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Laplace Vector Fireld for Vekua Differential Equation

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1. Introduction

In [1] I. Vekua has considered an elliptic system of partial differential equations

$$\begin{aligned} u'_x - v'_y &= a(x,y)u + b(x,y)v + f(x,y) \\ u'_y + v'_x &= c(x,y)u + d(x,y)v + g(x,y), \end{aligned} \tag{1.1}$$

where $a(x,y), b(x,y), c(x,y), d(x,y), f(x,y), g(x,y)$ are continuous real functions of real arguments x and y in a simply connected domain T, that plays an important role in theoretical and practical problems of mechanics. The system (1.1) can be transformed to Vekua complex differential equation

$$U'_{\bar{z}} = MU + N\bar{U} + L, \tag{1.2}$$

where $M(z,\bar{z}) = \frac{a+d+ic-ib}{4}$, $N(z,\bar{z}) = \frac{a-d+ic+ib}{4}$,
$L(z,\bar{z}) = \frac{f+ig}{2}$, $U(z,\bar{z}) = u + iv$.

By a substitution $U = wU_0$, where w is a new unknown function and U_0 is a regular particular solution of the equation $U'_{\bar{z}} = MU$ (see [2]), the equation (1.2) is transformed into canonic form

$$w'_{\bar{z}} = A\bar{w} + B \tag{1.3}$$

where $A = \frac{N\bar{U}_0}{U_0}$, $B = \frac{L}{U_0}$.

2. Of a vector field for canonic Vekua complex differential equation

A. Bilimovic [3] has developed a geometrical theory of generalized analytic functions $w(z, \bar{z}) = u(x, y) + iv(x, y)$ based on the deviation vector from the analyticness

$$\vec{B} = grad\ u + \vec{k} \times grad\ v = (u'_x - v'_y)\vec{i} + (u'_y + v'_x)j \tag{2.1}$$

where $\vec{i}, \vec{j}, \vec{k}$ are three normal orts.

The vector field of $\vec{B}$ characterizes different classes of generalized analytic functions in geometric sense. In this paper the generalized analytic functions in the sense of Vekua are considered. For that, the differential equation (3.1) is considered, that can be written as a system of partial equations

$$\begin{aligned} u'x - v'y &= a(x, y)u + b(x, y)v + c(x, y) \\ u'y + v'x &= b(x, y)u - a(x, y)v + d(x, y) \end{aligned} \tag{2.2}$$

where $A = \frac{a+ib}{2}$, $B = \frac{c+id}{2}$, $w'_{\bar{z}} = \frac{1}{2}(u'_x - v'_y) + \frac{i}{2}(u'_y + v'_x)$.

From (2.1) it is clear that

$$div\vec{B} = \nabla^2 u, \quad rot\vec{B} = \vec{k}\nabla^2 v. \tag{2.3}$$

These relations enable a classification of generalized Vekua analytic functions (2.2). For $\nabla^2 u = 0, \quad \nabla^2 v = 0$ the vector field of of $\vec{B}$ is Laplace, for $\nabla^2 u = 0, \quad \nabla^2 v \neq 0$ the vector field is a solenoid, for $\nabla^2 u \neq 0, \quad \nabla^2 v = 0$ the vector field is a potential, and finally, for $\nabla^2 u \neq 0, \quad \nabla^2 v \neq 0$ the vector field of $\vec{B}$ is a complex.

Generally, in the case when a class of complex functions or a class of differential equations that is generalization of the analytic functions, the introduction of vector field classification and discussion of its different types has more reasons and goals. Behind the geometric interpretation, this classification enables three consecutive generalizations, from the class of analytic functions to the most generalized goal class. So, the Laplace vector field corresponds to the first and the simplest generalization. The solenoid and potential fields are a strong generalization, and the complex vector field corresponds to the most generalized case.

In this paper only the Laplace vector field is considered.

3. Laplace vector field for canonic Vekua complex differential equation

Relations $div\vec{B} = \nabla^2 u = 0, \quad rot\vec{B} = \vec{k}\nabla^2 v = 0$ lead to

$$u''_{xx} + u''_{yy} = 0, \quad v''_{xx} + v''_{yy} = 0. \tag{3.1}$$

The sum of two equations, the first obtained from (2.2) differentiated by x and the second differentiated by y, by using the first relation from (3.1) gives

$$(u'_x - v'_y)a + (v'_x + u'_y)b = -(a'_x + b'_y)u + (a'_y - b'_x)v - c'_x - d'_y. \tag{3.2}$$

Similarly,

$$(u'_x - v'_y)b - (v'_x + u'_y)a = (a'_y - b'_x)u + (a'_x + b'_y)v + c'_y - d'_x \tag{3.3}$$

is obtained.

By the substitution of $u'_x - v'_y$ and $v'_x + u'_y$ from (2.2) in (3.2) and (3.3) the following equalities are obtained

$$\begin{aligned} &(a^2 + b^2 + a'_x + b'_y)u + (b'_x - a'_y)v = -ac - bd - d'_y - c'_x \\ &(b'_x - a'_y)u + (a^2 + b^2 - a'_x - b'_y)v = ad - bc - d'_x + c'_y. \end{aligned} \tag{3.4}$$

According to D. Mitrinovic [4], two cases can be distinguished:

I) Equalities in (3.4) are not identically satisfied. The values of u and v are solutions of the linear system (3.4)

$$u = \frac{C_1B_2 - C_2B_1}{A_1B_2 - A_2B_1}, \quad v = \frac{A_1C_2 - A_2C_1}{A_1B_2 - A_2B_1},$$

where the coefficient are:

$$\begin{aligned} &A_1 = a^2 + b^2 + a'_x + b'_y, \quad A_2 = b'_x - a'_y, \\ &B_1 = b'_x - a'_y, \quad B_2 = a^2 + b^2 - b'_y - a'_x, \\ &C_1 = -ac - bd - d'_y - c'_x, \quad C_2 = ad - bc - d'_x + c'_y. \end{aligned}$$

These values are particular solution of the elliptical system (2.2).

II) Equalities (3.4) are identically satisfied for all values of the variables u and v. In this case, the following conditions must be valid

$$\begin{aligned} &a^2 + b^2 + a'_x + b'_y = 0 \\ &b'_x - a'_y = 0 \\ &ac + bd - d'_y + c'_x = 0 \\ &a^2 + b^2 - (a'_x + b'_y) = 0 \\ &ad - bc - d'_x + c'_y = 0. \end{aligned} \tag{3.5}$$

The first and the fourth condition from (3.5) form a homogenous linear system where $(a^2 + b^2)$ and $(a'_x + b'_y)$ are variables. As the determinant of the system is $D = -2 \neq 0$, the system has only trivial solutions $a^2 + b^2 = 0$ and $a'_x + b'_y = 0$. This is valid for $a(x, y) = b(x, y) = 0$, when the second condition from (3.5) is satisfied also. The third and the fifth conditions from (3.5) are reduced to

$$d'_x = c'_y, \quad d'_y = -c'_x, \tag{3.6}$$

which means that the function

$$i\bar{C} = i(c - id) = d + ic = F(z)$$

is an analytic function.

Contrary, if it is supposed that $a(x, y) = b(x, y) = 0$ and $d(x, y)$ and $c(x, y)$ are harmonically conjugated, by its substitution in (2.2) and by differentiation, the equation

$$u''_{xx} + u''_{yy} = v''_{xx} + v''_{yy} = 0$$

is obtained.

So, the following can be formulated

Theorem 3.1. *The vector field of $\vec{B}$ for generalized Vekua analytic function* (1.3) *or* (2.2) *is Laplace if and only if $a(x, y) = b(x, y) = 0$ and the functions $d(x, y)$ and $c(x, y)$ are harmonically conjugated.*

Remark 3.1. In the case when the conditions from Theorem 3.1. are satisfied, the equation (1.3) is transformed into

$$w'_{\bar{z}} = \frac{i}{2}\bar{F} \tag{3.7}$$

with the general solution

$$w(z, \bar{z}) = \frac{i}{2}\int \bar{F} d\bar{z} + \Phi(z), \tag{3.8}$$

where $\Phi(z)$ is an arbitrary analytic function.

4. Laplace vector field for p-analytic functions

G. Plozij [5] introduced the following definition for p-analytic function: The function $f(z) = u+iv$ is named a p-analytic function with the characteristic $p = p(x, y)$ in domain Ω if it is defined on that domain and its real and imaginary part have continuous derivatives of the first order on x and y and satisfy the system of partial differential equations

$$u'_x = \frac{1}{p}v'_y, \quad u'_y = -\frac{1}{p}v'_x. \tag{4.1}$$

Polozij has shown that there is a deep connection between the solutions of the system (4.1) and the analytic functions. He has investigated some applications of p-analytic functions in different fields of mechanics also. The system (4.1) can be written as

$$u'_x - v'_y = \frac{1-p}{p}v'_y, \quad u'_y + v'_x = -\frac{1-p}{p}v'_x. \tag{4.2}$$

By the substitution $v = v_1 p$, v_1 is a new unknown function and the system (4.2) is transformed into

$$u'_x - v'_{1y} = \frac{1}{p}(p'_y v_1), \quad u'_y + v'_{1x} = -\frac{1}{p}(p'_x v_1)$$

or into a corresponding complex form

$$f'_{\bar{z}} = -\frac{p'_{\bar{z}}}{2p}(f - \bar{f}), \quad (f = u + iv_1). \tag{4.3}$$

By the new substitution $f = Up^{-\frac{1}{2}}$, $U = U(z, \bar{z})$ is a new unknown function, and the complex differential equation is transformed into canonic form

$$U'_{\bar{z}} = \frac{p'_{\bar{z}}}{2p}\bar{U}, \tag{4.4}$$

or into the following system of partial equations

$$\begin{aligned} u'_{1x} - u'_{2y} &= a(x,y)u_1 + b(x,y)u_2 \\ u'_{1y} + u'_{2x} &= b(x,y)u_1 - a(x,y)u_2, \end{aligned} \tag{4.5}$$

where $U = u_1 + iu_2$, $a = \frac{p'_x}{2p}$, $b = \frac{p'_y}{2p}$ and $div\vec{B} = \nabla^2 u_1$, $rot\vec{B} = \vec{k}\nabla^2 u_2$.

M. Čanak, L. Stefanovska and Lj. Protic has shown [6] the following

Theorem 4.1. *The vector field of $\vec{B}$ for the system (4.5) is Laplace ($div\ \vec{B} = 0$, $rot\ \vec{B} = \vec{0}$) if and only if $p = c = const$.*

Remark 4.1. In the upcoming text it is shown that p-analytic functions that correspond to the Laplace vector field are opposite for introducing of so called p-polyanalytic functions that can be used in the approximation theory.

5. Approximation of the non-analytic complex functions with p-analytic functions

The notion of p-polyanalytic function is given by M. Čanak [7]. In this paper the p-polyanalytic functions of the form

$$P_n(z,\bar{z}) = \sum_{k=0}^{n} \left(\frac{z-\bar{z}}{2i}\right)^k f_k(z,\bar{z}) \tag{5.1}$$

are considered, where $f_k(z,\bar{z})$ are arbitrary p-analytic function defined by (4.1), $(f = u + iv)$.

The complex form of the system is

$$pDf + i(1-p)Dv = 0, \quad (p \neq 0), \tag{5.2}$$

where $Df = (u'_x - v'_y) + i(u'_y + v'_y) = 2f'_{\bar{z}}$; $(f = u + iv)$, is the Kolosov differential operator known as an areolar derivative.

The characteristic $p = p(x,y)$ in (5.2) is a real function with real variables x and y. In the case when the corresponding vector field is Laplace, $p = const$ and the equation (5.2) is transformed into

$$D[pf + i(1-p)v] = 0. \tag{5.3}$$

Then

$$pf + i(1-p)v = \varphi(z)$$

or

$$\frac{p+1}{2}f + \frac{p-1}{2}\bar{f} = \varphi(z), \tag{5.4}$$

where $\varphi(z)$ is an arbitrary analytic function.

By conjugation on (5.4) and some calculations, the relation

$$\frac{p+1}{2p}\varphi - \frac{p-1}{2p}\bar{\varphi} = f(z,\bar{z}) \tag{5.5}$$

is obtained.

Let K be a set of continuous complex functions, P_c be a set of p-analytic functions with a constant characteristic (with corresponding Laplace field), and A be a set of analytic functions ($A \subset P_c \subset K$).On the basis of (5.4) and (5.5), the following operators on the set K are introduced:

$$R_c w = \frac{(c+1)w - (c-1)\bar{w}}{2c}, \quad R_c^{-1} w = \frac{(c+1)w + (c-1)\bar{w}}{2c}. \tag{5.6}$$

It is clear that the relation $R_c^{-1} R_c w = R_c R_c^{-1} w = w(z,\bar{z})$ is valid.

There is a unique p-analytic function $f(z,\bar{z})$ with a constant characteristic $p = c$ that corresponds to any analytic function $\varphi(z)$ by using the operator $R_c\varphi = \frac{c+1}{2c}\varphi - \frac{c-1}{2c}\bar{\varphi} = f(z,\bar{z})$. Consequently, a unique analytic function $\varphi(z)$ corresponds to any p_c-analytic function $f(z,\bar{z})$ by using the operator $R_c^{-1} f = \frac{c+1}{2} f + \frac{c-1}{2}\bar{f} = \varphi(z)$.

Let us consider the complex polynomials

$$F(z,\bar{z}) = \sum_{k=0}^{n} \left(\frac{z-\bar{z}}{2i}\right)^k f_{K_c}(z,\bar{z}), \quad \Phi(z,\bar{z}) = \sum_{k=0}^{n} \left(\frac{z-\bar{z}}{2i}\right)^k \varphi_k(z),$$

where f_{K_c} are arbitrary p-analytic functions with a constant characteristic $p = c$ and $\varphi_k(z)$ are arbitrary analytic functions. The function $F(z,\bar{z})$ is a p-polyanalytic function and $\Phi(z,\bar{z})$ is an ordinary analytic function. It can be remarked that the following relations hold:

$$R_c^{-1} F = R_c^{-1}\left[\sum_{k=0}^{n} \left(\tfrac{z-\bar{z}}{2i}\right)^k f_{K_c}\right] = \sum_{k=0}^{n} \left(\tfrac{z-\bar{z}}{2i}\right)^k R_c^{-1} f_{K_c},$$

$$R_c \Phi = R_c\left[\sum_{k=0}^{n} \left(\tfrac{z-\bar{z}}{2i}\right)^k \varphi_k\right] = \sum_{k=0}^{n} \left(\tfrac{z-\bar{z}}{2i}\right)^k R_c \varphi_k.$$

On the set of differentiable complex functions in the sense of Kolosov, we introduce a new operator A_c as a composition of three operators R_c, R_c^{-1} and D via the relation

$$A_c w(z,\bar{z}) = R_c D R_c^{-1} w. \tag{5.7}$$

This operator is a generalization of the Kolosov operator D and in the special case ($c = 1$) is reduced as such.

It is easy to show that the following properties of the operator hold:

(I) $A^{(0)}w = R_cR_c^{-1}w = w_1 A^{(1)}w = Aw \ldots A^{(k)}w = A[A^{(k-1)}w]$

(II) $A[w_1(z,\bar{z}) \pm w_2(z,\bar{z})] = A[w_1(z,\bar{z})] \pm A[w_2(z,\bar{z})]$

(III) $A[f_p(z,\bar{z}) = 0, \quad (f_p \in P_c)$

(IV) $A\left[\sum_{k=0}^{n} \left(\frac{z-\bar{z}}{2i}\right)^k f_{K_c}\right] = \sum_{k=1}^{n} K \left(\frac{z-\bar{z}}{2i}\right)^{k-1} R_c(i\varphi_k)$

Let $w = w(z,\bar{z})$ be a continuous and arbitrary times differentiable function in the sense of Kolosov and let us expand it in the complex power series

$$w(z,\bar{z}) = f_{0_c}(z,\bar{z}) + \left(\frac{z-\bar{z}}{2i}\right) f_{1_c}(z,\bar{z}) + \ldots + \left(\frac{z-\bar{z}}{2i}\right)^n f_{n_c}(z,\bar{z}) + \ldots \tag{5.8}$$

where $f_{0_c}, f_{1_c}, \ldots f_{n_c}$ are unknown p-analytic functions with a constant characteristic $p = c$.

At first let us introduce a mapping $\alpha_{g(z)}$ from the set K to the set A, ($g(z)$ is an arbitrary analytic function): The function $\Omega = \alpha_{g(z)}w$, ($w = w(z,\bar{z}) \in K$, $\Omega(z) \in A$) is obtained from the function $w = w(z,\bar{z})$ when $\bar{z}$ is substituted by $g(z)$, but the z is not substituted. In the function $\Omega = \alpha_{g(z)}w$ there is no variable $\bar{z}$, that generally means it is an analytic function. The geometric meaning of this operator is as follows: If $\bar{z} = g(z)$ is an equation of a simple smooth closed contour, then the functions $w = w(z,\bar{z})$ and $\alpha_{g(z)}w$ have the same boundary value on the mentioned contour.

If the operator α_z is applied on the both sides of (5.8), then the relation

$$\alpha_z w(z,\bar{z}) = \alpha_z f_{0_c}(z,\bar{z}) = \alpha_z \frac{(c+1)\varphi_0(z) - (c-1)\overline{\varphi_0(z)}}{2c} \tag{5.9}$$

is obtained. But only the subset $A_r \subset A$ whose Taylor series have only real coefficients is considered. In that case

$$\overline{\varphi(z)} = \overline{\sum_{k=1}^{\infty} a_k z^k} = \sum_{k=0}^{\infty} \overline{a_k z^k} = \sum_{k=0}^{\infty} a_k \bar{z}^k = \varphi(\bar{z})$$

and the relation (5.9) is transformed into

$$\alpha_z w(z,\bar{z}) = \alpha_z \frac{(c+1)\varphi_0(z) - (c-1)\varphi_0(\bar{z})}{2c} =$$

$$= \frac{(c+1)\varphi_0(z)}{2c} - \frac{(c-1)\varphi_0(z)}{2c} = \frac{\varphi_0(z)}{c},$$

that gives the relations $\varphi_0(z) = c\alpha_z w(z,\bar{z})$ and $f_{0_c} = R_c[c\alpha_z w(z,\bar{z})]$.

By applying the operator A and property (IV) on (5.8), the relations

$$Aw(z,\bar{z}) = R_c(i\varphi_1) + 2\left(\frac{z-\bar{z}}{2i}\right) R_c(i\varphi_2) + \ldots + n\left(\frac{z-\bar{z}}{2i}\right)^{n-1} R_c(i\varphi_n) + \ldots$$

and

$$\alpha_z Aw(z,\bar{z}) = \alpha_z R_c(i\varphi_1) = \alpha_z \frac{(c+1)i\varphi_1 - (c-1)\overline{i\varphi_1}}{2c} = \frac{i\varphi_1}{c}, \quad (i\varphi_1 \in A_r).$$

are obtained. From these relations, it is found that

$$\varphi_1 = \frac{c}{i}\alpha_z Aw(z,\bar{z}) \text{ and } f_{1_c} = R_c\left[\frac{c}{i}\alpha_z Aw(z,\bar{z})\right].$$

By the same procedure the relation

$$\alpha_z A^{(2)} w = \alpha_z 2R_c[i^2\varphi_2] = 2!i^2\frac{\varphi_2}{c}, \quad (\varphi_2 \in A_r)$$

is obtained from where

$$\varphi_2 = \frac{c}{2!i^2}\alpha_z A^{(2)} w, \quad f_{2_c} = R_c\left[\frac{c}{2!i^2}\alpha_z A^{(2)} w\right],$$

are obtained, or generally,

$$f_{k_c} = R_c\left[\frac{c}{k!i^k}\alpha_z A^{(k)} w\right], \quad (\varphi_{2k} \in A_r,\ i\varphi_{2k+1} \in A_r). \tag{5.10}$$

If f_{k_c} from (5.10) is substituted in (5.8), then the formal power series in areolar form of the function $w(z,\bar{z})$ is obtained via

$$w(z,\bar{z}) \approx \sum_{k=0}^{\infty} \left(\frac{z-\bar{z}}{2i}\right)^k R_c\left[\frac{c\alpha_z A^k w}{k!i^k}\right] \tag{5.11}$$

M. Čanak [8] has proved the following theorem

Theorem 5.1. *Let $w(z,\bar{z})$ be a given complex function that is arbitrary times differentiable in the interval $-\delta/2 \leq \mathrm{Im}z \leq \delta/2$ and its areolar derivatives* *are limited on it. Then the function $w(z,\bar{z})$ in δ-interval can be approximated by p-polyanalytic function on the n-th order as*

$$w(z,\bar{z}) \approx \sum_{k=0}^{n} \left(\frac{z-\bar{z}}{2i}\right)^k R_c \left[\frac{c\alpha_z A^{(k)} w}{k! i^k}\right]. \tag{5.12}$$

For the error estimation of the approximation, the Cauchy estimation can be used and $|R| \to 0$ when $n \to \infty$. In the case when the positive number $\varsigma = \sup \varsigma_k$, $(k = n+1, n+2, ...)$ exists, when ζ_k are majorants $|\alpha_z A^{(k)} w| \leq \varsigma_k$, $(k = n+1, n+2, ...)$, then the error can be estimated by

$$|R| \leq \sum_{k=n+1}^{\infty} \left(\frac{\delta}{2}\right)^k \frac{c\varsigma_k}{k!} \leq c\varsigma \sum_{k=n+1}^{\infty} \left(\frac{\delta}{2}\right)^k \frac{1}{k!} \leq c\varsigma \frac{(\delta/2)^{n+1}}{(n+1)!} \exp(\delta/2). \tag{5.13}$$

Remark 5.1. Non-analytic function $w(z,\bar{z})$ can be approximated via ordinary polyanalytic function and p-polyanalytic function. In the first case the error of the approximation [9] is

$$\varsigma \frac{(\delta/2)^{n+1}}{(n+1)!} \exp(\delta/2).$$

So, the second approximation (5.12)-(5.13) is better because the characteristic $p = c$ can be chosen to be arbitrary small and such to have an influence on error estimation.

Remark 5.2. In the case when $p = p(x,y)$, transition from (5.2) to (5.3) is not possible and the generalization of the theorem 5.1 is very complicated if the characteristic is a function. That is the reason why in this paper only the Laplace vector field is considered.

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