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On a Class of Functions Defined by Takahashi and Nunokawa

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Let $\mathcal{A}$ denote the class of analytic functions $f(z)$ in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ normalized so that $f(0) = f'(0) - 1 = 0$. Recently Takahashi and Nunokawa in their work: Takahashi, N.; Nunokawa, M. A certain connection between starlike and convex functions. *Appl. Math. Lett.* 16 (2003), no. 5, 653–655, introduced the following subclass of the class of starlike functions

$$\mathcal{STS}(\mu_1, \mu_2) = \left\{ f \in \mathcal{A} : \frac{\pi\mu_1}{2} < \arg \frac{zf'(z)}{f(z)} < \frac{\pi\mu_2}{2}, z \in \mathcal{U} \right\},$$

where $-1 \le \mu_1 < \mu_2 \le 1$.

Here this class is studied further by methods from the theory of differential subordinations and some simple sufficient conditions that imply $\mathcal{A} \subset \mathcal{STS}(\mu_1, \mu_2)$ are given. Also, comparison with some classical results is done.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of analytic functions $f(z)$ in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ and normalized so that $f(0) = f'(0) - 1 = 0$, i.e., of the form $f(z) = z + \sum_{i=2}^{\infty} a_i z^i$. Now, for $0 < \mu \le 1$, a function $f \in \mathcal{A}$ is said to be *strongly starlike of order* μ if

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\mu\pi}{2}, \quad z \in \mathcal{U}.$$

The class of all such functions is denoted with $\widetilde{S}^*(\mu)$. For $\mu = 1$ we receive the well known class of *starlike functions*,

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in \mathcal{U} \right\},$$

geometrically characterized with the property that f is a starlike function if and only if $f(\mathcal{U})$ is a starlike region, i.e.

$$w \in f(\mathcal{U}) \quad \Rightarrow \quad tw \in f(\mathcal{U}),\ t \in [0,1].$$

The last means that each point from a starlike region is visible from the origin.

Recently, Takahashi N. and Nunokawa M. in [1] introduced the following generalization of the class $\mathcal{S}^*(\mu)$,

$$\mathcal{S}^*(\mu_1,\mu_2) = \left\{ f \in \mathcal{A} : \frac{\mu_1\pi}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\mu_2\pi}{2}, z \in \mathcal{U} \right\},$$

where $-1 \le \mu_1 < 0 < \mu_2 \le 1$. Obviously $\mathcal{S}^*(\mu) = \mathcal{S}^*(-\mu,\mu)$ All of the above mentioned classes are subclasses of univalent functions in the unit disc $\mathcal{U}$ and, moreover, $\widetilde{S}^*(\mu) \subseteq \widetilde{S}^*(\mu_1,\mu_2) \subseteq S^*$.

In this paper we will give sufficient conditions over the expressions

$$f'(z) - (1-\gamma)\cdot\frac{f(z)}{z}, \quad zf''(z) + \gamma\cdot f'(z) \quad \text{and} \quad \frac{1-\gamma+\gamma zf''(z)/f'(z)}{zf'(z)/f(z)}$$

that will place $f(z)$ in $STS(\mu_1,\mu_2)$. Special cases $\gamma = 0$, $\gamma = 1$, $-\mu_1 = \mu_2 = 1$ and/or $-\mu_1 = \mu_2 = \mu$ are previously studied by P. Mocanu, M. Obradović, V. Singh, N. Tuneski in [2], [3], [4], [5] and [6].

The class $\mathcal{S}^*(\mu_1,\mu_2)$ will be studied using the theory of first order differential subordinations, and for that purpose we will recall the definition of subordination, and after that we will redefine the class $\mathcal{S}^*(\mu_1,\mu_2)$.

Let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and we write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disc $\mathcal{U}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. Specially, if $g(z)$ is univalent in $\mathcal{U}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

Valuable reference on this topic is [7]. The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [8] and [8]. Namely, if $\phi : \mathcal{C}^2 \to \mathcal{C}$ is analytic in a domain D, if $h(z)$ is univalent in $\mathcal{U}$, and if $p(z)$ is analytic in $\mathcal{U}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathcal{U}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{1.1}$$

The univalent function $q(z)$ is said to be a *dominant* of the differential subordination (1.1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1). If $\widetilde{q}(z)$ is a dominant of (1.1) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (1.1), then we say that $\widetilde{q}(z)$ is the *best dominant* of the differential subordination (1.1).

From the theory of first-order differential subordinations we will make use of the following lemma ([9]).

Lemma 1.1 *Let $q(z)$ be univalent in the unit disk $\mathcal{U}$, and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain $\mathcal{D}$ containing $q(\mathcal{U})$, with $\phi(\omega) \neq 0$ when $\omega \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

i) $Q(z) \in S^*$; *and*

ii) $\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0$, $z \in \mathcal{U}$.

If $p(z)$ is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subseteq \mathcal{D}$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{1.2}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.2).

2. Main results and consequences

In order to use Lemma 1.1, first we should redefine class $\mathcal{S}^*(\mu_1, \mu_2)$ in terms of subordination,

$$\mathcal{S}^*(\mu_1, \mu_2) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec q(z) \right\}.$$

For that purpose we need an univalent function $q(z)$ such that

$$q(0) = 1 \quad \text{and} \quad q(\mathcal{U}) = \left\{ w : \frac{\mu_1 \pi}{2} < \arg w < \frac{\mu_2 \pi}{2} \right\} \equiv \Omega.$$

Function $q(z)$ is not unique and one such function can be defined as

$$q(z) = q_1(g(z)),$$

where

$$g(z) = e^{i\varphi_d} \frac{|d| + z}{1 + |d|z}, \quad d = i \cdot \tan\left(-\frac{\varphi}{\psi} \cdot \frac{\pi}{4} \right), \quad \varphi_d = \arg d,$$

is an univalent function such that $g(\mathcal{U}) = \mathcal{U}$ and $g(0) = d$; and

$$q_1(z) = e^{i\varphi\pi/2} \left(\frac{1+z}{1-z} \right)^{\psi}, \quad \varphi = \frac{\mu_1 + \mu_2}{2}, \quad \psi = \frac{\mu_2 - \mu_1}{2}.$$

is an univalent function such that $q_1(\mathcal{U}) = \Omega$ and $q_1(d) = 1$. Therefore, $q(z)$ is univalent (as a composition of two univalent functions), $q(0) = 1$ and $q(\mathcal{U}) = \Omega$. Also, it is easy to check that $\operatorname{Re} d = 0$, $\overline{d} = -d$ and $|d|^2 = -d^2 < 1$.

Now, using Lemma 1.1 we prove the following theorem.

Theorem 2.1 *Let $p(z)$ be an analytic function in $\mathcal{U}$ such that $p(0) = 1$ and let $q(z) = q_1(g(z))$, $a \geq 0$ and $b > 0$. If*

$$ap(z) + bzp'(z) \prec aq(z) + bzq'(z) \equiv h(z) \tag{2.1}$$

then $p(z) \prec q(z)$.

Proof. We choose $\theta(\omega) = a\omega$ and $\phi(\omega) = b$. Then $q(z)$ is univalent (as shown in section 1), $\theta(\omega)$ and $\phi(\omega)$ are analytic in the domain $\mathcal{D} = \mathcal{C}$ which contains $q(\mathcal{U})$ and $\phi(\omega) \neq 0$ when $\omega \in q(\mathcal{U})$.

Further, we will prove that $Q(z) = zq'(z)\phi(q(z)) = bzq'(z)$ is a starlike function. We begin with

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} = zg'(z) \cdot \frac{q_1''(g(z))}{q_1'(g(z))} + 1 + \frac{zg''(z)}{g'(z)}.$$

Now, let $w = g(z)$. Then $|w| < 1$, $z = e^{-i\varphi_d} \cdot \frac{w-d}{1+dw}$ and

$$1 + \frac{zg''(z)}{g'(z)} = \frac{2}{1+|d|z} - 1 = 2 \cdot \frac{1+dw}{1+d^2} - 1.$$

This equality, together with

$$zg'(z) = (w-d) \cdot \frac{1+dw}{1+d^2} \quad \text{and} \quad \frac{q_1''(g(z))}{q_1'(g(z))} = 2 \cdot \frac{\psi + w}{1-w^2},$$

brings us to

$$\frac{zQ'(z)}{Q(z)} = 2 \cdot \left[\frac{\psi + w}{1-w^2} \cdot (w-d) + 1\right] \cdot \frac{1+dw}{1+d^2} - 1 \equiv H(w).$$

It can be verified that $H(d) = 1$,

$$\operatorname{Re} H(e^{is}) = 0 \quad \text{and} \quad \operatorname{Im} H(e^{is}) = \frac{\psi + \cos s}{\sin s}.$$

This is enough evidence that $\operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0$ for all $z \in \mathcal{U}$, i.e. $Q(z) \in \mathcal{S}^*$. Also,

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \frac{zQ'(z)}{Q(z)} + \frac{a}{b} > 0, \quad z \in \mathcal{U}.$$

Finally, since $p(z)$ is analytic in $\mathcal{U}$, $p(0) = q(0) = 1$ and $p(\mathcal{U}) \subseteq \mathcal{D} = \mathcal{C}$, the statemenet of the theorem follows directly from Lemma 1.1. ∎

Now we will use this theorem for obtaining several useful corollaries. In order to do that we should first show that $h(z)$ is an univalent function which will allow us to use the weaker definition of subordination. So, let define functions $Q_1(z) = \frac{Q(z)}{Q'(0)}$ and $h_1(z) = \frac{h(z)-h(0)}{h'(0)}$. They are both in the class $\mathcal{A}$, $Q_1(z)$ is starlike and

$$\operatorname{Re}\frac{zh_1'(z)}{Q_1(z)} = \frac{b}{a+b}\cdot\operatorname{Re}\frac{zh'(z)}{Q(z)} > 0, \quad z\in\mathcal{U}.$$

Thus, $h_1(z)$ is close-to-convex univalent function (see p. 10 from [7]). From here $h(z)$ is univalent, too.

Corollary 2.1 *Let $p(z)$ be an analytic function in $\mathcal{U}$ such that $p(0) = 1$, $a \geq 0$, $b > 0$, and let*

$$\delta_1 = \frac{\mu_1\pi}{2} + \arctan\frac{b\psi(1-3|d|)}{a(1-|d|^2)} < 0,$$

$$\delta_2 = \frac{\mu_2\pi}{2} + \arctan\frac{b\psi}{a(1-|d|)}, \qquad \lambda = \arcsin\frac{\min\{|\delta_1|,\delta_2\}}{a}.$$

(i) If $\delta_1 < \arg[ap(z) + bzp'(z)] < \delta_2$ for all $z \in \mathcal{U}$ then $p(z) \prec q(z)$, i.e., $\frac{\mu_1\pi}{2} < \arg p(z) < \frac{\mu_2\pi}{2}$ for all $z\in\mathcal{U}$.

(ii) If $|ap(z) + bzp'(z)| < \lambda$ for all $z\in\mathcal{U}$ then $p(z) \prec q(z)$.

Proof. (i) In order to prove this part of the corollary it is enough to show that subordination (2.1) holds. So, let $w = g(z)$ ($\Rightarrow |w| < 1$), $w = e^{is}$ and $t = \cot\frac{s}{2}$. Then

$$h(e^{is}) = e^{i\pi\varphi/2}\cdot(it)^{\psi}\cdot\left[a + \frac{2b\psi}{1-|d|^2}\cdot\left(|d| + (1-|d|)\cdot\frac{1+t^2}{4t}\right)\right],$$

and further,

$$\arg h(z) \geq \delta_2 \quad \text{for all} \quad t \geq 0,$$

and

$$\arg h(z) \leq \delta_1 \quad \text{for all} \quad t \leq 0.$$

Therefore, $\{w : \delta_1 < \arg w < \delta_2\} \subseteq h(\mathcal{U})$ and having in mind that $h(z)$ is an univalent function, by the weaker definition of subordination, (2.1) follows.

(ii) This part follows from (i) by simple trigonometry and the fact that $h(0) = a$. ■

In the special case when $\mu_2 = -\mu_1 = \mu > 0$ we receive

Corollary 2.2 *Let $p(z)$ be an analytic function in $\mathcal{U}$ such that $p(0) = 1$, $a \geq 0$, $b > 0$ and let*

$$\delta_1 = -\frac{\mu\pi}{2} + \arctan\frac{b\mu}{a} \quad \text{and} \quad \delta_2 = \delta_1 + \mu\pi, \ \lambda = \arcsin\frac{\min\{|\delta_1|, \delta_2\}}{a}.$$

(i) If $\delta_1 < \arg[ap(z) + bzp'(z)] < \delta_2$ for all $z \in \mathcal{U}$ then $p(z) \prec q(z)$, i.e., $|\arg p(z)| < \frac{\mu\pi}{2}$ for all $z \in \mathcal{U}$.

(ii) If $|ap(z) + bzp'(z)| < \lambda$ for all $z \in \mathcal{U}$ then $p(z) \prec q(z)$, i.e., $|\arg p(z)| < \frac{\mu\pi}{2}$ for all $z \in \mathcal{U}$.

The following example exhibits some concrete conclusions that can be obtained from the Corollary 2.1 by specifying respectively

- $p(z) = \frac{f(z)}{z}$, $a = \gamma$ and $b = 1$;
- $p(z) = \frac{f(z)}{z}$, $a = \gamma$ and $b = 1$; and
- $p(z) = \frac{f(z)}{z}$, $a = 1$ and $b = \gamma$.

Example 2.1 Let $f(z) \in \mathcal{A}$, $\gamma \geq 0$ and let δ_1, δ_2 and λ be as in Corollary 2.1.

(i) If $\delta_1 < \arg\left[f'(z) - (1-\gamma)\cdot\frac{f(z)}{z}\right] < \delta_2$ for all $z \in \mathcal{U}$ then

$$\frac{\mu_1\pi}{2} < \arg\frac{f(z)}{z} < \frac{\mu_2\pi}{2}, \quad z \in \mathcal{U}.$$

(ii) If $\left|f'(z) - (1-\gamma)\cdot\frac{f(z)}{z}\right| < \lambda$ for all $z \in \mathcal{U}$ then

$$\frac{\mu_1\pi}{2} < \arg\frac{f(z)}{z} < \frac{\mu_2\pi}{2}, \quad z \in \mathcal{U}.$$

(iii) If $\delta_1 < \arg[f''(z) + \gamma f'(z)] < \delta_2$ for all $z \in \mathcal{U}$ then

$$\frac{\mu_1\pi}{2} < \arg f'(z) < \frac{\mu_2\pi}{2}, \quad z \in \mathcal{U}.$$

(iv) If $|f''(z) + \gamma f'(z)| < \lambda$ for all $z \in \mathcal{U}$ then

$$\frac{\mu_1\pi}{2} < \arg f'(z) < \frac{\mu_2\pi}{2}, \quad z \in \mathcal{U}.$$

(v) If

$$\delta_1 < \arg\left[\gamma + \frac{1-\gamma+\gamma z f''(z)/f'(z)}{zf'(z)/f(z)}\right] < \delta_2$$

for all $z \in \mathcal{U}$ then

$$\frac{\mu_1 \pi}{2} < \arg\frac{f(z)}{zf'(z)} < \frac{\mu_2 \pi}{2}, \quad z \in \mathcal{U},$$

i.e., $f(z) \in STS(-\mu_2, -\mu_1)$.

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