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Schwarz Function for Vekua Complex Differential Equation

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The Schwarz function is very efficient in the theory of complex interpolation and approximation in solving boundary value problems. In this article, application of the Schwarz function in the theory of the Vekua complex differential equation is considered.

1. Introduction

Schwarz functions appear in the case of transforming the equation of a simple, smooth, closed (or non-closed) real function $L : F(x,y) = 0$ into a complex form. By using conjugate complex variables

$$\begin{aligned} z = x + iy, \quad \bar{z} = x - iy \\ x = \tfrac{1}{2}(z + \bar{z}), \quad y = \tfrac{1}{2i}(z - \bar{z}) \end{aligned} \tag{1.1}$$

the equation of real function becomes

$$F(x,y) = F\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = G(z,\bar{z}) = 0.$$

This equation under certain conditions can be expressed via $\bar{z}$ in the form

$$\bar{z} = S(z), \tag{1.2}$$

where $S(z)$ is an analytic function of complex variable in some domain Ω. For example, the complex form of some real curves is as follows:

a) A line crossing points z_1 and z_2:

$$\bar{z} = S(z) = \left(\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}\right)(z - z_2) + \bar{z}_2; \tag{1.3}$$

b) A circle with radius r and a center in the point z_0:

$$\bar{z} = S(z) = \frac{r^2}{z - z_0} + \bar{z}_0; \tag{1.4}$$

c) An elipse $(x^2/y^2) + (y^2/b^2) = 1, (a > b)$:

$$\bar{z} = S(z) = \frac{a^2 + b^2}{a^2 - b^2} z + \frac{2ab}{b^2 - a^2}\sqrt{z^2 + b^2 - a^2}; \tag{1.5}$$

d) A hyperbola $x^2 - y^2 = a^2$:

$$\bar{z} = S(z) = \sqrt{2a^2 - z^2}; \tag{1.6}$$

e) An equation of conic sections $ax^2 + 2bxy + cy^2 = 1$:

$$z^2(a - c - 2bi) + 2z\bar{z}(a + c) + \bar{z}^2(a - c + 2bi) = 4. \tag{1.7}$$

The complex equations (1.3) - (1.7) are self-conjugated, which means that by its conjugation the same equation is obtained. The functions $S(z)$ on the right side in mentioned equations are Schwarz functions for given curves.

Let L be a simple, smooth and closed contour. An analytic function is unique defined if the value in every point on the contour is defined. The Schwarz function for the curve L can be defined as a unique analytic function $S(z)$, that in every point z on the curve L its value is $\bar{z}$.

Let $g(z)$ be an analytic function such that the complex equation

$$\bar{z} = g(z) \tag{1.8}$$

describes a closed or non-closed contour, or a set of isolated points. Further, the set of points in the complex plane defined by (1.8) will be called K-contour [1].

In general case, it is clear that an arbitrary analytic function $g(z)$ cannot be a Schwartz function, because the condition of self-conjugation must be satisfied. But the condition of self-conjugation is not necessity, because there are self-conjugated functions that do not represent real curves. For example, $G(z, \bar{z}) = z\bar{z} + 1 = 0$ is a self-conjugated function, but it is not a real curve.

Let f be a conformal transformation that transforms a real segment $a \leq t \leq b$ into a simple smooth curve L. The condition of necessity and sufficiency for the analytic function $S(z)$ to be a Schwarz function [2] is

$$S = \overline{f} f^{-1}. \tag{1.9}$$

In the well known monograph [3], I. Vekua has considered in many details an elliptic system of partial differential equations

$$\begin{aligned} u'_x - v'_y &= a(x,y)u + b(x,y)v + c(x,y) \\ u'_y + v'_x &= b(x,y) - a(x,y)v + d(x,y), \end{aligned} \tag{1.10}$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$ are given continuous real functions of real arguments x and y in a simply connected domain T. This system takes an important role in theoretical and practical problems of mechanics. If the second equation in (1.10) is multiplied by i and after addition to the first equation, the following canonic Vekua complex differential equation is obtained

$$w'_{\bar{z}} = A\bar{w} + B, \tag{1.11}$$

where

$$A = \frac{a+ib}{2}, \quad B = \frac{c+id}{2},$$

and

$$w'_{\bar{z}} = \frac{1}{2}(u'_x - v'_y) + \frac{i}{2}(u'_y + v'_x).$$

It is clear that the general idea of Schwarz functions is expressing the contour equation in a complex form $\bar{z} = g(z)$. Naturally, this idea can be generalized into the general relation $\bar{w} = g(w)$, where $w = w(z,\bar{z})$ is a non-analytic complex function. In the second part of the paper, this relation is used to determinate the particular solution of the Vekua differential equation (1.11). In the third part, the notion of α-interpolation is introduced and instead of interpolation nods the interpolation contours are used represented by equations $\bar{z} = g_i(z)$, $(i = 0,1,2,...,n)$. So, the approximation problem of the non-analytic function $w = w(z,\bar{z})$ is solved via α-interpolation polynomial of the n-th degree. Finally, in the fifth part, the Vekua equation (1.11) is correlated with the theory of elastic shells and the Hilbert boundary value problem is considered with some mechanic interpretations that lead to possibility of application of α-interpolation.

2. Application of CRC-method to determinate a particular solution of Vekua differential equation

The method CRC (Complex Representation of Contour) is leaded by Čanak [1] and it uses the operator $\alpha_g(z)$, a complex presentation of a contour as $\bar{z} = g(z)$ and Schwarz functions. The idea of the application this method for solving Vekua differential equation is as follows: The general solution of the equation (1.11) given by Vekua practically can not be used because it contains infinite series with double singular integrals of Cauchy type, which is generally difficult for solving. The reason of this is existence of unknown function in equation (1.11) under the conjugation. Because of that, at first the Vekua equation (1.11) is reduced to an ordinary auxiliary differential equation. According the idea of Schwarz functions, the general relation $\bar{w} = g(w)$ is used. So, simultaneously with the Vekua equation

$$w'_{\bar{z}} = A\bar{w} + B \tag{1.11}$$

the auxiliary differential equation

$$w'_{\bar{z}} = Ag(w) + B \tag{2.1}$$

is used. The unknown function $g(w)$ is selected such that the function (2.1) is a finite integrable and convenient for solving.

In practice, usually the most frequent selection is $g(w) = Cw + D$, i.e. the linear relation

$$\bar{w} = Cw + D \tag{2.2}$$

where the coefficients $C = C(z, \bar{z})$ and $D = D(z, \bar{z})$ must be in a correlation with the coefficients of the Vekua equation. The relation (2.2) is useful because it conjugation is

$$w = \bar{C}\bar{w} + \bar{D} \tag{2.3}$$

and the unique solution of the system (2.2)-(2.3) is

$$w = \frac{\bar{C}D + \bar{D}}{1 - C\bar{C}}, \quad (C\bar{C} \neq 1). \tag{2.4}$$

For the Vekua equation (1.1) and on the base of (2.2), let the relation

$$A\bar{w} + B = \frac{w}{\bar{z}} \tag{2.5}$$

be introduced. The relation (1.11) is transformed into auxiliary differential equation

$$w'_{\bar{z}} = \frac{w}{\bar{z}} \tag{2.6}$$

with the solution

$$w = \bar{z}Q(z) \tag{2.7}$$

given by S. Fempl [4], $Q(z)$-is an arbitrary analytic function. By its substitution in (2.5), the relation

$$Az\overline{Q(z)} + B = Q(z) \tag{2.8}$$

is obtained.

The function $Q = Q(z)$ determinates the equality (2.8) to be identity. This equality can be written as

$$\overline{Q(z)} = \frac{Q(z) - B}{Az}. \tag{2.9}$$

As the left side of (2.9) is conjugated analytic function, the right side must be conjugated also. It is satisfied if the coefficient $A = A(z, \bar{z})$ is

$$A = \frac{Q(z) - B}{z\overline{Q(z)}}. \tag{2.10}$$

The following theorem is formulated

Theorem 2.1. *The Vekua complex differential equation (1.11) has particular solution $w = z\overline{Q(z)}$ for every value of the coefficient $B = B(z, \bar{z})$, if the coefficient A is in the form of (2.10).*

Example. Let us find a particular solution of the Vekua differential equation

$$w'_{\bar{z}} = \frac{z^2 - e^{z-3\bar{z}}}{z\bar{z}^2}\bar{w} + e^{z-3\bar{z}}. \tag{2.11}$$

Solution. The conditions of the theorem 2.1 are satisfied and in this case the coefficients are: $Q(z) = z^2$, $B(z, \bar{z}) = e^{z-3\bar{z}}$. The following equality is introduced

$$\frac{z^2 - e^{z-3\bar{z}}}{z\bar{z}^2}\bar{w} + e^{z-3\bar{z}} = \frac{w}{\bar{z}}.$$

The general solution of the auxiliary differential equation $w'_{\bar{z}} = \frac{w}{\bar{z}}$ is $w = \bar{z}Q(z)$ and a particular solution of the equation (2.11) is $w_p = \bar{z}z^2$.

3. An application of CRC-method on an interpolation problem

Definition 3.1. Let $g(z)$ be an analytic function in some domain Ω and $w = w(z, \bar{z})$ be a continuous complex function, which in the domain can be developed into a convergent power series by z and $\bar{z}$. Then the compound function $w(z, g(z))$ is an analytic function for which the notion $\alpha_{g(z)}w$ is used. The operator $\alpha_{g(z)}$ transforms the set of continuous complex functions $w = w(z, \bar{z})$ into a set of analytic functions and has geometrical meaning as follows: If $\bar{z} = g(z)$ is an equation of a closed contour, then the functions $w = w(z, \bar{z})$ and $\alpha_{g(z)}w$ have the same limit value on the mentioned contour.

Problem P. Let $L_0 : \bar{z} = g_0(z)$, $L_1 : \bar{z} = g_1(z), \ldots, L_n : \bar{z} = g_n(z)$ are given contours and all analytic functions $g_i(z), i = 0, 1, ..., n$ are different. Let $s_0(z), s_1(z), , ..., s_n(z)$ be given analytic functions and $w(z, \bar{z})$ be a given non-analytic function. The problem is to find interpolation polynomials $P_n(z, \bar{z})$ of n-th order that satisfy the interpolation conditions:

$$\begin{aligned} &\alpha_{g_0(z)}P_n = \alpha_{g_0(z)}w = s_0(z),\ \alpha_{g_1(z)}P_n = \alpha_{g_1(z)}w = s_1(z), ..., \\ &\alpha_{g_n(z)}P_n = \alpha_{g_n(z)}w = s_n(z). \end{aligned} \tag{3.1}$$

Let us construct a so called areolar ψ-differences:

$$\frac{\alpha_{g_1}w - \alpha_{g_0}w}{g_1 - g_0} = \frac{s_1(z) - s_0(z)}{g_1(z) - g_0(z)} = \psi(g_0, g_1)$$

$$\vdots$$

$$\frac{\alpha_{g_n}w - \alpha_{g_{n-1}}w}{g_n - g_{n-1}} = \frac{s_n(z) - s_{n-1}(z)}{g_n(z) - g_{n-1}(z)} = \psi(g_{n-1}, g_n) \tag{3.2}$$

$$\vdots$$

$$\frac{\psi(g_{n-1}, g_n) - \psi(g_{n-2}, g_{n-1})}{g_n - g_{n-2}} = \psi(g_{n-2}, g_{n-1}, g_n)$$

$$\vdots$$

$$\frac{\psi(g_1, g_2, ..., g_n) - \psi(g_0, g_1, ..., g_{n-1})}{g_n - g_0} = \psi(g_0, g_1, ..., g_{n-1}, g_n).$$

and let us construct a sequence of functions as follows:

$$\psi(\bar{z}, g_0) = \frac{\alpha_{g_0} w - w(z, \bar{z})}{g_0 - \bar{z}}, \ \psi(\bar{z}, g_0, g_1) = \frac{\psi(g_0, g_1) - \psi(\bar{z}, g_0)}{g_1 - \bar{z}} \tag{3.3}$$

$$\psi(\bar{z}, g_0, g_1, ..., g_n) = \frac{\psi(g_0, ..., g_n) - \psi(\bar{z}, g_0, ..., g_{n-1})}{g_n - \bar{z}}.$$

The interpolation polynomial

$$\begin{aligned} P_n(z, \bar{z}) = \varphi_0(z) + \varphi_1(z)(\bar{z} - g_0) + \varphi_2(z)(\bar{z} - g_0)(\bar{z} - g_1) + ... + \\ + \varphi_n(z)(\bar{z} - g_0)(\bar{z} - g_1)...(\bar{z} - g_{n-1}) \end{aligned} \tag{3.4}$$

is required, where $\varphi_0(z), \varphi_1(z), ..., \varphi_n(z)$ are unknown analytic functions. By a substitution of the first interpolation condition $\alpha_{g_0(z)} P_n = s_0(z)$ into (3.4), the relation $\varphi_0(z) = s_0(z)$ is obtained. By the same procedure, by a substitution of interpolation conditions (3.4), the relations

$$\varphi_1(z) = \frac{s_1(z) - s_0(z)}{g_1(z) - g_0(z)} = \psi(g_0, g_1),$$

$$\begin{aligned} &\varphi_2(z) = \psi(g_0, g_1, g_2), \\ &\vdots \\ &\varphi_n(z) = \psi(g_0, g_1, ..., g_n) \end{aligned} \tag{3.5}$$

are obtained.

The polynomial that is required is obtained by the substitution of (3.5) into (3.4) and it is

$$\begin{aligned} P_n(z, \bar{z}) = s_0(z) + (\bar{z} - g_0)\psi(g_0, g_1) + (\bar{z} - g_0)(\bar{z} - g_1)\psi(g_0, g_1, g_2) + \\ + ... + (\bar{z} - g_0)(\bar{z} - g_1)...(\bar{z} - g_{n-1})\psi(g_0, g_1, ..., g_n). \end{aligned} \tag{3.6}$$

The formulas (3.3) can be written as

$$\begin{aligned} &w(z, \bar{z}) = \alpha_{g_0} w + (\bar{z} - g_0)\psi(\bar{z}, g_0) \\ &w(z, \bar{z}) = \alpha_{g_0} w + (\bar{z} - g_0)\psi(g_0, g_1) + (\bar{z} - g_0)(\bar{z} - g_1)\psi(\bar{z}, g_0, g_1) \\ &\vdots \\ &w(z, \bar{z}) = \alpha_{g_0} w + (\bar{z} - g_0)\psi(g_0, g_1) + (\bar{z} - g_0)(\bar{z} - g_1)\psi(g_0, g_1, g_2) + \\ &+ ... + (\bar{z} - g_0)(\bar{z} - g_1)...(\bar{z} - g_{n-1})\psi(g_0, g_1, ..., g_n) + \\ &+ (\bar{z} - g_0)(\bar{z} - g_1)...(\bar{z} - g_n)\psi(\bar{z}, g_0, g_1, ..., g_n). \end{aligned} \tag{3.7}$$

If in (3.7) the value of $\bar{z}$ is substituted by $g_0(z), g_1(z), ..., g_n(z)$, the relations

$$\alpha_{g_0(z)}w = s_0(z),\ \alpha_{g_1(z)}w = s_1(z), ..., \alpha_{g_n(z)}w = s_n(z)$$

are valid. It means that by substitution of the last term, the polynomial (3.6) is obtained such that on the given contours $\bar{z} = g_i(z)$, $i = 0, 1, 2, ..., n$, it has the same boundary values as the function $w(z, \bar{z})$ does, so it is an approximation of the function. The estimation error is the estimation of the last term in (3.7).

The application of this result is shown in the next part of the paper, and the previous interpolation is applied on Vekua differential equation and on some of its applications.

4. Hilbert boundary value problem for the equation of elastic shell voltage

It is known [3] that the equations of momentless position of the elastic shell voltage are

$$\text{(4.1)}\qquad \begin{aligned} &\frac{1}{\sqrt{f}}\frac{\partial\sqrt{f}T^{\alpha\beta}}{\partial x^{\alpha}} + \Gamma^{\beta}_{\alpha\lambda}T^{\alpha\lambda} = X^{\beta}, \quad (\beta = 1, 2)\\ &\pi_{\alpha\beta}T^{\alpha\beta} = Z, \end{aligned}$$

where $T^{11}, T^{12} = T^{21}, T^{22}$ are opposite-variant components of the voltage tensor, and X^1, X^2, Z are given functions of a point (x^1, x^2) from the middle shell surface.

The complex function of the voltage position of the elastic shell is the function

$$\text{(4.2)}\qquad U = f\sqrt[4]{K}(T^{11} - iT^{12}),$$

where K is the main curve from the surface. The value f is given via the relation

$$\text{(4.3)}\qquad f = f_{11}f_{22} - f_{12}^2 > 0$$

where $f_{\alpha\beta} = \tau_\alpha\tau_\beta$ and $\pi_{\alpha\beta} = n\tau_{\alpha\beta}$ are symmetric covariant tensors of second order, $\tau(x^1, x^2)$ is a surface radius vector of the surface, and $\tau_1 = \frac{\partial\tau}{\partial x^1}$, $\tau_2 = \frac{\partial\tau}{\partial x^2}$ are basic vectors of the coordinate system (x^1, x^2) and n is the ort from the surface normal.

Vekua has shown [3] that the equation (4.1) can be reduced on complex differential equation

$$\text{(4.4)}\qquad U'_{\bar{z}} - A(z, \bar{z})\bar{U} = F(z, \bar{z}),$$

where

$$A(z,\bar{z}) = \Gamma^2_{12} + i\Gamma^1_{12} - \frac{1}{4K}\frac{\partial K}{\partial \bar{z}} - \frac{1}{\sqrt{f}}\frac{\partial \sqrt{f}}{\partial \bar{z}} \tag{4.5}$$

and

$$F(z,\bar{z}) = \frac{1}{2} f \sqrt[4]{K}\left[X^1 - iX^2 - \frac{z}{D}(\Gamma^1_{22} - i\Gamma^2_{22}) + \frac{i}{\sqrt{f}}\frac{\partial}{\partial y}\frac{\sqrt{f}Z}{D}\right]. \tag{4.6}$$

To determine the unknown function U, it is necessary the function to satisfy some additional condition on the given closed contour L. Usually in practice, the following boundary value condition is used

$$\mathrm{Re}[g(t)U] = \gamma(t), \quad (g = \alpha - i\beta,\ |g| \neq 0), \tag{4.7}$$

where $g(t)$, $\gamma(t)$ are given functions of a point $t \in L$.

The following condition has the same form

$$N_\nu \cos\sigma(t) + S_\nu \sin\sigma(t) = \gamma(t), \tag{4.8}$$

where $\sigma(t)$, $\gamma(t)$ are given real functions of a point $t \in L$, and N_ν, S_ν are normal and tangent forces that affect the surface cross section that has a normal $\vec{\nu}$. The mechanical meaning of this condition is: in every point t from the contour L from the domain T, there is a force (projection of a force vector) that affects into direction $\cos\sigma(t)$ and $\sin\sigma(t)$.

If everywhere on L the condition $\sigma(t) = 0$ is valid, then the boundary value condition (4.8) becomes

$$N_\nu = \gamma(t), \tag{4.9}$$

which means that in every point of the contour a normal force affects.

If everywhere on L the condition $\sigma(t) = \frac{\pi}{2}$ is valid, then boundary value condition (4.8) becomes

$$S_\nu = \gamma(t), \tag{4.10}$$

which means that in every point of the contour a tangent force affects.

In this paper, the boundary value problem (4.9) is considered for the equation (4.4) and for shells with a positive Gauss curve. Boundary value problem (4.10) can be discussed in the same way.

The boundary value problem (4.9) can be written in the following form ([3], p.80-81)

$$\mathrm{Re}[(\nu_1 + i\nu_2)^2 U] = \gamma_0(t), \tag{4.11}$$

where ν_1, ν_2 are covariant components of the ort $\vec{\nu}$, that in the point (x^1, x^2) cross the middle surface and lie into the tangent plane and

$$\gamma_0(t) = (\gamma - N_\nu^0) f \sqrt[4]{K}. \tag{4.12}$$

But the condition (4.11) can be written in the form

$$\begin{aligned} &pU_1 + qU_2 = \gamma_0(t) \\ &(p = \nu_1^2 - \nu_2^2, \quad q = -2\nu_1\nu_2, \quad U = U_1 + iU_2) \end{aligned} \tag{4.13}$$

that is a type of Hilbert boundary value problem.

Generally, there is not a procedure for effective solving of the mentioned problem. Because of that, in this paper some important cases in practice will be shown when the solution can be found in the finite form.

I Case: Let

$$A(z, \bar{z}) = \Gamma_{12}^2 + i\Gamma_{12}^1 - \frac{1}{4K}\frac{\partial K}{\partial \bar{z}} - \frac{1}{\sqrt{f}}\frac{\partial \sqrt{f}}{\partial \bar{z}} = 0. \tag{4.14}$$

In this case, the equation (4.4) is transformed into

$$U'_{\bar{z}} = F(z, \bar{z}) \tag{4.15}$$

and has a general solution

$$U = \int F(z, \bar{z}) d\bar{z} + Q(z) = U_0(z, \bar{z}) + Q(z), \tag{4.16}$$

where $Q(z)$ is an arbitrary analytic function. By using the notations

$$U_0(z, \bar{z}) = u_0(x, y) + iv_0(x, y), \quad Q(z) = q_1(x, y) + iq_2(x, y),$$

the boundary value problem (4.13) is transformed into

$$p(u_0 + q_1) + q(v_0 + q_2) = \gamma_0(t),$$

or after some calculations

$$pq_1 + qq_2 = \gamma_0(t) - pu_0 - qv_0, \tag{4.17}$$

the Hilbert boundary value problem to determinate an analytic function $Q(z) = q_1 + iq_2$ is obtained by a known solving procedure [5]. By a determination of an unknown analytic function $Q(z)$, the function $U = U_1 + iU_2$ is determined that is a solution of the boundary value problem.

II Case: Let one of covariant components of the vector $\vec{\nu}$, i.e. ν_1 or ν_2 be zero. Then the boundary condition (4.13) becomes

$$U_1 = \frac{\gamma_0(t)}{p} \tag{4.18}$$

and the solution of the equation (4.4) is searched whose real component U_1 on the contour L is $\frac{\gamma_0(t)}{p}$.

Consideration becomes very interesting if the solution of the problem is complex function whose real or imaginary part is a harmonic function with solenoid or potential vector field. The class of differentiable complex functions whose real part is a harmonic function with corresponding solenoid vector field is denoted by U_{Rh}. By harmonic extending a unique harmonic function $U_1(x, y) = \alpha(x, y)$ can be determined that on L has a value (4.18). To determine the unknown imaginary part $U_2(x, y)$ of the solution, the equation (4.4) is written as

$$\frac{1}{2}\left(\frac{\partial \alpha}{\partial x} - \frac{\partial U_2}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial \alpha}{\partial y} - \frac{\partial U_2}{\partial x}\right) - (a_1 + ia_2)(\alpha - iU_2) = f_1 + if_2,$$

and by separating the real and imaginary part

$$\begin{array}{l} \frac{\partial \alpha}{\partial x} - \frac{\partial U_2}{\partial y} - 2a_1\alpha - 2a_2U_2 = 2f_1 \\ \frac{\partial \alpha}{\partial y} - \frac{\partial U_2}{\partial x} - 2a_2\alpha + 2a_1U_2 = 2f_2 \end{array} \tag{4.19}$$

are obtained. Eliminating U_2 from (4.19), the partial equation

$$a_2\frac{\partial U_2}{\partial x} - a_1\frac{\partial U_2}{\partial y} = \beta(x, y) \tag{4.20}$$

is obtained where $\beta = 2(f_1a_1 + f_2a_2) + 2\alpha(a_1^2 + a_2^2) - a_1\frac{\partial \alpha}{\partial x} - a_2\frac{\partial \alpha}{\partial y}$.
System of ordinary differential equations

$$\frac{dx}{a_2(x, y)} = \frac{dy}{-a_1(x, y)} = \frac{dU_2}{\beta(x, y)} \tag{4.21}$$

corresponds to the equation (4.20)
Let us suppose that the first integrals of this system are

$$\varphi_1(x, y, U_2) = c_1, \quad \varphi_2(x, y, U_2) = c_2. \tag{4.22}$$

Then the general solution of (4.20) is

$$G(\varphi_1, \varphi_2) = 0, \tag{4.23}$$

where G is an arbitrary differential function. Let us suppose that (4.23) can be expressed in the form

$$U_2 = \psi\{x, y, \varphi[\xi(x, y)]\}, \tag{4.24}$$

where φ is an arbitrary differential function and ψ, ξ are known functions. By substitution of (4.24) into the first equation of (4.19), the following differential of the first order with φ as an unknown function

$$\frac{\partial \alpha}{\partial x} - \left(\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial \varphi}\varphi'_\xi \frac{\partial \xi}{\partial y}\right) - 2a_1\alpha - 2a_2\psi\{x, y, \varphi[\xi(x, y)]\} = 2f_1 \tag{4.25}$$

is obtained. If it is possible to solve this equation, then by substituting the value of φ into (4.24), the imaginary part of the solution of the boundary value problem can be obtained.

Example 4.1. Find the solution of the Vekua differential equation

$$U'_{\bar{z}} - (a + ia)\bar{U} = \frac{i}{2} - x(a + ia) \tag{4.26}$$

that belongs to the class U_{Rh} and its real part on the unit circle $L : x^2 + y^2 = 1$, $(|z|) = 1)$ is $\cos t$, $(\mathrm{Re}U|_L = \cos t)$.

Solution. By solving the Dirichlet boundary value problem for unit circle, it is found that the required harmonic function $U_1(x, y)$, $(U = U_1 + iU_2)$ is $U_1 = x$. Then the system (4.19) is transformed into

$$\begin{aligned} 1 - U'_{2_y} - 2aU_2 = 0 \\ U'_{2_x} + 2aU_2 = 1 \end{aligned} \tag{4.27}$$

and by addition of these equations, the following partial differential equation

$$U'_{2_x} - U'_{2_y} = 0 \tag{4.28}$$

is obtained. Its general solution is

$$U_2(x, y) = \varphi(x + y), \tag{4.29}$$

where φ is an arbitrary differentiable function. By its substitution in the second equation of (4.27), the ordinary differential equation

$$\varphi'(x+y) + 2a\varphi(x+y) - 1 = 0 \tag{4.30}$$

is obtained with a general solution

$$\varphi(x,y) = ce^{-2a(x+y)} + \frac{1}{2a} = U_2(x,y). \tag{4.31}$$

Finally, the solution of the boundary value problem is $U = U_1 + U_2$.

III Case: Let be $\nu_1 = \pm\nu_2$, then the boundary condition (4.13) is

$$U_2 = \frac{\gamma_0(t)}{q}, \tag{4.32}$$

i.e. let us find the solution of the equation (4.4) whose imaginary part U_2 on the contour L takes the value $\frac{\gamma_0(t)}{q}$. This case is equivalent as the previous one.

Remark 4.1. It is shown that voltage position of the elastic shell can be described via generalized analytic function of the Vekua type where the α-interpolation is very useful. So, if on the middle surface of the shell $n+1$ different closed contours are selected and if the values of the function $w(z,\bar{z})$ on these contours are given, then by using α-interpolation this function is approximated by a polynomial that is convenient for numerical treatment. The calculation is simpler in the case of rotational shells, because for the contours the system of circles $\bar{z} = a_i/z$ may be chosen ($g_i = a_i/z$-are Schwarz functions) and than the $\alpha_{a_i/z}$ interpolation can be applied.

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