be the parameter appearing in (2.5) which depends on the sign $\pm$ accompanying the operator A^ε in (3.1), $\mathbf{H}$ being $\mathbf{H} \equiv H^\varepsilon \times H^\varepsilon$. For the positive sign in (3.1) we have inequality (2.7) depending on the norm of A^ε, while for the negative sign we have (2.17).

Also, let $(u^\varepsilon, \lambda^\varepsilon)$ be a quasimode with remainder r_ε of the operator A^ε arising in (3.1). Let r_ε^* be $r_\varepsilon^* > r_\varepsilon$, and let us assume that the interval $[\lambda^\varepsilon -$

$r_\varepsilon^*, \lambda^\varepsilon + r_\varepsilon^*]$ only contains discrete spectrum of A^ε. Then, on account of Lemma 2.1, we consider $\{\lambda^\varepsilon_{i(r_\varepsilon^*)+k}\}_{k=1,2,\cdots,I(r_\varepsilon^*)}$ the set of eigenvalues of A^ε in the interval $[\lambda^\varepsilon - r_\varepsilon^*, \lambda^\varepsilon + r_\varepsilon^*]$ for some index $i(r_\varepsilon^*)$ and $I(r_\varepsilon^*) \geq 1$, and let the associated eigenfunctions be $\{u^\varepsilon_{i(r^*)+k}\}_{k=1,2,\cdots,I(r_\varepsilon^*)}$ satisfying the orthonormality condition $(u^\varepsilon_{i(r^*)+k}, u^\varepsilon_{i(r^*)+j})_{H^\varepsilon} = \delta_{k,j}$, for $k, j = 1, 2, \cdots, I(r_\varepsilon^*)$. Let $u^{\varepsilon *} \in H^\varepsilon$,

$$\|u^{\varepsilon *}\|_{H^\varepsilon} = 1,\ u^{\varepsilon *} \in \left[u^\varepsilon_{i(r_\varepsilon^*)+1}, u^\varepsilon_{i(r_\varepsilon^*)+2}, \cdots, u^\varepsilon_{i(r_\varepsilon^*)+I(r_\varepsilon^*)}\right],\ u^{\varepsilon *} = \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k^\varepsilon u^\varepsilon_{i(r_\varepsilon^*)+k} \tag{3.2}$$

and satisfying

$$\|u^\varepsilon - u^{\varepsilon *}\|_{H^\varepsilon} = \|u^\varepsilon - \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k^\varepsilon u^\varepsilon_{i(r_\varepsilon^*)+k}\|_{H^\varepsilon} \leq \frac{2r_\varepsilon}{r_\varepsilon^*}. \tag{3.3}$$

On account of the normalization for the eigenfunctions and for $u^{\varepsilon *}$, obviously the α_k^ε are constants such that $|\alpha_k^\varepsilon| \leq 1$ for $k = 1, 2, \cdots I(r_\varepsilon^*)$.

Since it must be assumed that A^ε has only discrete spectrum in the interval $[\lambda^\varepsilon - r_\varepsilon^*, \lambda^\varepsilon + r_\varepsilon^*]$, it can obviously be assumed that $\lambda^\varepsilon - r_\varepsilon^* > 0$, $\forall \varepsilon > 0$. Additionally, for the results in this section, we assume that $r_\varepsilon \to 0$ and $r_\varepsilon^* \to 0$ as $\varepsilon \to 0$, that there is a constant $\delta > 0$ such that $\lambda^\varepsilon - r_\varepsilon^* > \delta$ for sufficiently small ε, and, finally that $r_\varepsilon^* > r_\varepsilon$ and $\lim_{\varepsilon\to 0}(r_\varepsilon/r_\varepsilon^*) = 0$.

Under the hypotheses above for the operator A^ε and the quasimode $(u^\varepsilon, \lambda^\varepsilon)$, we can rewrite statements and proofs in Sections 2.1–2.3 with minor modifications and we have the results stated below. In this respect, we observe that bounds in Theorems 2.1-2.4 and Propositions 2.1-2.4 are valid and keep the dependence on ε of the parameters λ, r, * and ω, while $\widetilde{C}$ is a constant independent of ε.

Theorem 3.1 *Consider the equation* $\dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} + A^\varepsilon \mathbf{u}^\varepsilon = 0$ *and the initial data* $\varphi^\varepsilon = 0$ *and* $\psi^\varepsilon = u^\varepsilon$. *Then, for any* $t > 0$, *the solution* $\mathbf{u}^\varepsilon(t)$ *of (3.1) satisfies*

$$\left\| \frac{\sin(\sqrt{\lambda^\varepsilon}\, t)}{\sqrt{\lambda^\varepsilon}} u^\varepsilon - \mathbf{u}^\varepsilon(t) \right\|_{H^\varepsilon} \leq \widetilde{C} \max\left(\frac{r_\varepsilon}{r_\varepsilon^*}(C_\varepsilon + t),\ \frac{r_\varepsilon}{r_\varepsilon^* \sqrt{\lambda^\varepsilon}},\ \frac{\sqrt{r_\varepsilon^*}}{\sqrt{\lambda^\varepsilon}}(t+1) \right) \tag{3.4}$$

and

$$\left\| \cos(\sqrt{\lambda^\varepsilon}\, t) u^\varepsilon - \frac{d\mathbf{u}^\varepsilon}{dt}(t) \right\|_{H^\varepsilon} \leq \widetilde{C} \max\left(\frac{r_\varepsilon}{r_\varepsilon^*}(C_\varepsilon + t),\ \frac{r_\varepsilon}{r_\varepsilon^*},\ \sqrt{r_\varepsilon^*}\, t \right), \tag{3.5}$$

for sufficiently small ε (namely, $\varepsilon < \varepsilon_0$ with ε_0 independent of t). Here C_ε and $\widetilde{C}$ are constants independent of λ^ε, t, r_ε and r_ε^*; C_ε is the constant $C_\varepsilon = (1+\|A^\varepsilon\|_{\mathcal{L}(H^\varepsilon)})^{1/2}$; λ^ε, r_ε and r_ε^* are the constants appearing in (3.2)-(3.3) while $\widetilde{C}$ is a certain constant which is also independent of ε.

Theorem 3.2 *Consider the equation $\dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} + A^\varepsilon \mathbf{u}^\varepsilon = 0$ and the initial data $\varphi^\varepsilon = u^\varepsilon$ and $\psi^\varepsilon = 0$. Then, for any $t > 0$, the solution $\mathbf{u}^\varepsilon(t)$ of (3.1) satisfies*

$$\left\|\cos(\sqrt{\lambda^\varepsilon}\, t)u^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{H^\varepsilon} \leq \widetilde{C} \max\left(\frac{r_\varepsilon}{r_\varepsilon^*}(C_\varepsilon + t),\, \frac{r_\varepsilon}{r_\varepsilon^*},\, \sqrt{r_\varepsilon^*}\, t\right) \tag{3.6}$$

and

$$\left\|\sqrt{\lambda^\varepsilon}\sin(\sqrt{\lambda^\varepsilon}\, t)u^\varepsilon + \frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|_{H^\varepsilon} \leq \widetilde{C} \max\left(\frac{r_\varepsilon}{r_\varepsilon^*}(C_\varepsilon + t),\, \frac{r_\varepsilon}{r_\varepsilon^*}\sqrt{\lambda^\varepsilon},\, \sqrt{r_\varepsilon^*}(\sqrt{\lambda^\varepsilon}\, t + 1)\right) \tag{3.7}$$

for sufficiently small ε, and constants r_ε, r_ε^*, C_ε and $\widetilde{C}$ as in Theorem 3.1.

Theorem 3.3 *Assume all the hypotheses above for the operator A^ε and the quasimode $(u^\varepsilon, \lambda^\varepsilon)$ and consider the equation $\dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} - A^\varepsilon \mathbf{u}^\varepsilon = 0$.*

i). For the initial data $\varphi^\varepsilon = u^\varepsilon$, $\psi^\varepsilon = -\sqrt{\lambda^\varepsilon} u^\varepsilon$, for any positive t and sufficiently small ε (namely, $\varepsilon < \varepsilon_0$ with ε_0 independent of t), the solution $\mathbf{u}^\varepsilon(t)$ of (3.1) satisfies

$$\|e^{-\sqrt{\lambda^\varepsilon}\, t}u^\varepsilon - \mathbf{u}^\varepsilon(t)\|_{H^\varepsilon} \leq \widetilde{C} \max\left(\left(\frac{r_\varepsilon}{r_\varepsilon^*}(1 + \sqrt{\lambda^\varepsilon}) + \sqrt{r_\varepsilon^*}\right) e^{\omega_\varepsilon t},\, \sqrt{r_\varepsilon^*}\, t\right) \tag{3.8}$$

and

$$\|\sqrt{\lambda^\varepsilon} e^{-\sqrt{\lambda^\varepsilon}\, t}u^\varepsilon + \frac{d\mathbf{u}^\varepsilon}{dt}(t)\|_{H^\varepsilon} \leq \widetilde{C} \max\left(\left(\frac{r_\varepsilon}{r_\varepsilon^*}(1 + \sqrt{\lambda^\varepsilon}) + \sqrt{r_\varepsilon^*}\right) e^{\omega_\varepsilon t},\right.$$

$$\left.\left(\frac{r_\varepsilon}{r_\varepsilon^*}\sqrt{\lambda^\varepsilon},\, (\sqrt{\lambda^\varepsilon} + \sqrt{r_\varepsilon^*})\sqrt{r_\varepsilon^*}\, t + \sqrt{r_\varepsilon^*}\right)\right). \tag{3.9}$$

ii). For the initial data $\varphi^\varepsilon = u^\varepsilon$, $\psi^\varepsilon = \sqrt{\lambda^\varepsilon} u^\varepsilon$, for any positive t and sufficiently small ε (namely, $\varepsilon < \varepsilon_0$ with ε_0 independent of t), the solution $\mathbf{u}^\varepsilon(t)$ of (3.1) satisfies

$$\left\|e^{\sqrt{\lambda^\varepsilon}\, t}u^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{H^\varepsilon} \leq$$

$$\text{(3.10)}\ \widetilde{C}\max\left(\left(\frac{r_\varepsilon}{r_\varepsilon^*}(1+\sqrt{\lambda^\varepsilon})+\sqrt{r_\varepsilon^*}\,\right)e^{\omega_\varepsilon t}\,,\frac{r_\varepsilon}{r_\varepsilon^*}\,e^{\sqrt{\lambda^\varepsilon}\,t}\,,\ \sqrt{r_\varepsilon^*}\,t\,e^{(\sqrt{\lambda^\varepsilon}+\sqrt{r_\varepsilon^*})\,t}\right)$$

and

$$\left\|\sqrt{\lambda}e^{\sqrt{\lambda^\varepsilon}\,t}u^\varepsilon-\frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|_{H^\varepsilon}\leq\widetilde{C}\ \max\left(\left(\frac{r_\varepsilon}{r_\varepsilon^*}(1+\sqrt{\lambda^\varepsilon})+\sqrt{r_\varepsilon^*}\,\right)e^{\omega_\varepsilon t},\right.$$

$$\text{(3.11)}\quad\left.\left(\frac{r_\varepsilon}{r_\varepsilon^*}\sqrt{\lambda^\varepsilon}\,e^{\sqrt{\lambda^\varepsilon}\,t},\ \sqrt{\lambda^\varepsilon}+\sqrt{r_\varepsilon^*}\,\right)\sqrt{r_\varepsilon^*}\,te^{(\sqrt{\lambda^\varepsilon}+\sqrt{r_\varepsilon^*})\,t}+\sqrt{r_\varepsilon^*}e^{\sqrt{\lambda^\varepsilon}\,t}\right).$$

In inequalities (3.8)-(3.11), $\widetilde{C}$ are constants independent of λ^ε, t, r_ε and r_ε^*; ω_ε is the constant appearing in (2.17) for $A\equiv A^\varepsilon$ and $H\equiv H^\varepsilon$; λ^ε, r_ε and r_ε^* are the constants appearing in (3.2)-(3.3) while $\widetilde{C}$ is a certain constant which is also independent of ε.

Remark 3. 1. Note that the estimates in the inequalities (3.4)–(3.5), (3.6)–(3.7), and (3.8)–(3.11), establish the range of t for which the standing waves $(\sqrt{\lambda^\varepsilon})^{-1}\sin(\sqrt{\lambda^\varepsilon}t)u^\varepsilon$, $\cos(\sqrt{\lambda^\varepsilon}t)u^\varepsilon$ and $e^{\pm\sqrt{\lambda^\varepsilon}\,t}u^\varepsilon$ approach the respective solutions of (3.1), when the given initial data (along with λ^ε) are related to the quasimodes of A^ε as stated in the Theorems 3.1, 3.2 and 3.3 respectively. In addition, the same estimates and range of t hold for the discrepancies between $(\sqrt{\lambda^\varepsilon})^{-1}\sin(\sqrt{\lambda^\varepsilon}t)u^\varepsilon$, $\cos(\sqrt{\lambda^\varepsilon}t)u^\varepsilon$ and $e^{\pm\sqrt{\lambda^\varepsilon}\,t}u^\varepsilon$ and the elementary solutions of the evolution problems when the initial data are $\varphi^\varepsilon=0$, $\psi^\varepsilon=\sum_{k=1}^{I(r_\varepsilon^*)}\alpha_k^\varepsilon u_{i(r_\varepsilon^*)+k}^\varepsilon$; $\varphi^\varepsilon=\sum_{k=1}^{I(r_\varepsilon^*)}\alpha_k^\varepsilon u_{i(r_\varepsilon^*)+k}^\varepsilon$, $\psi^\varepsilon=0$; and $\varphi^\varepsilon=\sum_{k=1}^{I(r_\varepsilon^*)}\alpha_k^\varepsilon u_{i(r_\varepsilon^*)+k}^\varepsilon$, $\psi^\varepsilon=\pm\sum_{k=1}^{I(r_\varepsilon^*)}\sqrt{\lambda_{i(r_\varepsilon^*)+k}^\varepsilon}\alpha_k^\varepsilon u_{i(r_\varepsilon^*)+k}^\varepsilon$, respectively (see also Remark 2.2).

As a sample, the precise range of t is stated in Section 4 for an operator arising in the homogenization of a Steklov type eigenvalue problem (cf. (4.25), (4.26) and (4.31)). Also, the precise bounds for the discrepancies between the solutions and the standing waves are provided depending on the relations between $(r_\varepsilon/r_\varepsilon^*)$, $(r_\varepsilon/r_\varepsilon^*)t$, $\sqrt{r_\varepsilon^*}$, $\sqrt{\lambda^\varepsilon r_\varepsilon^*}\,t$, $\sqrt{r_\varepsilon^*}\,t$, $(r_\varepsilon/r_\varepsilon^*)\sqrt{r_\varepsilon^*}$, $(r_\varepsilon/r_\varepsilon^*)\sqrt{\lambda^\varepsilon}$, $r_\varepsilon/(r_\varepsilon^*\sqrt{\lambda^\varepsilon})$, $\sqrt{r_\varepsilon^*/\lambda^\varepsilon}\,t$, and on the products of some of these factors by $\|A^\varepsilon\|_{\mathcal{L}(H^\varepsilon)}^{1/2}$ or by the exponential functions $e^{\sqrt{\lambda^\varepsilon}\,t}$, $e^{\sqrt{r_\varepsilon^*}\,t}$ and $e^{\omega_\varepsilon t}$. ■

The results in Section 2.4 can also be extended for a first order evolution problem associated with the operator A^ε. Namely, under the assumptions (2.37) and (2.38), which are satisfied for certain $\omega^\pm\equiv\omega_\varepsilon^\pm$ and $\xi_0^\pm\equiv\xi_{0,\varepsilon}^\pm$, we consider the problem

$$\text{(3.12)}\qquad\left\{\begin{aligned}\frac{d\mathbf{u}^\varepsilon}{dt}\pm A^\varepsilon\mathbf{u}^\varepsilon&=0\\ \mathbf{u}^\varepsilon(0)&=\varphi^\varepsilon\end{aligned}\right.$$

for a given initial data $\varphi^\varepsilon \in H^\varepsilon$ related to quasimodes of $(u^\varepsilon, \lambda^\varepsilon)$ of the operators A^ε which satisfy all the hypotheses in (3.2)-(3.3). Then, from Theorem 2.4 we derive the results in Theorem 3.4 below.

Theorem 3.4 *Assume all the hypotheses above for the operator A^ε and the quasimode $(u^\varepsilon, \lambda^\varepsilon)$ and consider the initial data $\varphi^\varepsilon = u^\varepsilon$ in problem (3.12). Then, for any positive t and sufficiently small ε (namely, $\varepsilon < \varepsilon_0$ with ε_0 independent of t), the solution $\mathbf{u}^\varepsilon(t)$ of (3.12) satisfies*

$$\|e^{-\lambda^\varepsilon t}u^\varepsilon - \mathbf{u}^\varepsilon(t)\|_{H^\varepsilon} \leq \widetilde{C} \max\left(\frac{r_\varepsilon}{r_\varepsilon^*}, r_\varepsilon^* t\right) \tag{3.13}$$

when the sign + accompanies the operator, and

$$\left\|e^{\lambda^\varepsilon t}u^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{H^\varepsilon} \leq \widetilde{C} \max\left(\frac{r_\varepsilon}{r_\varepsilon^*}e^{\omega_\varepsilon^- t}, \frac{r_\varepsilon}{r_\varepsilon^*}e^{\lambda^\varepsilon t}, r_\varepsilon^* t\, e^{(\lambda^\varepsilon + r_\varepsilon^*)t}\right). \tag{3.14}$$

In inequalities (3.13)–(3.14), ω_ε^- and $\widetilde{C}$ are constants independent of λ^ε, t, r_ε and r_ε^*; ω_ε^- is the constant appearing in (2.40) for $A \equiv A^\varepsilon$ and $H \equiv H^\varepsilon$; λ^ε, r_ε and r_ε^* are the constants appearing in (3.2)-(3.3), while $\widetilde{C}$ is a certain constant which is also independent of ε.

Remark 3. 2. Note that bounds in Theorems 2.1-2.4 and Propositions 2.2–2.4 as well as those in Theorems 3.1-3.4 rely on bounds (2.24) and (2.25) where the corresponding $|f(t)|$ and $|f'(t)|$ have been replaced by their respective bounds, which may be worse for small ε. ■

Remark 3. 3. In connection with Remark 3.2 we also note that bound (2.7) used in Theorems 2.1 and 2.2 can be improved using (2.36), but (2.7) is kept uniform when changing the initial data in both theorems as we outline at the end of Section 2.1. ■

4 The boundary homogenization problem

In this section we apply the results in Section 3 for operators arising in a spectral boundary homogenization problem. First, we introduce the homogenization problem and the associated spectral local problem; then, for the sake of completeness, in Section 4.1 we gather certain results on the quasimodes constructed in Pérez [14]. Finally, in Section 4.2 we use these results to construct standing waves approaching solutions of time-dependent problems and we derive the bounds for the discrepancies.

Let Ω be an open bounded domain of $\mathbb{R}^{2+}$ with a Lipschitz boundary $\partial\Omega$. This boundary $\partial\Omega$ is assumed to be in contact with the line $\{x_2 = 0\}$, $\partial\Omega = \overline{\Sigma} \cup \overline{\Sigma}_f \cup \overline{\Gamma}_\Omega$ where the part of $\partial\Omega$ in contact $\{x_2 = 0\}$ is assumed to be the union of Σ_f and Σ, $\Sigma \neq \emptyset$ and $\overline{\Sigma}_f = (\overline{\Omega} \cap \{x_2 = 0\}) \setminus \Sigma$. Without any restriction, we can assume that the segment Σ is centered at zero and that, in the case where $\Sigma_f \neq \emptyset$, $\overline{\Sigma} \cap \overline{\Gamma}_\Omega = \emptyset$.

For fixed ε, $\varepsilon \in (0,1)$, we consider Σ to contain the union of segments Σ_k^ε of length ε which we define as follows: For $k = 0, \pm 1, \pm 2, \pm 3, \cdots, \pm N_\varepsilon$, let T_k^ε (Σ_k^ε, G_k^ε, respectively) be the homothetic T^1 (Σ^1, G^1, respectively), of ratio ε; centered at the point $\tilde{x}_k = (k\varepsilon P, 0)$. Here, T^1 and Σ^1 are segments centered at the origin, T^1 strictly contained in Σ^1, $G^1 = \Sigma^1 \times (0,\infty)$, ε is a small parameter that we shall make go to zero, P is a fixed number, $P > 0$, and $2N_\varepsilon + 1$ denotes the number of Σ_k^ε contained in Σ, $N_\varepsilon = O(\varepsilon^{-1})$. For each fixed k, the change of variable

$$y = \frac{x - \tilde{x}_k}{\varepsilon} \tag{4.1}$$

transforms T_k^ε, Σ_k^ε and G_k^ε into T^1, Σ^1 and G^1 respectively. If no confusion arises, in what follows we shall write $\bigcup T^\varepsilon$ to denote $\bigcup_{i=-N_\varepsilon}^{N_\varepsilon} T_i^\varepsilon$.

We refer to Pérez [14] for further geometrical considerations and for the proofs of the results that we state below and in Section 4.1.

Let $\mathbf{V}^\varepsilon$ denote the space completion of $\{v \in \mathcal{D}(\overline{\Omega}) \,/\, v = 0 \text{ on } \partial\Omega \setminus \bigcup T^\varepsilon\}$ with the norm

$$\|v\|^2_{\mathbf{V}^\varepsilon} = \int_\Omega |\nabla v|^2 \, dx \,, \tag{4.2}$$

whose elements vanish on $\partial\Omega \setminus \bigcup T^\varepsilon$, and, for sufficiently small ε, they satisfy

$$\int_\Sigma u^2 \, dx_1 = \int_{\bigcup T^\varepsilon} u^2 \, dx_1 \leq \varepsilon C_P \int_\Omega |\nabla u|^2 \, dx \,, \quad \forall u \in \mathbf{V}^\varepsilon, \tag{4.3}$$

where C_P is a well-determined constant independent of ε and u (we can take C_P related to the Poincaré constant for the elements of $\{U \in H^1(\Sigma^1 \times (0,1)) \,/\, U = 0 \text{ on } T^1\}$).

Let us consider the spectral problem

$$\left\{ \begin{array}{rcl} -\Delta u^\varepsilon & = & 0 \text{ in } \Omega \,, \\ u^\varepsilon & = & 0 \text{ on } \partial\Omega \setminus \bigcup T^\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial x_2} + \beta^\varepsilon u^\varepsilon & = & 0 \text{ on } \bigcup T^\varepsilon \,, \end{array} \right. \tag{4.4}$$

whose variational formulation reads: Find β^ε and $u^\varepsilon \in \mathbf{V}^\varepsilon$, $u^\varepsilon \neq 0$, satisfying

$$\int_\Omega \nabla u^\varepsilon . \nabla v \, dx = \beta^\varepsilon \int_{\bigcup T^\varepsilon} u^\varepsilon v \, dx_1 \,, \quad \forall v \in \mathbf{V}^\varepsilon \,. \tag{4.5}$$

For each fixed ε, and for each $v^\varepsilon \in \mathbf{V}^\varepsilon$, the Riesz representation theorem identifies the element of $(\mathbf{V}^\varepsilon)'$ defined by $\int_\Sigma v^\varepsilon v\, dx_1$, $\forall v \in \mathbf{V}^\varepsilon$, with an element of $\mathbf{V}^\varepsilon$. Therefore, the problem (4.5) can be written as an eigenvalue problem for a non-negative, self-adjoint, compact operator $\mathbf{A}^\varepsilon$ on the space $\mathbf{V}^\varepsilon$ as follows: Find μ^ε ($\mu^\varepsilon = 1/\beta^\varepsilon$) and $u^\varepsilon \in \mathbf{V}^\varepsilon$, $u^\varepsilon \neq 0$ satisfying $\mathbf{A}^\varepsilon u^\varepsilon = \mu^\varepsilon u^\varepsilon$, where

$$(4.6) \qquad < \mathbf{A}^\varepsilon u, v >= \int_{\bigcup T^\varepsilon} uv\, dx_1, \quad \forall u, v \in \mathbf{V}^\varepsilon$$

(cf. Lobo&Pérez [10], Pérez [14] and Remark 4.8 to compare).

The operator $\mathbf{A}^\varepsilon$ has the eigenvalue 0 with corresponding eigenspace $Ker(\mathbf{A}^\varepsilon) = \{u \in \mathbf{V}^\varepsilon \,/\, u = 0 \text{ on } \bigcup T^\varepsilon\} \equiv H_0^1(\Omega)$, and the rest of the spectrum, which is discrete, is denoted by $\{(\beta_i^\varepsilon)^{-1}\}_{i=1}^\infty$, where $\{\beta_i^\varepsilon\}_{i=1}^\infty$ are the set of eigenvalues with finite multiplicity of (4.5), $\beta_i^\varepsilon \to \infty$ as $i \to \infty$; the convention of repeated eigenvalues is used. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the set of associated eigenfunctions which are assumed to be orthonormal in $\mathbf{V}^\varepsilon$. They form an orthonormal basis in the space complement orthogonal to $Ker(\mathbf{A}^\varepsilon)$ in $\mathbf{V}^\varepsilon$, $Ker(\mathbf{A}^\varepsilon)^\perp$, which satisfies:

$$(4.7) \; Ker(\mathbf{A}^\varepsilon)^\perp \subset \{u \in H^1(\Omega) \,/\, \Delta u = 0 \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega \setminus \bigcup T^\varepsilon\}.$$

The minimax principle characterizes the i-th eigenvalue of problem (4.5); in this case (see Pérez [14]), it allows us to assert that $\beta_i^\varepsilon = O(\varepsilon^{-1})$. In addition, the limit behavior of the re-scaled eigenvalues $\beta_i^\varepsilon \varepsilon$ and the associated eigenfunctions is involved with the eigenelements of the local problem, namely problem (4.8), as stated in Theorem 4.1 below (see also Remark 4.1).

The eigenvalue local problem in the half-band G^1 is: Find $(\beta^0, V^0) \in \mathbb{R}^+ \times \mathbf{V}^1$, $V^0 \neq 0$, satisfying

$$(4.8) \qquad \int_{G^1} \nabla_y V^0 . \nabla_y V\, dy = \beta^0 \int_{T^1} V^0 V\, dy_1\,, \quad \forall V \in \mathbf{V}^1\,.$$

Here y is the *local variable* defined by (4.1), and $\mathbf{V}^1$ denotes the space completion of $\{V \in \mathcal{D}(\overline{G^1}),\, V = 0 \text{ on } \Sigma^1 \setminus T^1, V(y_1, y_2) \text{ is } y_1 - \textit{periodic} \text{ in } G^1\}$ with the gradient norm in $L^2(G^1)$. It is known that (4.8) has a discrete spectrum and we consider the set of eigenelements $\{\beta_i^0, V_i^0\}_{i=1}^\infty$, where we assume that the eigenfunctions have norm 1 in $\mathbf{V}^1$ (cf. Pérez [14] for more details).

4.1 On the quasimodes for (4.6)

For each eigenfunction V^0 of (4.8), $\|V^0\|_{\mathbf{V}^1} = 1$, let $w^\varepsilon(x)$ be the function defined by

$$(4.9) \qquad w^\varepsilon(x_1, x_2) = V^0(y_1, y_2) \quad \text{for } (x_1, x_2) \in G_0^\varepsilon = \varepsilon G^1$$

and extended by periodicity to all the half-bands G_i^ε such that the corresponding Σ_i^ε are contained in Σ.

Let us consider the cutoff function η^ε,

$$\eta^\varepsilon(x) = \eta\left(x_2 \delta_\varepsilon{}^{-1}\right), \tag{4.10}$$

where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ and η is a smooth function with a compact support, $supp\,(\eta') \subset [\frac{1}{3}, \frac{2}{3}]$,

$$\eta \in C^1(\mathbb{R}), \quad 0 \le \eta \le 1\,, \quad \eta(t) = 1 \text{ for } t \le \frac{1}{3} \quad \text{and} \quad \eta(t) = 0 \text{ for } t \ge \frac{2}{3}\,.$$

For each fixed $V^0(y)$ solution of (4.8), the order function δ_ε can be taken to be $\delta_\varepsilon = \tilde{k}\varepsilon|ln\,\varepsilon|$, where $\tilde{k}$ is a well determined constant which depends on V^0 (see Pérez [14] and Panasenko&Pérez [12] for the construction of η^ε and proofs).

Let us denote by $w^\varepsilon\eta^\varepsilon \equiv w^\varepsilon(x)\eta^\varepsilon(x)$ the function $V^0(x/\varepsilon)$ extended by periodicity to all the Σ_i^ε contained in Σ and multiplied by the function $\eta^\varepsilon(x)$ which is only dependent on x_2. $w^\varepsilon\eta^\varepsilon$ is a periodic function of the x_1 variable which vanishes on $\Sigma \setminus \bigcup T^\varepsilon$.

For any fixed intervals (a,b) and (c,d) contained in Σ, with (a,b) strictly contained in (c,d), let ψ be a function

$$\psi \in C_0^\infty(\mathbb{R}), \quad 0 \le \psi \le 1, \quad \psi(x_1) = 1 \text{ if } x \in [a,b], \quad \psi(x_1) = 0 \text{ if } x_1 \notin (c,d)\,. \tag{4.11}$$

Then, we define the *boundary layer function* $w^\varepsilon\eta^\varepsilon\psi$, concentrating its support in a thin layer near Σ,

$$(w^\varepsilon\eta^\varepsilon\psi)(x) = w^\varepsilon(x_1,x_2)\eta^\varepsilon(x_2)\psi(x_1)\,. \tag{4.12}$$

Obviously, $w^\varepsilon\eta^\varepsilon\psi \in \mathbf{V}^\varepsilon$ where now the function $\eta^\varepsilon\psi \in C_0^\infty(\mathbb{R}^2)$ takes the value 1 in the rectangle $[a,b] \times [0,(1/3)\tilde{k}\varepsilon\,|ln\,\varepsilon|]$ and vanishes outside the rectangle $[c,d] \times [0,(2/3)\tilde{k}\varepsilon\,|ln\,\varepsilon|]$. Hence, $w^\varepsilon\eta^\varepsilon\psi$ is also a strongly oscillating function in the thin layer where it concentrates its support.

Finally, let us introduce the constant α^ε such that $\|\alpha^\varepsilon(w^\varepsilon\eta^\varepsilon\psi)\|_{\mathbf{V}^\varepsilon} = 1$, that is,

$$\alpha^\varepsilon = \left(\int_\Omega |\nabla(w^\varepsilon\eta^\varepsilon\psi)|^2\,dx\right)^{-1/2}. \tag{4.13}$$

Theorem 4.1 *Let (β^0, V^0) be any eigenelement of (4.8), V^0 with norm 1 in $\mathbf{V}^1$ (that is, $\int_{G^1} |\nabla_y V^0|^2\,dy = 1$). There exists a sequence d^ε, $d^\varepsilon \to 0$, as $\varepsilon \to 0$, such that there are eigenvalues β^ε of (4.5) with $\varepsilon\beta^\varepsilon \in [\beta^0 - d^\varepsilon, \beta^0 + d^\varepsilon]$ (or equivalently, such that $(\beta^\varepsilon)^{-1} \in [\varepsilon(\beta^0)^{-1} - r^\varepsilon, \varepsilon(\beta^0)^{-1} + r^\varepsilon]$ for $r^\varepsilon = O(d^\varepsilon\varepsilon)$).*

In addition, there are $\tilde{u}^\varepsilon$, with $\int_\Omega |\nabla\tilde{u}^\varepsilon|^2\,dx = 1$, $\tilde{u}^\varepsilon$ in the eigenspace of all the eigenfunctions u^ε of (4.5) associated with the eigenvalues β^ε such that $\varepsilon\beta^\varepsilon \in [\beta^0 - \tilde{d}^\varepsilon, \beta^0 + \tilde{d}^\varepsilon]$ (or equivalently, such that $(\beta^\varepsilon)^{-1} \in [\varepsilon(\beta^0)^{-1} - \tilde{r}^\varepsilon, \varepsilon(\beta^0)^{-1} + \tilde{r}^\varepsilon]$ for $\tilde{r}^\varepsilon = O(\tilde{d}^\varepsilon\varepsilon)$, $\varepsilon(\beta^0)^{-1} > \tilde{r}^\varepsilon$), with $\tilde{d}^\varepsilon \to 0$ and $d^\varepsilon/\tilde{d}^\varepsilon \to 0$ as $\varepsilon \to 0$, (or equivalently, $\tilde{r}^\varepsilon \to 0$ and $r^\varepsilon/\tilde{r}^\varepsilon \to 0$ as $\varepsilon \to 0$), and $\tilde{u}^\varepsilon$ satisfying:

$$(4.14)\qquad \int_\Omega |\nabla(\tilde{u}^\varepsilon - \alpha^\varepsilon w^\varepsilon \eta^\varepsilon \psi)|^2\,dx \le C\left(\frac{r^\varepsilon}{\tilde{r}^\varepsilon}\right)^2,$$

where C is a constant independent of ε, α^ε is the constant defined by (4.13) and the function $w^\varepsilon\eta^\varepsilon\psi$ is defined in (4.12) from (4.9), (4.10) and (4.11). The sequences d^ε and r^ε can be taken as follows:

$$(4.15)\qquad d^\varepsilon = K_1|ln\,\varepsilon|^{-1/2}, \qquad \textit{and} \qquad r^\varepsilon = K_2\varepsilon|ln\,\varepsilon|^{-1/2}$$

where K_1, K_2 are certain constants independent of ε. Also, sequences $\tilde{d}^\varepsilon$ and $r^\varepsilon/\tilde{r}^\varepsilon = d^\varepsilon/\tilde{d}^\varepsilon$ can be chosen in order to get either smaller intervals $[\beta^0 - \tilde{d}^\varepsilon, \beta^0 + \tilde{d}^\varepsilon]$ or improved bounds (4.14).

Moreover, considering $\varepsilon(\beta^0)^{-1} > \tilde{r}^\varepsilon$ and (4.15), possible choices of $\tilde{r}^\varepsilon$ are

$$(4.16)\qquad \tilde{r}^\varepsilon = K_3\varepsilon|ln\,\varepsilon|^{-\beta}, \qquad \textit{with } K_3 \textit{ any constant and } \ 0 < \beta < \frac{1}{2}.$$

In particular, $d^\varepsilon/\tilde{d}^\varepsilon = r^\varepsilon/\tilde{r}^\varepsilon = O(|ln\,\varepsilon|^{-1/4})$ is one of these possible choices.

We refer to Pérez [14] for the proof of Theorem 4.1 and for further results obtained from the statements in this theorem that we gather in the following remark.

Remark 4. 1. As a matter of fact, applying results in Theorem 4.1 allows us to assert that each eigenvalue β^0 of (4.8) is an accumulation point of the re-scaled eigenvalues $\varepsilon\beta^\varepsilon$ of (4.5). As regards the eigenfunctions, for any eigenfunction V^0 associated with the eigenvalue β^0 of (4.8), and sufficiently small ε, functions $\alpha^\varepsilon(w^\varepsilon\eta^\varepsilon\psi)$ are the so-called *quasimodes* of (4.5), approaching linear combinations of eigenfunctions $\tilde{u}^\varepsilon$ in small intervals as stated in Theorem 4.1. The norm used for the approach is (4.2), and considering the support of $(w^\varepsilon\eta^\varepsilon\psi)$, we can assert that these eigenfunctions $\{\tilde{u}^\varepsilon\}_{\varepsilon>0}$ concentrate asymptotically their support in a thin layer of width $O(\varepsilon|ln\,\varepsilon|)$ around a part of the boundary Σ (in which the $supp(\psi)$ is contained) and they vanish outside. In this thin layer they are also strongly oscillating functions.

In addition, for fixed i, it has been shown in Pérez [14] that $\lim_{\varepsilon\to 0}\beta_i^\varepsilon\varepsilon = \beta_1^0$ where β_1^0 is the first eigenvalue of (4.8), but this result should be combined with others in order to obtain an idea on the asymptotic behavior for the eigenfunction associated with β_1^ε and for the eigenfunctions of (4.5) individually (see also Nazarov&Pérez [11] and remarks in Pérez [14]). ∎

4.2 On evolution problems associated with (4.6)

In this section, we deal with the evolution problems associated with the operator $\mathbf{A}^\varepsilon$ in (4.6) and the standing waves from the quasimodes in Section 4.1. For initial data $(\varphi^\varepsilon, \psi^\varepsilon) \in \mathbf{V}^\varepsilon \times \mathbf{V}^\varepsilon$ we consider the problem

$$(4.17) \qquad \begin{cases} \dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} \pm \mathbf{A}^\varepsilon \mathbf{u}^\varepsilon &= 0 \\ \mathbf{u}^\varepsilon(0) &= \varphi^\varepsilon \\ \dfrac{d\mathbf{u}^\varepsilon}{dt}(0) &= \psi^\varepsilon \end{cases}$$

$\mathbf{A}^\varepsilon$ satisfies the hypotheses in Section 3 for $H^\varepsilon \equiv \mathbf{V}^\varepsilon$, and here we compute the associated ω^ε arising in (2.5) or, equivalently, in (2.7).

From $A \equiv \mathbf{A}^\varepsilon$ we define the operator $\mathcal{A}$ in (2.3) on the space $\mathbf{H} = \mathbf{V}^\varepsilon \times \mathbf{V}^\varepsilon$. For $(u,v) \in \mathbf{V}^\varepsilon \times \mathbf{V}^\varepsilon$, let us consider $\bar{\mathbf{u}} = (u,v)$ and take into account the definition of $\mathbf{A}^\varepsilon$ in (4.6) and the inequality (4.3). Then, we can write

$$(4.18) \quad \|\mathcal{A}\bar{\mathbf{u}}\|_{\mathbf{H}} = \|(\pm v,\, \mathbf{A}^\varepsilon u)\|_{\mathbf{H}} \le \|v\|_{\mathbf{V}^\varepsilon} + \varepsilon C_P \|u\|_{\mathbf{V}^\varepsilon} \le (1+\varepsilon C_P)\|\bar{\mathbf{u}}\|_{\mathbf{H}},$$

for the constant C_P in (4.3). Consequently, for both signs, we can take ω in (2.5) to be $\omega_\varepsilon = (1+\varepsilon C_P)$ which is bounded by any fixed constant $\omega > 1$ independent of ε, for sufficiently small ε (without any restriction we can assume $\omega = 2$).

Thus, for each sign $\pm$, we have a unique solution $\mathbf{u}^\varepsilon(t)$ of (4.17), $\mathbf{u}^\varepsilon \in C^2([0,T];\mathbf{V}^\varepsilon)$ for any positive T, which satisfies the inequality (cf. (2.17))

$$\|\mathbf{u}^\varepsilon(t)\|^2_{\mathbf{V}^\varepsilon} + \left\|\frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|^2_{\mathbf{V}^\varepsilon} \le 4e^{2(1+\varepsilon C_P)t}\left(\|\varphi^\varepsilon\|^2_{\mathbf{V}^\varepsilon} + \|\psi^\varepsilon\|^2_{\mathbf{V}^\varepsilon}\right) \le$$

$$(4.19) \qquad 4e^{2\omega t}\left(\|\varphi^\varepsilon\|^2_{\mathbf{V}^\varepsilon} + \|\psi^\varepsilon\|^2_{\mathbf{V}^\varepsilon}\right),$$

for any $t \ge 0$ and ω above. Considering (2.7), for the positive sign in (4.17) we also have

$$(4.20) \quad \|\mathbf{u}^\varepsilon(t)\|^2_{\mathbf{V}^\varepsilon} + \left\|\frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|^2_{\mathbf{V}^\varepsilon} \le 2(1+\varepsilon C_P + t^2)\left(\|\varphi^\varepsilon\|^2_{\mathbf{V}^\varepsilon} + \|\psi^\varepsilon\|^2_{\mathbf{V}^\varepsilon}\right), \quad \forall t > 0.$$

Hence, we are in the framework of Section 3, and, we consider the quasimodes $W^\varepsilon(x) = \alpha^\varepsilon(w^\varepsilon \eta^\varepsilon \psi)(x)$ constructed in Theorem 4.1, from (4.9)–(4.13) and (β^0, V^0) an eigenelement of (4.8), V^0 of norm 1 in $\mathbf{V}^1$. On account of Remark 3.2 and (2.24), we state results in Theorems 3.1–3.3 for the operator $\mathbf{A}^\varepsilon$ and the initial data which we describe below.

Let us consider problem (4.17) with the positive sign accompanying the operator, namely, the equation

$$\frac{d^2\mathbf{u}^\varepsilon}{dt^2} + \mathbf{A}^\varepsilon \mathbf{u}^\varepsilon = 0$$

and the initial data $(\varphi^\varepsilon, \psi^\varepsilon) \in \mathbf{V}^\varepsilon \times \mathbf{V}^\varepsilon$. It is self-evident that for $(\varphi^\varepsilon, \psi^\varepsilon) = (u^\varepsilon_{i(\varepsilon)}, 0)$ ($(\varphi^\varepsilon, \psi^\varepsilon) = (0, u^\varepsilon_{i(\varepsilon)})$ respectively), where $(\beta^\varepsilon_{i(\varepsilon)}, u^\varepsilon_{i(\varepsilon)})$ is an eigenelement of (4.5), the solution of (4.17) is

$$\mathbf{u}^\varepsilon(t) = \cos\left(\sqrt{1/\beta^\varepsilon_{i(\varepsilon)}}\, t\right) u^\varepsilon_{i(\varepsilon)}, \ \left(\mathbf{u}^\varepsilon(t) = \sqrt{\beta^\varepsilon_{i(\varepsilon)}}\, \sin\left(\sqrt{1/\beta^\varepsilon_{i(\varepsilon)}}\, t\right) u^\varepsilon_{i(\varepsilon)}, \text{ respect.}\right)$$

When $(\varphi^\varepsilon, \psi^\varepsilon)$ are the quasimodes constructed above, namely, $(\varphi^\varepsilon, \psi^\varepsilon) = (W^\varepsilon, 0)$ ($(\varphi^\varepsilon, \psi^\varepsilon) = (0, W^\varepsilon)$, respectively) the relation of the solutions of (4.17) and the standing waves

$$\cos\left(\sqrt{\varepsilon/\beta^0}\, t\right) W^\varepsilon, \quad \left(\sqrt{\beta^0/\varepsilon}\, \sin\left(\sqrt{\varepsilon/\beta^0}\, t\right) W^\varepsilon \text{ respectively}\right)$$

is provided by the following theorems.

Theorem 4.2 *With the notations in Theorem 4.1, for the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (W^\varepsilon, 0)$, the solution $\mathbf{u}^\varepsilon(t)$ of (4.17) with the positive sign in the equation satisfies*

$$\left\|\cos\left(\sqrt{\varepsilon/\beta^0}\, t\right) W^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{H^1(\Omega)} \leq C_1 \max\left(\frac{r_\varepsilon}{\tilde{r}_\varepsilon}(1+t), \sqrt{\tilde{r}_\varepsilon}\, t\right), \tag{4.21}$$

$$\left\|\sqrt{\varepsilon/\beta^0}\, \sin\left(\sqrt{\varepsilon/\beta^0}\, t\right) W^\varepsilon + \frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|_{H^1(\Omega)} \leq$$

$$C_2 \max\left(\frac{r_\varepsilon}{\tilde{r}_\varepsilon}(1+t), \sqrt{\varepsilon}\sqrt{\tilde{r}_\varepsilon}\, t + \sqrt{\tilde{r}_\varepsilon}\right), \tag{4.22}$$

where C_1 and C_2 are constants independent of t and ε, and, r_ε and $\tilde{r}_\varepsilon$ are the order functions in Theorem 4.1.

Theorem 4.3 *With the notations in Theorem 4.1, for the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (0, W^\varepsilon)$, the solution $\mathbf{u}^\varepsilon(t)$ of (4.17) with the positive sign in the equation satisfies*

$$\left\|\sqrt{\beta^0/\varepsilon}\, \sin\left(\sqrt{\varepsilon/\beta^0}\, t\right) W^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{H^1(\Omega)} \leq$$

$$C_1 \max\left(\frac{r_\varepsilon}{\widetilde{r}_\varepsilon}(1+t),\, \frac{r_\varepsilon}{\widetilde{r}_\varepsilon}\, t,\, \frac{\sqrt{\widetilde{r}_\varepsilon}}{\sqrt{\varepsilon}}(t+1)\right), \tag{4.23}$$

$$\left\| \cos\left(\sqrt{\varepsilon/\beta^0}\; t\right) W^\varepsilon - \frac{d\mathbf{u}^\varepsilon}{dt}(t) \right\|_{H^1(\Omega)} \le C_2 \max\left(\frac{r_\varepsilon}{\widetilde{r}_\varepsilon}(1+t),\, \sqrt{\widetilde{r}_\varepsilon}\, t\right), \tag{4.24}$$

where C_1 and C_2 are constants independent of t and ε, and, r_ε and $\widetilde{r}_\varepsilon$ are the order functions in Theorem 4.1,

Remark 4. 2. Note that the bounds in Theorem 4.2 (Theorem 4.3, respectively) establish the range of t in which the standing wave $\cos\left(\sqrt{\varepsilon/\beta^0}\; t\right) W^\varepsilon$ ($\sqrt{\beta^0/\varepsilon}\sin\left(\sqrt{\varepsilon/\beta^0}\; t\right) W^\varepsilon$, respectively) approaches the solution of (4.17) for given initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (W^\varepsilon, 0)$ $((\varphi^\varepsilon, \psi^\varepsilon) = (0, W^\varepsilon)$, respectively) quasi-modes of the operator $\mathbf{A}^\varepsilon$ defined in (4.6).

Namely, considering r_ε and $\widetilde{r}_\varepsilon$ given by (4.15) and (4.16), and the constants β and γ satisfying $\frac{1}{3} < \beta < \frac{1}{2}$ and $0 < \gamma < 1$, for

$$t \in \left[0, |\ln \varepsilon|^{\frac{1-2\beta}{2}\gamma}\right], \tag{4.25}$$

we have the bounds

$$\left\| \cos\left(\sqrt{\frac{\varepsilon}{\beta^0}}\; t\right) W^\varepsilon - \mathbf{u}^\varepsilon(t) \right\|_{H^1(\Omega)} \le C\,|\ln(\varepsilon)|^{-\widetilde{\gamma}}$$

$$\left\| \sqrt{\frac{\varepsilon}{\beta^0}} \sin\left(\sqrt{\frac{\varepsilon}{\beta^0}}\; t\right) W^\varepsilon + \frac{d\mathbf{u}^\varepsilon}{dt}(t) \right\|_{H^1(\Omega)} \le C\,|\ln(\varepsilon)|^{-\widetilde{\gamma}}$$

$$\left\| \sqrt{\frac{\beta^0}{\varepsilon}} \sin\left(\sqrt{\frac{\varepsilon}{\beta^0}}\; t\right) W^\varepsilon - \mathbf{u}^\varepsilon(t) \right\|_{H^1(\Omega)} \le C\,|\ln(\varepsilon)|^{-\widetilde{\gamma}}$$

$$\left\| \cos\left(\sqrt{\frac{\varepsilon}{\beta^0}}\; t\right) W^\varepsilon - \frac{d\mathbf{u}^\varepsilon}{dt}(t) \right\|_{H^1(\Omega)} \le C\,|\ln(\varepsilon)|^{-\widetilde{\gamma}}$$

with $\widetilde{\gamma} = \frac{(1-2\beta)(1-\gamma)}{2} > 0$, and C a certain constant independent of t and ε. ∎

Remark 4. 3. In connection with Remark 4.2, it should be noted that the range of t can be enlarged if we need to compute the discrepancy between the solution and the standing wave only on the part of the boundary Σ. Indeed, this is a consequence of estimate (4.3) satisfied by the elements of $\mathbf{V}^\varepsilon$ which implies that considering the norm of $L^2(\Sigma)$ on the left hand side of estimates (4.21)–(4.24), the right hand side is multiplied by $\sqrt{\varepsilon}$. Consequently, the discrepancies in $L^2(\Sigma)$ between standing waves and solutions in Remark 4.2 remain valid for

$$t \in \left[0,\, (\sqrt{\varepsilon}\,)^{-1} |\ln \varepsilon|^{\frac{1-2\beta}{2}\gamma}\right]. \tag{4.26}$$

Comparing intervals (4.25) and (4.26) leads us to assert that the standing waves on the boundary Σ remain close to the solutions of (4.17) for a longer period of time. ∎

Let us consider problem (4.17) with the negative sign accompanying the operator, namely, the equation

$$\frac{d^2\mathbf{u}^\varepsilon}{dt^2} - \mathbf{A}^\varepsilon \mathbf{u}^\varepsilon = 0$$

and the initial data $(\varphi^\varepsilon, \psi^\varepsilon) \in \mathbf{V}^\varepsilon \times \mathbf{V}^\varepsilon$. Obviously, for $(\varphi^\varepsilon, \psi^\varepsilon)$ the pair $(u^\varepsilon_{i(\varepsilon)},\ \pm\sqrt{1/\beta^\varepsilon_{i(\varepsilon)}}\, u_{i(\varepsilon)})$, the solution of (4.17) is given by

$$\mathbf{u}^\varepsilon(t) = e^{\pm\sqrt{1/\beta^\varepsilon_{i(\varepsilon)}}\, t}\, u^\varepsilon_{i(\varepsilon)}$$

while for $(\varphi^\varepsilon, \psi^\varepsilon) = (W^\varepsilon, \pm\sqrt{\varepsilon/\beta^0}\, W^\varepsilon)$, the connection between the standing wave $e^{\pm\sqrt{\varepsilon/\beta^0}\, t}\, W^\varepsilon$ and the solution of $\mathbf{u}^\varepsilon(t)$ of problem (4.17) is given by the following theorem.

Theorem 4.4 *With the notations in Theorem 4.1, the solution $\mathbf{u}^\varepsilon(t)$ of (4.17) with the negative sign in the equation satisfies:*

i). For the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (W^\varepsilon, -\sqrt{\varepsilon/\beta^0}\, W^\varepsilon)$, for any positive t and sufficiently small ε (namely, $\varepsilon < \varepsilon_0$ with ε_0 independent of t),

$$\|e^{-\sqrt{\varepsilon/\beta^0}\, t}\, W^\varepsilon - \mathbf{u}^\varepsilon(t)\|_{H^1(\Omega)} \le \widetilde{C}_1 \max\left((\frac{r_\varepsilon}{\widetilde{r}_\varepsilon} + \sqrt{\widetilde{r}_\varepsilon})\, e^{\omega t}\,,\, \sqrt{\widetilde{r}_\varepsilon}\, t \right) \tag{4.27}$$

and

$$\left\| \sqrt{\varepsilon/\beta^0}\, e^{-\sqrt{\varepsilon/\beta^0}\, t}\, W^\varepsilon + \frac{d\mathbf{u}^\varepsilon}{dt}(t) \right\|_{H^1(\Omega)} \le$$

$$\widetilde{C}_2 \max\left((\frac{r_\varepsilon}{\widetilde{r}_\varepsilon} + \sqrt{\widetilde{r}_\varepsilon})\, e^{\omega t}\,, (\sqrt{\varepsilon} + \sqrt{\widetilde{r}_\varepsilon})\sqrt{\widetilde{r}_\varepsilon}\, t + \sqrt{\widetilde{r}_\varepsilon} \right). \tag{4.28}$$

ii). For the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (W^\varepsilon, \sqrt{\varepsilon/\beta^0}\, W^\varepsilon)$, for any positive t and sufficiently small ε (namely, $\varepsilon < \varepsilon_0$ with ε_0 independent of t),

$$\left\| e^{\sqrt{\varepsilon/\beta^0}\, t}\, W^\varepsilon - \mathbf{u}^\varepsilon(t) \right\|_{H^1(\Omega)} \le$$

$$\widetilde{C}_3 \max\left((\frac{r_\varepsilon}{\widetilde{r}_\varepsilon} + \sqrt{\widetilde{r}_\varepsilon})\, e^{\omega t}\,,\ \sqrt{\widetilde{r}_\varepsilon}\, t\, e^{(\sqrt{\varepsilon/\beta^0}+\sqrt{\widetilde{r}_\varepsilon})\, t}\right) \tag{4.29}$$

and

$$\left\| \sqrt{\varepsilon/\beta^0}\, e^{\sqrt{\varepsilon/\beta^0}\, t}\, W^\varepsilon - \frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|_{H^1(\Omega)} \leq \widetilde{C}_4 \max\left((\frac{r_\varepsilon}{\widetilde{r}_\varepsilon} + \sqrt{\widetilde{r}_\varepsilon})\, e^{\omega t}\,,\right.$$

$$\left. (\sqrt{\varepsilon} + \sqrt{\widetilde{r}_\varepsilon})\sqrt{\widetilde{r}_\varepsilon}\, t\, e^{(\sqrt{\varepsilon/\beta^0}+\sqrt{\widetilde{r}_\varepsilon})\, t} + \sqrt{\widetilde{r}_\varepsilon} e^{\sqrt{\varepsilon/\beta^0}\, t}\right). \tag{4.30}$$

The constants ω, $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, \widetilde{C}_4$ appearing in (4.27)–(4.30) are constants independent of t and ε, with ω appearing in (4.19), and, r_ε and $\widetilde{r}_\varepsilon$ are the order functions in Theorem 4.1.

Remark 4. 4. Similarly to Remark 4.2, with the notation in Theorem 4.4, from (4.27)–(4.30) we derive uniform bounds for the discrepancies of the standing waves of the type $e^{\pm\sqrt{\varepsilon/\beta^0}\, t}\, W^\varepsilon$ and the solutions $\mathbf{u}^\varepsilon(t)$ of (4.17) when the initial data $(\varphi^\varepsilon, \psi^\varepsilon)$ are given by $(W^\varepsilon, \pm\sqrt{\varepsilon/\beta^0}\, W^\varepsilon)$. Namely, for

$$t \in \left[0, \frac{1-2\beta}{2\omega}\gamma \ln(|\ln \varepsilon|)\right], \tag{4.31}$$

we have

$$\left\| e^{\pm\sqrt{\varepsilon/\beta^0}\, t}\, W^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{H^1(\Omega)} \leq C|\ln(\varepsilon)|^{-\widetilde{\gamma}}$$

and

$$\left\| \sqrt{\varepsilon/\beta^0}\, e^{\pm\sqrt{\varepsilon/\beta^0}\, t}\, W^\varepsilon \pm (-1)\frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|_{H^1(\Omega)} \leq C|\ln(\varepsilon)|^{-\widetilde{\gamma}},$$

where γ and $\widetilde{\gamma}$ are the same constants introduced in Remark 4.2, $0 < \beta < \frac{1}{2}$, and C is a constant independent of t and ε. ∎

Remark 4. 5. On account of Theorem 3.4, the kind of bounds in Theorem 4.4 with the suitable modifications can be obtained when a first order evolution problem (3.12) is considered for the operator $\mathbf{A}^\varepsilon$ defined by (4.6) and φ^ε the quasimode W^ε constructed in Theorem 4.1 from (λ^0, V^0) an eigenelement of (4.8). Namely, considering (4.3) and (4.18), under the assumptions in this section for the operator and the initial data, the solution $\mathbf{u}^\varepsilon(t)$ of (3.12) satisfies

$$\|e^{-(\varepsilon/\beta^0)\, t} W^\varepsilon - \mathbf{u}^\varepsilon(t)\|_{H^1(\Omega)} \leq \widetilde{C}_1 \max\left(\frac{r_\varepsilon}{\widetilde{r}_\varepsilon}, \widetilde{r}_\varepsilon\, t\right)$$

when the positive sign accompanies the operator, and

$$\left\| e^{(\varepsilon/\beta^0)\, t} W^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{H^1(\Omega)} \leq \widetilde{C}_2 \max\left(\frac{r_\varepsilon}{\widetilde{r}_\varepsilon}\, e^{(\varepsilon C_P) t}\,,\ \frac{r_\varepsilon}{\widetilde{r}_\varepsilon}\, e^{(\varepsilon/\beta^0)\, t}\,,\ \widetilde{r}_\varepsilon\, t\, e^{(\varepsilon/\beta^0+\widetilde{r}_\varepsilon)\, t}\right)$$

otherwise, where $\widetilde{C}_1, \widetilde{C}_2$ are constants independent of t and ε, and r_ε and $\widetilde{r}_\varepsilon$ are the order functions in Theorem 4.1. ∎

Remark 4. 6. We emphasize that according to formulas (4.9)-(4.13), the standing waves constructed in Theorems 4.2 - 4.4 and Remark 4.5 concentrate asymptotically their support along a small neighborhood of any chosen segment on the boundary Σ where the strongly alternating conditions are imposed. The width of the band where the support is concentrated is $O(\varepsilon|ln\varepsilon|)$. Consequently, we can construct approaches to solutions of first and second order evolution problems that concentrate asymptotically their support along a line for long periods of time (cf. Remarks 4.2 - 4.5). ∎

Remark 4. 7. As regards the inequality (4.19), we note that, for the positive sign in $\pm\mathbf{A}^\varepsilon$, multiplying the equation $(4.17)_1$ by $\dfrac{d\mathbf{u}^\varepsilon}{dt}$ and integrating between 0 and t for any positive t, on account of $(4.17)_2$ and $(4.17)_3$, we can write an *energy equality*

$$\left\|\frac{d\mathbf{u}^\varepsilon}{dt}\right\|^2_{\mathbf{V}^\varepsilon} + \int_\Sigma |\mathbf{u}^\varepsilon|^2\, dx_1 = \|\psi^\varepsilon\|^2_{\mathbf{V}^\varepsilon} + \int_\Sigma (\varphi^\varepsilon)^2\, dx_1\,, \quad \forall t \geq 0, \tag{4.32}$$

which avoids time-dependent exponential functions accompanying the initial data. Instead, formula (4.32) does not provide bounds for $\|\nabla\mathbf{u}^\varepsilon(t)\|^2_{L^2(\Omega)}$ in terms of the initial data, providing only estimates for the derivative of $\mathbf{u}^\varepsilon$ in $H^1(\Omega)$ and for $\mathbf{u}^\varepsilon$ in $L^2(\Sigma)$.

This is in good agreement with the fact that the norm in the space product $\mathbf{H} = \mathbf{V}^\varepsilon \times \mathbf{V}^\varepsilon$ is not provided by the *energy of the system.* In addition, for the negative sign in $\pm\mathbf{A}^\varepsilon$, there appears a negative sign affecting the integrals on Σ in (4.32) which renders this equality useless. ∎

Remark 4. 8. It should be noted that the original spectral problem in Pérez [14], which has the variational formulation (4.5), is written in terms of harmonic functions in Ω, with the Steklov boundary condition $\frac{\partial u^\varepsilon}{\partial x_2} + \beta^\varepsilon u^\varepsilon = 0$ on $\bigcup T^\varepsilon$; namely in terms of partial derivatives in the spatial variable x. This problem has a discrete spectrum, while when considering the spectral problem in terms of the operator (4.6), the eigenvalue 0 is added with an associated eigenspace $H^1_0(\Omega)$ which, along with the orthogonal space (4.7), completes $\mathbf{V}^\varepsilon$.

However, when considering the evolution problem (4.17) associated with the operator (4.6), we may not obtain the solution $\mathbf{u}^\varepsilon(t)$ as a harmonic function unless we restrict ourselves to initial data $(\varphi^\varepsilon, \psi^\varepsilon)$ which are harmonic functions, namely $\varphi^\varepsilon, \psi^\varepsilon \in Ker(\mathbf{A}^\varepsilon)^\perp$ (see (4.7) and Lobo&Pérez [10] to compare).

Indeed, multiplying the equation (4.17) by $v^\varepsilon \in \mathbf{V}^\varepsilon$, we have

$$\left\langle \frac{d^2\mathbf{u}^\varepsilon}{dt^2}, v^\varepsilon \right\rangle_{\mathbf{V}^\varepsilon} \pm \langle \mathbf{A}^\varepsilon \mathbf{u}^\varepsilon, v^\varepsilon \rangle = 0$$

Taking $v^\varepsilon = v \in H_0^1(\Omega)$ we can write:

$$\int_\Omega \nabla \left(\frac{d^2\mathbf{u}^\varepsilon}{dt^2} \right) . \nabla v \, dx = 0 , \quad \forall v \in H_0^1(\Omega),$$

and therefore,

$$\Delta \left(\frac{d^2\mathbf{u}^\varepsilon}{dt^2} \right) = 0 \quad \text{in } \mathcal{D}'(\Omega)$$

but this does not ensure that $\mathbf{u}^\varepsilon$ is a harmonic function. In order to ensure this, we must restrict problem (4.17) to the subspace $Ker(\mathbf{A}^\varepsilon)^\perp$ of $\mathbf{V}^\varepsilon$ (with the same scalar product), whose elements are harmonic functions in Ω, this being the space of the initial data. Therefore, all the properties for the solutions of (4.17) in $\mathbf{V}^\varepsilon$ hold for the solutions in $Ker(\mathbf{A}^\varepsilon)^\perp$. In other words, with the initial data $\varphi^\varepsilon, \psi^\varepsilon \in Ker(\mathbf{A}^\varepsilon)^\perp$ we have a unique solution of problem (4.17) in $Ker(\mathbf{A}^\varepsilon)^\perp$ and this is the only solution of (4.17) in $\mathbf{V}^\varepsilon$. ∎

Remark 4. 9. According to Remark 4.8, in order to have a physical meaning of the solutions of (4.17) in terms of the solutions of wave equations (similarly, for first order equations in terms of heat equations), we must look for solutions of (4.17) which are harmonic functions and this restricts the initial data to harmonic functions in $\mathbf{V}^\varepsilon$. This happens for instance if the initial data are eigenfunctions of $\mathbf{A}^\varepsilon$ associated with positive eigenvalues but not for the given initial data which are the quasimodes W^ε related with the eigenelements of (4.8) (λ^0, V^0).

In this framework, it should be noted that all the results of the Theorems 4.2-4.4 and Remarks 4.2 – 4.5 hold if we replace W^ε by its projection on $\widetilde{W}^\varepsilon$ into the space $Ker(\mathbf{A}^\varepsilon)^\perp$ (cf. (4.7)). This result is a consequence of the fact that the eigenfunctions approaching the quasimodes are already in $Ker(\mathbf{A}^\varepsilon)^\perp$ and consequently, estimates (4.14) and the fact that $\|\widetilde{W}^\varepsilon\|_{\mathbf{V}^\varepsilon} \leq 1$ is not a restriction in the proofs of the theorems. The only exception worth mentioning is that we lose the property of the concentration of the support of the standing waves and consequently of the solutions of the evolution problems along a line: see Lobo&Pérez [10] in this respect. ∎

Remark 4. 1 0 . In the framework of Remarks 4.6. - 4. 9., another point is that when looking at the low frequency vibrations associated with problem (4.5), the frequency of vibration is expected to be in fact very large, namely, of the order of $O(\varepsilon^{-1})$ instead of the order $O(\varepsilon)$ as we obtain in this paper considering the evolution problem associated with the operator $\mathbf{A}^\varepsilon$; $\mathbf{A}^\varepsilon$ has the eigenvalues which are the inverse of the eigenvalues of the original problem (4.5). Another formulation of the evolution problem in terms of unbounded operators on the fractionary Sobolev spaces on the boundary of the T^ε is given in Lobo&Pérez [10] using the results in Pérez [15]. With this new alternative evolution problem in Lobo&Pérez [10], we lose the concentration of the support of the standing waves that we construct as well as that of the solutions of the evolution problems (cf. also Lobo&Pérez [9] and Lobo&Pérez [10] in this respect). ■

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Received: 23.08.2010