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Special Relativity Based on the $SO(3, C)$ Structural Group and 3-Dimensional Time

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In this paper we consider an extended model of the Special Relativity via a principal bundle with structure group $SO(3, \mathbf{C})$ over the base $B = \mathbf{R}^3$. From this viewpoint the 4-vector of velocity is replaced now by a 3×3 orthogonal Hermitian matrix. It is introduced 1-dimensional time parameter and it is parallel to the velocity vector. Starting tom the structure group $SO(3, \mathbf{C})$ the Lorentz transformations are deduced. So this paper gives a wider view of the Special Relativity and it gives a relationship between the two approaches.

1. Introduction

When we consider parallel transport of a 4-vector of velocity, the displaced 4-vector is again a 4-vector of velocity. But if we consider the 4-vector of velocity as a Lorentz boost, then its parallel displacement may not be a boost but may contain a space rotation, and can simultaneously give information for both the velocity and space rotation of the considered body. The change of the angular velocity is studied by parallel displacement of the spin vector separately from the velocity vector. This is the main motivation for the present paper where we present a new model, using the 3-dimensional time. We consider in this paper only linear transformations as in the Special Relativity (SR). The study of gravitation and inertial forces, where the mentioned anomaly disappears, is left for a forthcoming paper. In this paper the matrices of the group $O_+^\uparrow(1,3)$ will be considered for imaginary time coordinate ict.

Albert Einstein and Henri Poincare many years ago thought about 3-dimensional time, such that the space and time would be of the same dimension. At present time some of the authors [1-5,7-9] propose multidimensional time in order to give explanation of the quantum mechanics. In [6] it is also proposed 3-dimensional time and replacement of the Lorentz transformation with vector Lorentz transformations.

2. Basic results

Let us denote by x, y, and z the coordinates in our 3-dimensional space. Having in mind that the unit component $O_+^\uparrow(1,3)$ of the Lorentz group is isomorphic to $SO(3,\mathbf{C})$, we assume that in a chosen moment the set of all moving frames can be considered as a principal bundle over $\mathbf{R}^3$ with structural Lie group $SO(3,\mathbf{C})$, i.e. $\mathbf{R}^3 \times SO(3,\mathbf{C})$. This bundle will be called *space-time bundle.* If we consider another moment, the same frames will be rearranged, but they will also form the same set. The space-time bundle can be parameterized by the following 9 coordinates $\{x,y,z\}$, $\{x_s,y_s,z_s\}$, $\{x_t,y_t,z_t\}$, such that the first 6 coordinates parameterize the subbundle with the fiber $SO(3,\mathbf{R})$. So this approach in the SR will be called 3+3+3-dimensional model. Indeed, to each body are related 3 coordinates for the position, 3 coordinates for the space orientation and 3 coordinates to its velocity.

Firstly, we consider the analog of the Lorentz boosts from the 3+1-dimensional space-time. The next few assumptions are in accordance with the structure of the group $SO(3,\mathbf{C})$. The coordinates x_s,y_s,z_s,x_t,y_t,z_t are functions of x, y, and z, and assume that the Jacobi matrices

$$(2.1)\qquad V=\begin{bmatrix}\frac{\partial x_s}{\partial x} & \frac{\partial x_s}{\partial y} & \frac{\partial x_s}{\partial z}\\ \frac{\partial y_s}{\partial x} & \frac{\partial y_s}{\partial y} & \frac{\partial y_s}{\partial z}\\ \frac{\partial z_s}{\partial x} & \frac{\partial z_s}{\partial y} & \frac{\partial z_s}{\partial z}\end{bmatrix}\quad\text{and}\quad V^*=\begin{bmatrix}\frac{\partial x_t}{\partial x} & \frac{\partial x_t}{\partial y} & \frac{\partial x_t}{\partial z}\\ \frac{\partial y_t}{\partial x} & \frac{\partial y_t}{\partial y} & \frac{\partial y_t}{\partial z}\\ \frac{\partial z_t}{\partial x} & \frac{\partial z_t}{\partial y} & \frac{\partial z_t}{\partial z}\end{bmatrix}$$

are respectively symmetric and antisymmetric. Further, let us denote $X = x_s+ix_t$, $Y=y_s+iy_t$, $Z=z_s+iz_t$, such that the Jacobi matrix $\mathcal{V}=\left[\frac{\partial(X,Y,Z)}{\partial(x,y,z)}\right]$ is Hermitian and $\mathcal{V}=V+iV^*$.

The antisymmetric matrix V^* depends on 3 variables and its general form can be written as

$$(2.2)\qquad V^*=\frac{-1}{c\sqrt{1-\frac{v^2}{c^2}}}\begin{bmatrix}0 & v_z & -v_y\\ -v_z & 0 & v_x\\ v_y & -v_x & 0\end{bmatrix}.$$

From (2.2) we can join to V^* a 3-vector $\vec{v}=(v_x,v_y,v_z)$, which transforms as a 3-vector. Namely, let we choose an orthogonal 3×3 matrix P, which determines a space rotation on the base $B = \mathbf{R}^3$, applying to the coordinates x, y, z. Then this transformation should also be applied to both sets of coordinates $\{x_s, y_s, z_s\}$ and $\{x_t, y_t, z_t\}$. Hence the matrix V^* maps into $PV^*P^{-1} = PV^*P^T$, which corresponds to the 3-vector $P \cdot \vec{v}$. Thus $\vec{v} \mapsto P \cdot \vec{v}$, and $\vec{v}$ is a 3-vector.

It is natural to assume that $\mathcal{V}$ should be presented in the form

$$\mathcal{V} = e^{iA} = \cos A + i \sin A.$$

Assume that A is an antisymmetric real matrix, which is given by

$$A = \begin{bmatrix} 0 & -k\cos\gamma & k\cos\beta \\ k\cos\gamma & 0 & -k\cos\alpha \\ -k\cos\beta & k\cos\alpha & 0 \end{bmatrix},$$

where $\vec{v} = c(\cos\alpha, \cos\beta, \cos\gamma)\tanh(k)$ and $(\cos\alpha, \cos\beta, \cos\gamma)$ is a unit vector of the velocity vector. As a consequence we obtain

$$\sin A = \frac{-1}{c\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 0 & v_z & -v_y \\ -v_z & 0 & v_x \\ v_y & -v_x & 0 \end{bmatrix}, \tag{2.3}$$

i.e. that $V^* = \sin A$ is given by (2.2), while the symmetric 3×3 matrix $\cos A$ is given by

$$(\cos A)_{ij} = V_4 \delta_{ij} + \frac{1}{1+V_4} V_i V_j, \tag{2.4}$$

where $(V_1, V_2, V_3, V_4) = \frac{1}{ic\sqrt{1-\frac{v^2}{c^2}}}(v_x, v_y, v_z, ic)$.

From (2.1) and (2.2) the time vector in this special case is given by

$$(x_t, y_t, z_t) = \frac{\vec{v}}{c\sqrt{1-\frac{v^2}{c^2}}} \times (x, y, z) + (x_t^0, y_t^0, z_t^0), \tag{2.5}$$

where (x_t^0, y_t^0, z_t^0) does not depend on the basic coordinates. The coordinates x_t, y_t, z_t are independent and they cover the Euclidean space $\mathbf{R}^3$ or an open subset of it. But the Jacobi matrix $\left[\frac{\partial(x_t,y_t,z_t)}{\partial(x,y,z)}\right]$ is a singular matrix as antisymmetric matrix of order 3, where the 3-vector of velocity maps into zero vector. So the quantity $(x_t, y_t, z_t) \cdot \vec{v}$ does not depend on the basic coordinates and hence we assume that it determines the 1-dimensional time t measured from the basic coordinates. For example, if velocity is parallel to the z-axis, then z_t does not depend on the basic coordinates because $\frac{\partial z_t}{\partial x} = \frac{\partial z_t}{\partial y} = \frac{\partial z_t}{\partial z} = 0$ and hence z_t is proportional with the time from the basic coordinate system. Further, one can easily verify that $(1-\frac{v^2}{c^2})^{-1/2}(\vec{v} \times (x, y, z)) = (1-\frac{v^2}{c^2})^{-1/2}(\vec{v} \times (\cos A)^{-1}(x', y', z')) =$

$\vec{v} \times (x', y', z')$ for simultaneous points in basic coordinates. So the formula (2.5) becomes

$$(x_t, y_t, z_t) = \frac{\vec{v}}{c} \times (x', y', z') + \vec{c} \cdot \Delta t, \tag{2.6}$$

where $\vec{c}$ is the velocity of light, which has the same direction as $\vec{v}$, i.e. $\vec{c} = \frac{\vec{v}}{v} \cdot c$. Notice that for two points which rest ($v = 0$) and which are considered at the same moment, i.e. $\Delta t = 0$ in the basic coordinates, it is $x_t = y_t = z_t = 0$.

3. Local isomorphism between $O_+^\uparrow(1,3)$ and $SO(3,\mathbf{C})$

Let us consider the following mapping $F : O_+^\uparrow(1,3) \to SO(3,\mathbf{C})$ given by

$$\begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 - \frac{1}{1+V_4}V_1^2 & -\frac{1}{1+V_4}V_1V_2 & -\frac{1}{1+V_4}V_1V_3 & V_1 \\ -\frac{1}{1+V_4}V_2V_1 & 1 - \frac{1}{1+V_4}V_2^2 & -\frac{1}{1+V_4}V_2V_3 & V_2 \\ -\frac{1}{1+V_4}V_3V_1 & -\frac{1}{1+V_4}V_3V_2 & 1 - \frac{1}{1+V_4}V_3^2 & V_3 \\ -V_1 & -V_2 & -V_3 & V_4 \end{bmatrix}$$

$$\mapsto M \cdot (\cos A + i \sin A), \tag{3.1}$$

where $\cos A$ and $\sin A$ are given by (2.4) and (2.3). This is well defined because the decomposition of any matrix from $O_+^\uparrow(1,3)$ as product of space rotation and a boost is unique. Moreover, it is a bijection. Although it is known that the groups $O_+^\uparrow(1,3)$ and $SO(3,\mathbf{C})$ are isomorphic, in the following theorem is constructed an effectively such an isomorphism [10].

Theorem 1. *The mapping (3.1) defines (local) isomorphism between the groups $O_+^\uparrow(1,3)$ and $SO(3,\mathbf{C})$.*

Indeed, the mapping

$$\begin{bmatrix} 0 & c & -b & ix \\ -c & 0 & a & iy \\ b & -a & 0 & iz \\ -ix & -iy & -iz & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & c+iz & -b-iy \\ -c-iz & 0 & a+ix \\ b+iy & -a-ix & 0 \end{bmatrix}$$

defines an isomorphism between the Lie algebras $o(1,3)$ and $o(3,\mathbf{C})$. This isomorphism induces local isomorphism between $O_+^\uparrow(1,3)$ and $SO(3,\mathbf{C})$, and it induces (local) isomorphism between the two groups. Further it is proved that this (local) isomorphism is given by (3.1).

If we want to find the composition of two space-time transformations which determine space rotations and velocities, there are two possibilities which lead to the same result: to multiply the corresponding two matrices from $SO(3,\mathbf{C})$ or from $O_+^\uparrow(1,3)$. Since the result is the same, the three dimensionality of the time is difficult to detect, and we feel like the time is 1-dimensional. The essential difference in using these two methods is the following. The Lorentz transformations give relationship between the coordinates of a 4-vector with respect to two different inertial coordinate systems as it is well known. So they show how the coordinates of a considered 4-vector change by changing the base space. On the other side, the matrices of the isomorphic group $SO(3,\mathbf{C})$ show how the space rotation and velocity change between two bodies, using the chosen base space, by consideration of changes in the fiber. So we have a duality in the Special Relativity. The use of the group $SO(3,\mathbf{C})$ alone is not sufficient, because their matrices are only Jacobi matrices free from any motion.

4. Preparation for the main result

(i) Our final goal is to deduce the Lorentz transformations using the group $SO(3,\mathbf{C})$. We assume that there is no effective motion, but simply rotation for an imaginary angle. Such a transformation will be called *passive motion.* The examination of observation of a moving body can easily be done in the following way.

Let us assume that $v = v_x$, while $v_y = v_z = 0$. In this case the matrix $V = \cos A$ determined by (2.4) is given by $V = \cos A = \mathrm{diag}\Big(1, \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Big)$. Hence there is no length contraction in the direction of motion (x-direction), while the lengths in any direction orthogonal to the direction of motion (yz-plane) are observed to be larger $\left(1-\frac{v^2}{c^2}\right)^{-1/2}$ times. Notice that if we multiply all these length coefficients by $\sqrt{1-\frac{v^2}{c^2}}$ we obtain the prediction from the SR.

If there is an *active motion*, i.e. there is change of the basic coordinates, we see from the previous discussion that *all of the previously described observed lengths in any direction additionally should be multiplied by the coefficient* $\sqrt{1-\frac{v^2}{c^2}}$. Hence the observations for lengths for passive and active motions together is in agreement with the classical known results. Since the previous conclusion is deduced by comparison with the consequences from the Lorentz transformations, and our goal is to deduce the Lorentz transformations, we should accept the previous conclusion axiomatically.

(ii) The previous conclusion for the spatial lengths can be supported by the following conclusion about time intervals. While the observation of lengths may be done in different directions, the observation of time flow does not depend on the direction, but only on velocity. The time observed in a moving system is slower for coefficient $\sqrt{1-\frac{v^2}{c^2}}$ for active motions. It is a consequence of the relativistic law of adding collinear velocities and it is presented by the following theorem, which is proved in [10].

Theorem 2. *Assume that the relativistic law of summation of collinear velocities is satisfied, and assume that the observed time in a moving inertial coordinate system with velocity v is observed to be multiplied with* $f(\frac{v}{c})$, *where f is a differentiable function and the first order Taylor development of f does not contain linear summand of* v/c. *Then, f must be* $f(\frac{v}{c}) = \sqrt{1-\frac{v^2}{c^2}}$.

Since the 1-dimensional time direction is parallel to the velocity vector, there is no change in the observation of the time vector which corresponds to the passive motion. So the observed change for the time vector considered in the previous theorem comes only from the active motion.

Using the Theorem 2 and the assumption that the 1-dimensional time is a quotient between the 3-vector of displacement and the 3-vector of velocity, the following conclusion is deduced in [10]. Let the initial and the end point of a 4-vector $\vec{r'}$ be simultaneous in one coordinate system S'. Then these two points in another coordinate system differ for time

$$\delta t = \frac{\frac{\vec{r'}\vec{v}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{4.1}$$

where $\vec{v}$ is the velocity vector. Notice that (4.1) is also a consequence from the Lorentz transformations.

(iii) The base manifold $\mathbf{R}^3$ is 3-dimensional. It is convenient to consider it as a subset of $\mathbf{C}^3$, consisting of $(x, y, z, ct_x, ct_y, ct_z)$, where $ct_x = ct_y = ct_z = 0$ at a chosen initial moment, and call it complex base. The change of the coordinates can be done via the 6×6 real matrix $\begin{bmatrix} M\cos A & -M\sin A \\ M\sin A & M\cos A \end{bmatrix}$, where M is a space rotation. It acts on the 6-dimensional vectors $(\Delta x, \Delta y, \Delta z, 0, 0, 0)$ of the introduced complex base. Multiplying the vectors of the complex base $(\Delta x, \Delta y, \Delta z, 0, 0, 0)$ from left with this matrix, we obtain 3-dimensional base subspaces as they are viewed from the observer who rests with respect to the chosen complex base. Moreover, the pair $((\Delta x, \Delta y, \Delta z, 0, 0, 0), G) \in \mathbf{R}^6 \times SO(3, \mathbf{C})$ viewed for moving and rotated base space determined by the matrix $P \in SO(3, \mathbf{C})$ is given by $(P(\Delta x, \Delta y, \Delta z, 0, 0, 0)^T, PGP^T) \in \mathbf{R}^6 \times SO(3, \mathbf{C})$.

(iv) Until now we considered mainly the passive motions, while our goal is to consider active motion in the basic coordinates. The active motion is simply translation in the basic space, caused by the flow of the time. So besides the complex rotations of $SO(3,\mathbf{C})$ we should consider also translations in $\mathbf{C}^3$. Now $(\Delta ct_x, \Delta ct_y, \Delta ct_z)$ for the basic coordinates is not more a zero vector. The time which can be measured in basic coordinates is $\Delta t = \Big[(\Delta t_x)^2 + (\Delta t_y)^2 + (\Delta t_z)^2\Big]^{1/2}$. In case of motion of a point with velocity $\vec{v}$ we have translation in the basic coordinates for the vector $\vec{v}\Delta t + i\vec{c}\Delta t$. The space part $\vec{v}\Delta t$ is obvious, while the time part $\vec{c}\Delta t$ follows from (2.6). An orthogonal complex transformation may be applied, if previously the basic coordinates are translated.

5. Lorentz transformations as transformations on $\mathbf{C}^3$

For the sake of simplicity we will omit the symbol "Δ" for space coordinates. So we assume that the initial point of the considered space-time vector has coordinates equal to zero. Assume that x, y, z are basic coordinates. Let the coordinates x_s, y_s, z_s are denoted by x', y', z' and let us denote $\vec{r} = (x,y,z)$ and $\vec{r'} = (x', y', z')$. It is of interest to see the form of the Lorentz boosts as transformations in $\mathbf{C}^3$, while the space rotations are identical in both cases.

Theorem 3. *The following transformation in* $\mathbf{C}^3$

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \begin{bmatrix} \vec{r'} \\ \vec{c}t' + \frac{\vec{v}\times\vec{r'}}{c} \end{bmatrix} = \begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix} \begin{bmatrix} \vec{r} + \vec{v}(t+\delta t) \\ \vec{c}(t+\delta t) \end{bmatrix} \tag{5.1}$$

via the group $SO(3,\mathbf{C})$ *is equivalent to the transformation of a Lorentz boost determined by the isomorphism (3.1).*

Before we prove the theorem we give the following comments. The coefficient $\beta = (1 - \frac{v^2}{c^2})^{-1/2}$ is caused by the active motion (i). Obviously we have translation in the basic coordinates for vector $(\vec{v}(t+\delta t), \vec{c}(t+\delta t))$, where δt is defined by (4.1). On the other side, according to (2.6) in the moving system we have the time vector $\vec{v}\times\vec{r'}/c$, which disappears in basic coordinates ($\vec{v} = 0$).

P r o o f. Notice that if we consider a space rotation P, which applies to all triples, the system (5.1) remains covariant. Indeed, $\vec{r}, \vec{r'}, \vec{v}, \vec{c}, \vec{v}\times\vec{r'}$ transform as vectors, t and δt, which is defined by (4.1), transform as scalars, while $\cos A$ and $\sin A$ transform as tensors. Hence, if we multiply from left with $\begin{bmatrix} P & O \\ O & P \end{bmatrix}$ the both sides of (5.1), we obtain

$$\beta \begin{bmatrix} P\vec{r'} \\ P\vec{c}t' + \frac{(P\vec{v})\times(P\vec{r'})}{c} \end{bmatrix} = \begin{bmatrix} P\cos AP^T & -P\sin AP^T \\ P\sin AP^T & P\cos AP^T \end{bmatrix} \begin{bmatrix} P\vec{r} + P\vec{v}(t+\delta t) \\ P\vec{c}(t+\delta t) \end{bmatrix}$$

and since $P(\cos A)P^T = \cos(PAP^T)$ and $P(\sin A)P^T = \sin(PAP^T)$, the covariance of (5.1) is proved. So it is sufficient to apply such a transformation P which maps vector $\vec{v}$ into $(v,0,0)$ and to prove the theorem in this special case.

Notice that both left and right side of (5.1) are linear functions of x,y,z,t, x',y',z',t', and so after some transformations it can be simplified. Then the first three equations of (5.1) reduce to the following three equations respectively

$$x' = \frac{x+vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z.$$

Further, using these three equations, the fourth equation of (5.1) reduces to

$$t' = \frac{t+\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}},$$

while the 5-th and the 6-th equations are identically satisfied. ∎

According to Theorem 3 the well known 4-dimensional space-time is not fixed in 6 dimensions, but changes with the direction of velocity. Namely this 4-dimensional space-time is generated by the basic space vectors and the velocity vector from the imaginary part of the complax base.

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