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Adaptive Asymptotic Stabilization of a Bioprocess Model with Unknown Kinetics

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We consider a nonlinear model of an anaerobic wastewater treatment process, in which biodegradable organic is decomposed to produce methane. The model, described by a four-dimensional dynamic system, is known to be practically validated and reliable. We propose a feedback control law for asymptotic stabilization of the closed-loop system towards a previously chosen operating point, presented as a linear combination of the substrate concentrations. This feedback depends only on online measurable quantities. A model-based numerical extremum seeking algorithm is applied to stabilize the dynamics towards the maximum methane output flow rate. Computer simulations are reported to illustrate the theoretical results.

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1. Introduction

We consider a model of an anaerobic digestion process, described by the following nonlinear system of ordinary differential equations ([2], [12], [13]):

$$\frac{ds_1}{dt} = u(s_1^i - s_1) - k_1\mu_1(s_1)x_1 \tag{0.1}$$

$$\frac{dx_1}{dt} = (\mu_1(s_1) - \alpha u)x_1 \tag{0.2}$$

$$\frac{ds_2}{dt} = u(s_2^i - s_2) + k_2\mu_1(s_1)x_1 - k_3\mu_2(s_2)x_2 \tag{0.3}$$

$$\frac{dx_2}{dt} = (\mu_2(s_2) - \alpha u)x_2 \tag{0.4}$$

$$Q = k_4\mu_2(s_2)x_2. \tag{0.5}$$

Table 1: Definition of the model variables and parameters

s_1	concentration of chemical oxygen demand (COD) [g/l]
s_2	concentration of volatile fatty acids (VFA) [mmol/l]
x_1	concentration of acidogenic bacteria [g/l]
x_2	concentration of methanogenic bacteria [g/l]
u	dilution rate [day^{-1}]
s_1^i	influent concentration s_1 [g/l]
s_2^i	influent concentration s_2 [mmol/l]
k_1	yield coefficient for COD degradation [g COD/(g x_1)]
k_2	yield coefficient for VFA production [mmol VFA/(g x_1)]
k_3	yield coefficient for VFA consumption [mmol VFA/(g x_2)]
k_4	coefficient [l^2/g]
$\mu_{\max}$	maximum acidogenic biomass growth rate [day^{-1}]
μ_0	maximum methanogenic biomass growth rate [day^{-1}]
k_{s_1}	saturation parameter associated with s_1 [g COD/l]
k_{s_2}	saturation parameter associated with s_2 [mmol VFA/l]
k_I	inhibition constant associated with s_2 [(mmol VFA/l)$^{1/2}$]
α	proportion of dilution rate reflecting process heterogeneity
Q	methane gas flow rate

The state variables s_1, s_2 and x_1, x_2 denote substrate and biomass concentrations, respectively: s_1 represents the organic substrate, characterized by its chemical oxygen demand (COD), s_2 denotes the volatile fatty acids (VFA), x_1 and x_2 are the acidogenic and methanogenic bacteria respectively. The parameter $\alpha \in [0, 1]$ represents the proportion of bacteria that are affected by the dilution; $\alpha = 0$ and $\alpha = 1$ correspond to an ideal fixed bed reactor and to an ideal continuous stirred tank reactor, respectively (cf. e. g. [1], [2], [4], [5], [12], [13], [18]). The dilution rate u is considered as a control variable. We assume that the methane flow rate Q is the measurable output. The definition of the model parameters is given in Table 1. The functions $\mu_1(s_1)$ and $\mu_2(s_2)$ model the specific growth rates of the microorganisms.

It is known [5], [13] that the model (0.1)–(0.4) describes a two-stage process in a continuously stirred tank-reactor, based on two main reactions: (a) acidogenesis, where the organic substrate (denoted by s_1) is degraded into volatile fatty acids (VFA, denoted by s_2) by acidogenic bacteria (x_1); (b) methanogenesis, where VFA are degraded into methane CH_4 and carbon dioxide CO_2 by methanogenic bacteria (x_2). In [8], the asymptotic stabilizability of this model is studied under the assumption that the acidogenesis (first stage, described by equations (0.1)–(0.2)) has been already stabilized to some operating point s_1^*; then a nonlinear adaptive feedback is proposed, which stabilizes asymptotically the closed-loop second stage dynamics (methanogenic phase) towards a previously chosen reference point s_2^*, such that (s_2^*, x_2^*) is an equilibrium point of (0.3)–(0.4). Further, a numerical extremum seeking algorithm is applied to steer the (two-dimensional) system to an equilibrium point, where maximum methane output flow rate is achieved.

Here we propose a new feedback law, that stabilizes simultaneously the whole system (0.1)–(0.4). The feedback depends on online measurable quantities, the so called biochemical oxygen demand (BOD), which is a linear combination of the substrate concentrations s_1 and s_2; the latter are online measurable by using real sensors or numerical estimators (cf. e. g. [6], [11] and the references therein). The extremum seeking algorithm from [8], [9] is adapted to the four-dimensional system to maximize the methane rate outlet.

The paper is organized as follows. Section 2 presents the main and new result on asymptotic stabilization of the control system (0.1)–(0.4) towards a previously chosen operating point (called also reference or set point). In order to prove that the closed-loop system is asymptotically stable, suitable Lyapunov functions are constructed explicitly. The numerical extremum seeking algorithm is applied in Section 3 to stabilize the dynamics towards the equilibrium point where maximum production of biogas (methane) is achieved. Computer simulations illustrating the theoretical results, are reported in Section 4.

2. Adaptive asymptotic stabilization

Let us consider the model (0.1)–(0.4) as a control system, where s_1, x_1, s_2 and x_2 are the state variables and u is the control. For biological evidence we assume that all phase variables as well as the control take only nonnegative values. We shall construct an adaptive stabilizing controller under the following assumptions:

Assumption A1: The methane gas flow rate Q and a linear combination $\frac{k_2}{k_1}s_1 + s_2$ of s_1 and s_2, called the biochemical oxygen demand (BOD), are online measurable outputs.

Assumption A2: $\mu_i(s_i)$ is defined for $s_i \in [0, +\infty)$, $\mu_i(0) = 0$ and $\mu_i(s_i) > 0$ for $s_i > 0$; $\mu_i(s_i)$ is continuously differentiable and bounded for all $s_i \in [0, +\infty)$, $i = 1, 2$.

Let us fix an operating (reference) point $\bar{s}$,

$$\bar{s} \in (0, s^i) \quad \text{with} \quad s^i := \frac{k_2}{k_1}s_1^i + s_2^i.$$

Assume that the following so-called regulability condition [12] is fulfilled: there exists a point $\bar{s}_1$ such that

$$\mu_1(\bar{s}_1) = \mu_2\left(\bar{s} - \frac{k_2}{k_1}\bar{s}_1\right), \quad \bar{s}_1 \in \left(0, s_1^i\right). \tag{0.6}$$

Define further

$$\bar{s}_2 = \bar{s} - \frac{k_2}{k_1}\bar{s}_1, \quad \bar{x}_1 = \frac{s_1^i - \bar{s}_1}{\alpha k_1}, \quad \bar{x}_2 = \frac{s_2^i - \bar{s}_2 + \alpha k_2 \bar{x}_1}{\alpha k_3} = \frac{s^i - \bar{s}}{\alpha k_3}. \tag{0.7}$$

It is straightforward to see that the point $\zeta(\bar{s}) = \bar{\zeta} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$ is an equilibrium point for (0.1)–(0.4). Our goal is to construct an adaptive feedback law to asymptotically stabilize the system (0.1)–(0.4) to $\zeta(\bar{s})$. Denote further by

$$Q(\bar{\zeta}) = k_4\, \mu(\bar{s}_2)\, \bar{x}_2 \tag{0.8}$$

the static characteristic of the model, which is defined on the set of all steady states $\zeta(\bar{s})$ parameterized by $\bar{s}$. We shall also show that the adaptive feedback law can be applied so that to stabilize the control system (0.1)–(0.4) to an equilibrium point where the static characteristic of the model is maximal.

Denote $s = \dfrac{k_2}{k_1}s_1 + s_2$ and define the following sets

$$\begin{aligned}
\Omega_0 &= \{(s_1, x_1, s_2, x_2)|\ s_1 > 0,\ x_1 > 0,\ s_2 > 0,\ x_2 > 0\},\\
\Omega_1 &= \left\{(s_1, x_1, s_2, x_2)|\ s_1 + k_1 x_1 \le \frac{s_1^i}{\alpha},\ s + k_3 x_2 \le \frac{s^i}{\alpha}\right\},\\
\Omega_2 &= \left\{(s_1, x_1, s_2, \bar{x}_2)|\ \frac{k_2}{k_1}s_1 + s_2 = \bar{s}\right\},\\
\Omega &= \Omega_0 \cap \Omega_1.
\end{aligned}$$

The last assumption imposed on the considered model is the following:

Assumption A3: Let $\dfrac{d}{ds_1}\mu_1(s_1) + \dfrac{k_2}{k_1}\cdot\dfrac{d}{ds_1}\mu_2\left(\bar{s} - \dfrac{k_2}{k_1}s_1\right) > 0$ be satisfied on the set $\Omega \cap \Omega_2$.

Remark 1. Assumption A3 is technical. It is remarkable that this assumption is fulfilled whenever μ_1 and μ_2 are the Monod and the Haldane model functions and the values of the parameters are determined through off-line measurements (cf. [1], [2]).

The main result of this section is the following

Theorem 1. *Let us fix an arbitrary operating point* $\bar{s} \in (0, s^i)$. *Let Assumptions* A1, A2 *and* A3 *be satisfied. Then there exists a feedback control law that asymptotically stabilizes the control system (0.1)–(0.4) to the point* $\zeta(\bar{s}) = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$ *for each starting point* ζ_0 *from the set* Ω_0.

Proof. Let us fix an arbitrary point $\zeta_0 \in \Omega_0$ and a positive value $u_0 > 0$ for the control. According to Lemma 1 from [12] there exists $T > 0$ such that the value of the corresponding trajectory of (0.1)–(0.4) for $t = T$ belongs to the set Ω. For that reason we can consider (without loss of generality) the control system (0.1)–(0.4) with starting point belonging to the set Ω.

Following [2] and [17], we extend the system (0.1)–(0.4) by adding the differential equation

$$\frac{d\beta}{dt} = -C(\beta - \beta^-)(\beta^+ - \beta)k_4\, \mu_2(s_2)\, x_2\, (s - \bar{s}), \tag{0.9}$$

where $C > 0$ is an arbitrary constant. Denote

$$\bar{\beta} = \frac{1}{\alpha k_4 \bar{x}_2} = \frac{k_3}{k_4(s^i - \bar{s})} \tag{0.10}$$

and let $\beta^- > 0$ and $\beta^+ > 0$ be arbitrary real numbers such that $\bar{\beta} \in (\beta^-, \beta^+)$.

We consider the control system (0.1)–(0.4) and (0.9) in the augmented state space (ζ, β) with $\zeta = (s_1, x_1, s_2, x_2)$ and define the following feedback control law

$$k(\zeta, \beta) := \beta\, k_4\, \mu_2(s_2)\, x_2. \tag{0.11}$$

Taking into account the expression for Q (see (0.5)), equation (0.9) and Assumption A1, we can state that the proposed feedback $k(\zeta, \beta)$ uses only online measurable quantities.

Let (Σ) denotes the closed-loop system obtained from (0.1)–(0.4) and (0.9) by substituting the control variable u by the feedback $k(\zeta, \beta)$. Define the following function:

$$V(\zeta, \beta) = (s - \bar{s} + k_3(x_2 - \bar{x}_2))^2 +$$
$$\Gamma\left(\int_{\bar{s}}^{s} \frac{v - \bar{s}}{s^i - v}\, dv + \frac{1}{C}\int_{\bar{\beta}}^{\beta} \frac{w - \bar{\beta}}{(w - \beta^-)(\beta^+ - w)}\, dw\right),$$

where the parameter $\Gamma > 0$ will be determined later. Clearly, the values of this function are nonnegative. If we denote by $\dot{V}(\zeta, \beta)$ the Lie derivative of the function V with respect to the right-hand side of the closed-loop system Σ at the point (ζ,β), then it is easy to see that for each point (ζ,β) from the set $\widetilde{\Omega} := \Omega \times (\beta^-,\beta^+)$, the following equality holds true:

$$\begin{aligned}\dot{V}(\zeta,\beta) &= -k(\zeta,\beta)\left(2+\Gamma\cdot\frac{k_3}{k_4\beta(s^i-s)(s^i-\bar{s}))}\right)(s-\bar{s})^2\\ &\quad -2(1+\alpha)k_3\cdot k(\zeta,\beta)(s-\bar{s})(x_2-\bar{x}_2)\\ &\quad -2\alpha k_3^2\cdot k(\zeta,\beta)(x_2-\bar{x}_2)^2.\end{aligned}$$

The boundedness of the set $\widetilde{\Omega}$ implies the existence of a sufficiently large constant $\Gamma > 0$ so that

(0.12) $$\dot{V}(\zeta,\beta)\le 0 \text{ for each point } (\zeta,\beta)\in\widetilde{\Omega}.$$

We set

$$\widetilde{\Omega}_2 := \{(\zeta,\beta)\in\widetilde{\Omega}:\ \dot{V}(\zeta,\beta)=0\}.$$

One can directly check that

$$\widetilde{\Omega}_2 := \left\{(s_1,x_1,s_2,\bar{x}_2,\bar{\beta})\in\widetilde{\Omega}:\ \frac{k_2}{k_1}s_1+s_2=\bar{s}\right\},$$

or equivalently

$$\widetilde{\Omega}_2 = \left\{\left(s_1,x_1,\bar{s}-\frac{k_2}{k_1}s_1,\bar{x}_2,\bar{\beta}\right)\in\widetilde{\Omega}\right\}.$$

Applying the LaSalle's invariance principle (cf. e. g. [16]), it follows that every solution of the system (Σ) starting from a point of $\widetilde{\Omega}$ is defined on the interval $[0,+\infty)$ and approaches the largest invariant set (with respect to (Σ)) which is contained in the set $\widetilde{\Omega}_2$. In fact one can directly check that the set $\widetilde{\Omega}_2$ is invariant with respect to the trajectories of (Σ). Using (0.7), (0.10) and (0.11), the dynamics of (Σ) on the set $\widetilde{\Omega}_2$ can be described by the following system

(0.13) $$\begin{aligned}\frac{ds_1}{dt} &= \frac{1}{\alpha}\chi(s_1)(s_1^i-s_1)-k_1\mu_1(s_1)x_1\\ \frac{dx_1}{dt} &= (\mu_1(s_1)-\chi(s_1))x_1,\end{aligned}$$

where $\chi(s_1) := \mu_2\left(\bar{s}-\frac{k_2}{k_1}s_1\right)$. Taking into account that $\bar{s}=\frac{k_2}{k_1}\bar{s}_1+\bar{s}_2$ and $s_1^i=\bar{s}_1+\alpha k_1\bar{x}_1$, (0.13) can be rewritten as follows:

$$\begin{aligned}\frac{ds_1}{dt} &= -\frac{1}{\alpha}\chi(s_1)\cdot(s_1-\bar{s}_1+\alpha k_1(x_1-\bar{x}_1))-k_1\left(\mu_1(s_1)-\chi(s_1)\right)\cdot x_1\\ \frac{dx_1}{dt} &= (\mu_1(s_1)-\chi(s_1))\cdot x_1.\end{aligned}$$

Consider the function

$$W(\zeta,\beta) = (s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1))^2 + \alpha(1-\alpha)k_1^2(x_1 - \bar{x}_1)^2.$$

Clearly, this function depends only on the variables s_1 and x_1 and takes only nonnegative values; moreover, for each point $(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta})$ from the set $\widetilde{\Omega}_2$, the following presentation holds true:

$$\text{(0.14)} \quad \begin{aligned} \dot{W}(s_1, x_1, \bar{s} - \tfrac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta}) &= -\frac{2}{\alpha}\chi(s_1)(s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1))^2 \\ &\quad - 2(1-\alpha)k_1 x_1(s_1 - \bar{s}_1)(\mu_1(s_1) - \chi(s_1)). \end{aligned}$$

The regulability condition (0.6) implies the following:

$$\begin{aligned} \mu_1(s_1) - \chi(s_1) &= \mu_1(s_1) - \mu_2\left(\bar{s} - \frac{k_2}{k_1}s_1\right) \\ &= \mu_1(s_1) - \mu_2\left(\bar{s}_2 - (s_1 - \bar{s}_1)\frac{k_2}{k_1}\right) \\ &= \mu_1(\bar{s}_1) + \int_{\bar{s}_1}^{s_1} \frac{d}{ds_1}\mu_1(\theta)\,d\theta - \mu_2(\bar{s}_2) \\ &\quad + \frac{k_2}{k_1}\int_{\bar{s}_1}^{s_1} \frac{d}{ds_2}\mu_2\left(\bar{s}_2 - (\theta - \bar{s}_1)\frac{k_2}{k_1}\right)d\theta \\ &= \int_{\bar{s}_1}^{s_1}\left(\frac{d}{ds_1}\mu_1(\theta) + \frac{k_2}{k_1}\frac{d}{ds_2}\mu_2\left(\bar{s}_2 - (\theta - \bar{s}_1)\frac{k_2}{k_1}\right)\right)d\theta, \end{aligned}$$

and by means of Assumption A3 it follows that

$$(s_1 - \bar{s}_1)\int_{\bar{s}_1}^{s_1}\left(\frac{d}{ds_1}\mu_1(\theta) + \frac{k_2}{k_1}\frac{d}{ds_2}\mu_2\left(\bar{s}_2 - (\theta - \bar{s}_1)\frac{k_2}{k_1}\right)\right)d\theta > 0.$$

From this inequality and from (0.14) we obtain that

$$\text{(0.15)} \qquad \dot{W}(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta}) < 0$$

for each point $(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2, \bar{\beta})$ from the set $\widetilde{\Omega}_2 \setminus \{(\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2, \bar{\beta})\}$.

To complete the proof we use an idea from [10] (cf. the proof of Theorem 3.1). First we shall remind some notions. Let us denote by $\phi(t,\zeta,\beta)$ the value of the trajectory of the closed-loop system (Σ) at time t starting from the point $(\zeta,\beta) \in \widetilde{\Omega}$. The positive limit set (or ω-limit set) of the solution $\phi(t,\zeta,\beta)$ of (Σ) is defined as

$$L^+(\zeta,\beta) = \Big\{(\tilde{\zeta},\tilde{\beta}) \mid \text{there exists a sequence } \{t_n\} \to +\infty \\ \text{with } (\tilde{\zeta},\tilde{\beta}) = \lim_{t_n\to+\infty}\phi(t_n,\zeta,\beta)\Big\}.$$

The negative limit set (or α-limit set) $L^-(\zeta,\beta)$ of the solution $\phi(t,\zeta,\beta)$ of (Σ) is defined in an analogous way using sequences $\{t_n\} \to -\infty$.

Let us fix an arbitrary point (ζ_0,β_0) from the set $\widetilde{\Omega}$. The invariance of the bounded set $\widetilde{\Omega}$ with respect to the trajectories of (Σ) implies that the ω-limit set $L^+(\zeta_0,\beta_0)$ is a nonempty compact connected invariant set. Moreover, the LaSalle's invariance principle implies that $L^+(\zeta_0,\beta_0)$ is a subset of $\widetilde{\Omega}_2$.

We shall prove that $L^+(\zeta_0,\beta_0) = \{(\bar{\zeta},\bar{\beta})\}$. Let us assume the contrary, i. e. there exists a point $(\zeta_\infty,\bar{\beta}) \in L^+(\zeta_0,\beta_0)$ with $\zeta_\infty \neq \bar{\zeta}$. Then $\varepsilon := \|(\zeta_\infty,\bar{\beta}) - (\bar{\zeta},\bar{\beta})\| > 0$.

The invariance of the set $L^+(\zeta_0,\beta_0)$ with respect to the trajectories of (Σ) implies that $\phi(-t,\zeta,\beta) \in L^+(\zeta_0,\beta_0)$ for each positive t and for each point $(\zeta,\beta) \in L^+(\zeta_0,\beta_0)$. In particular we have that $\phi(-t,\zeta_\infty,\bar{\beta}) \in L^+(\zeta_0,\beta_0)$ for each positive t and hence $L^-(\zeta_\infty,\bar{\beta}) \subseteq L^+(\zeta_0,\beta_0)$.

The inequality (0.15) implies the existence of a sequence $t_n \to +\infty$ such that

$$\lim_{t_n\to+\infty} \phi(t_n,\widehat{\zeta},\bar{\beta}) = (\bar{\zeta},\bar{\beta}), \quad \text{where } (\widehat{\zeta},\bar{\beta}) \text{ is an arbitrary point from } L^-(\zeta_\infty,\bar{\beta}).$$

On the other hand, the invariance of the set $L^-(\zeta_\infty,\bar{\beta})$ with respect to the trajectories of (Σ) implies that $\phi(t_n,\widehat{\zeta},\bar{\beta}) \in L^-(\zeta_\infty,\bar{\beta})$, $n = 1,2,\ldots$. Then the closeness of the set $L^-(\zeta_\infty,\bar{\beta})$ implies that $\lim_{t_n\to+\infty}\phi(t_n,\widehat{\zeta},\bar{\beta})$ also belongs to the set $L^-(\zeta_\infty,\bar{\beta})$. Thus we have obtained the following relation

$$(\bar{\zeta},\bar{\beta}) \in L^-(\zeta_\infty,\bar{\beta}) \subseteq L^+(\zeta_0,\beta_0). \tag{0.16}$$

Let $B((\bar{\zeta},\bar{\beta}),\varepsilon/3)$ be a closed ball centered at $(\bar{\zeta},\bar{\beta})$ with radius $\varepsilon/3$. The first inclusion of (0.16) implies the existence of a sufficiently large number $T > 0$ such that $\phi(-T,\zeta_\infty,\bar{\beta}) = (\zeta_1,\beta_1) \in B((\bar{\zeta},\bar{\beta}),\varepsilon/3)$. But this means that

$$\phi(T,\zeta_1,\beta_1) = (\zeta_\infty,\bar{\beta}). \tag{0.17}$$

The invariance of the set $\widetilde{\Omega}_2$ with respect to the trajectories of (Σ) implies

$$(\zeta_1,\beta_1) \in B((\bar{\zeta},\bar{\beta}),\varepsilon/3) \cap \widetilde{\Omega}_2. \tag{0.18}$$

Then (0.15), (0.17) and (0.18) contradict to the equality $\|(\zeta_\infty,\bar{\beta}) - (\bar{\zeta},\bar{\beta})\| = \varepsilon$. This contradiction shows that $(\bar{\zeta},\bar{\beta}) = L^+(\zeta_0,\beta_0)$ and completes the proof. ■

3. Extremum seeking

According to Assumption A1, the BOD concentration $s = \dfrac{k_2}{k_1}s_1 + s_2$ and the effluent methane flow rate Q are online measurable. Denote by $\bar{s} \in$

$(0, s^i)$ some reference point and consider $\zeta(\bar{s}) = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$ where $\bar{s}_1$, $\bar{x}_1$, $\bar{s}_2$ and $\bar{x}_2$ are computed according to (0.6) and (0.7). We assume that the static characteristic

$$Q(\bar{\zeta}) = k_4\, \mu_2(\bar{s}_2)\, \bar{x}_2,$$

which is defined on the set of all steady states $\zeta(\bar{s})$ has a maximum at a unique steady state point

$$\zeta_{\max} = (s_1^m,\, x_1^m,\, s_2^m,\, x_2^m), \qquad s_{\max} = \frac{k_2}{k_1} s_1^m + s_2^m \in (0, s^i),$$

that is $Q_{\max} = Q(\zeta_{\max})$.

Our goal now is to stabilize the dynamic system towards the (unknown) maximum methane flow rate $Q_{\max}$. For that purpose we write equation (0.9) in the form

$$\frac{d\beta}{dt}(t) = -C \cdot (\beta(t) - \beta^-) \cdot (\beta^+ - \beta(t)) \cdot Q(t) \cdot (s(t) - \bar{s}), \tag{0.19}$$

where $Q(t)$ denotes the methane flow rate measured at time t. It should be pointed out that not only $Q(t)$ but also all quantities in (0.19) are online measurable. Therefore, the values of its solution can be determined online as well. Since the solution of (0.19) depends on $\bar{s}$, we denote it by $\beta_{\bar{s}}(t)$, $t \in [0, +\infty)$. The last fact allows us to apply on-line the feedback control law

$$(s, Q, \beta_{\bar{s}}) \longmapsto k(s, Q, \beta_{\bar{s}}) = \beta_{\bar{s}}\, Q. \tag{0.20}$$

According to Theorem 1, this feedback will asymptotically stabilize the control system (0.1)–(0.4), (0.19) to the point $(\bar{\zeta}, \bar{\beta}_{\bar{s}})$ with $\bar{\beta}_{\bar{s}} = \dfrac{k_3}{k_4(s^i - \bar{s})}$.

To stabilize the dynamics (0.1)–(0.4), (0.19) towards $Q_{\max}$ by means of the feedback (0.20), we use a numerical iterative extremum seeking algorithm. The algorithm is based on the fact that Theorem 1 is valid for *any* reference point $\bar{s} \in (0, s^i)$. Thus we can construct a sequence of points $\bar{s}^{(1)}, \bar{s}^{(2)}, \ldots, \bar{s}^{(n)}, \ldots$ and generate in a proper way a sequence of values for the methane rate $Q^{(1)}$, $Q^{(2)}$, …, $Q^{(n)}$, …, which converges to $Q_{\max}$. The algorithm, which is first presented in [8] for a two-dimensional model, is adapted for the model considered here. The algorithm is carried out in two stages: on Stage I, an interval $[S] = [S^-, S^+]$ is found such that $[S^-, S^+] \subset (0, s^i)$ and $s_{\max} \in [S^-, S^+]$; on Stage II, the interval $[S]$ is refined using an elimination procedure based on a Fibonacci search technique [15]. Stage II produces the final interval $[S^-_{\max}, S^+_{\max}]$ such that

$[S^-_{\max}, S^+_{\max}] \subseteq [S^-, S^+] \subset (0, s^i)$, $s_{\max} \in [S^-_{\max}, S^+_{\max}]$ and $S^+_{\max} - S^-_{\max} \le \varepsilon$, where the tolerance $\varepsilon > 0$ is assumed to be specified by the user.

4. Numerical simulation

In the computer simulation, we consider for $\mu_1(s_1)$ and $\mu_2(s_2)$ the Monod and the Haldane model functions [3] for the specific growth rates:

$$\mu_1(s_1) = \frac{\mu_{\max} s_1}{k_{s_1} + s_1}, \qquad \mu_2(s_2) = \frac{\mu_0 s_2}{k_{s_2} + s_2 + \left(\dfrac{s_2}{k_I}\right)^2}.$$

Obviously, $\mu_1(s_1)$ and $\mu_2(s_2)$ satisfy Assumption A1. Moreover, Q also has a maximum at a unique steady state point.

The following values for the model coefficients [1], [2] are used in the simulation process

$$\begin{array}{lllllllll}
\alpha & = & 0.5, & k_1 & = & 10.53, & k_2 & = & 28.6, \\
k_3 & = & 1074, & k_4 & = & 675 & \mu_{\max} & = & 1.2, \\
k_{s_1} & = & 7.1, & \mu_0 & = & 0.74, & k_{s_2} & = & 9.28, \\
k_I & = & 16, & s_1^i & = & 7, & s_2^i & = & 70.
\end{array}$$

With the above coefficient values, the functions $\mu_1(s_1)$ and $\mu_2(s_2)$ satisfy Assumption A3.

Figure 1 shows the time profiles of the phase variables $s_1(t)$, $x_1(t)$, $s_2(t)$, $x_2(t)$ (plots (a) to (d) respectively), of the feedback $k(t)$ (plot (e)) and of $Q(t)$ (plot (f)). In the plots the symbol $\diamond$ denotes the initial values $s_1(0)$, $x_1(0)$, $s_2(0)$ and $x_2(0)$; the horizontal dash-line segments go through s_1^m, x_1^m, s_2^m, x_2^m, $u_{\max}$ and $Q_{\max}$ respectively, where

$$u_{\max} = k(\zeta_{\max}, \beta_{\max}), \quad \beta_{\max} = \frac{k_3}{k_4(s^i - s_{\max})}.$$

5. Conclusion

The paper is devoted to the stabilization of a four-dimensional nonlinear dynamic system, which models an anaerobic degradation of organic wastes and produces methane. A nonlinear adaptive feedback is proposed, which stabilizes asymptotically the dynamic system towards the (unknown) maximum methane production rate $Q_{\max}$. For that purpose, it is first shown that for any previously chosen reference point $\bar{s}$, representing the biochemical oxygen demand, the system can be asymptotically stabilized to an equilibrium point $(\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$, such

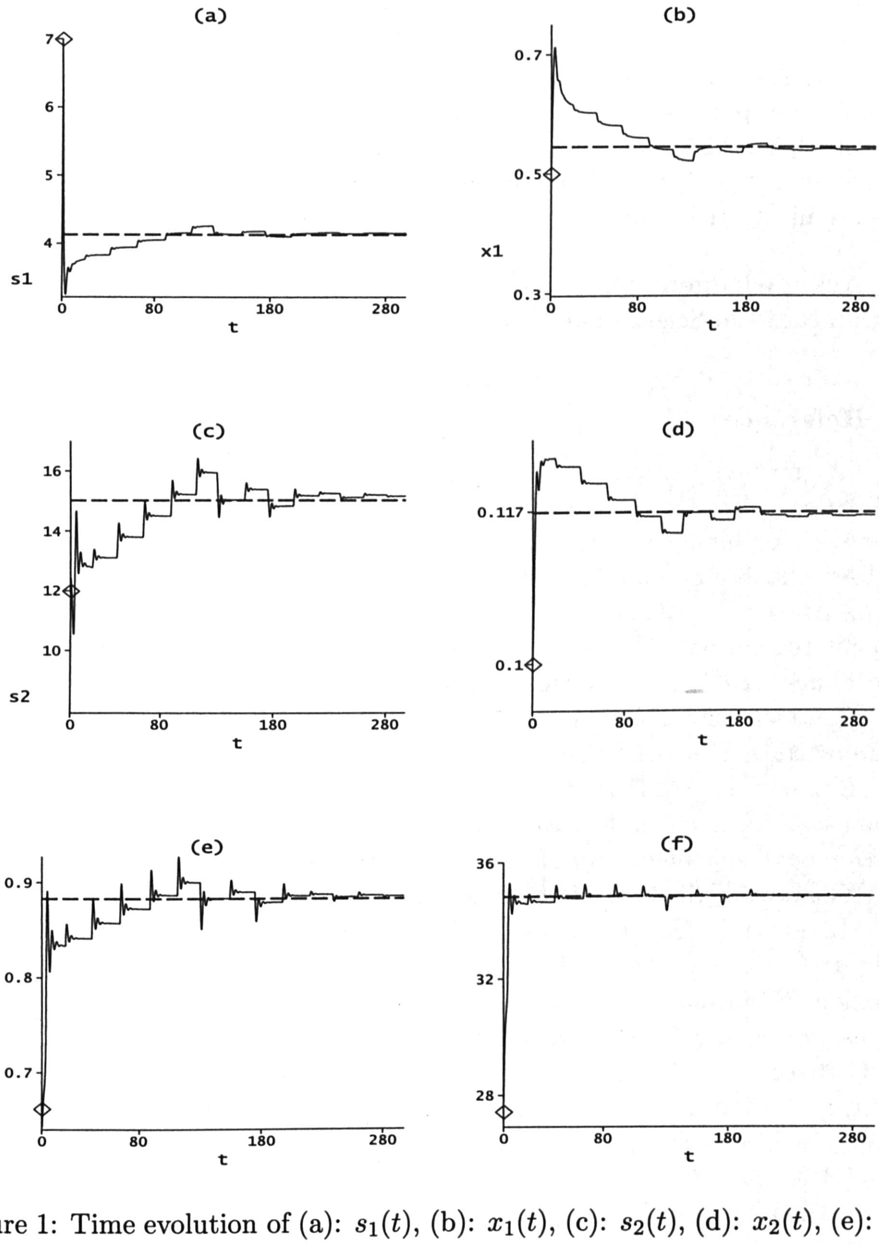

Figure 1: Time evolution of (a): $s_1(t)$, (b): $x_1(t)$, (c): $s_2(t)$, (d): $x_2(t)$, (e): $k(t)$ and (f): Q(t); the horizontal (dash) line segments go through s_1^m, x_1^m, s_2^m, x_2^m, $u_{\max}$ and $Q_{\max}$ respectively.

that $\bar{s} = \frac{k_2}{k_1}\bar{s}_1 + \bar{s}_2$. Further, an iterative numerical extremum seeking algorithm is applied to stabilize the closed-loop system into an interval $[S_{\max}]$, containing the equilibrium point $s_{\max}$ for which the methane output flow rate Q takes its maximum $Q_{\max}$. The interval $[S_{\max}]$ can be made as tight as desired depending on a tolerance $\varepsilon > 0$, which has to be specified by the user. The theoretical results are illustrated numerically.

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