Mathematica
Balkanica

New Series Vol. 25, 2011, Fasc. 3

Approximation of the Risk Process - a Survey

Elisaveta I. Pancheva, Ivan K. Mitov

An insurance model can be interpreted as a point process $\mathcal{N} = \{(T_k, X_k) : k = 1, 2, ...\}$ on a particular time-state space. The time components T_k mark the customers' claim arrival times and the state components X_k model the claim sizes. The basic idea of an approximation of the risk process associated with $\mathcal{N}$ is to normalize properly the time-state space in such a way that the resulting sequence of point processes $\{(T_{nk}, X_{nk}) : k = 1, 2, ...\}$, $n = 1, 2, ...$ is vaguely convergent. First an accompanying model with deterministic time points is considered and after that the general problem with random time points is investigated. The investigation is done under different assumptions on T_k and X_k so that different kinds of approximations arise.

1. Introduction

An insurance risk model $\mathcal{I}$ can be considered as determined if a data point process $\mathcal{N} = \{(T_k, X_k) : k = 1, 2, ...\}$ is given on the time-state space $\mathcal{S} = (0, \infty) \times (0, \infty)$ where

a) the state coordinates X_k represent the claim sizes. We assume that they are independent random variables (rv's).

b) The time coordinates T_k are interpreted as the claim arrival times. We assume that they are strictly increasing: $0 < T_1 < T_2 < ... < T_n \to \infty$.

c) Both sequences T_k and X_k are supposed to be independent.

With the point process $\mathcal{N}$ we associate three random processes:

- the counting process $N(t) := \sum_k \mathbf{1}_{\{T_k \le t\}} = \max\{k : T_k \le t\}$ which counts the number of claims in the interval $(0, t]$. (Here and latter $\mathbf{1}_A$ denotes the indicator of the event A.)
- the accumulated claim process $S(t) := \sum_{k=1}^{N(t)} X_k$;

- the risk reserve process $R(t) := u + ct - S(t)$, where $u = R(0)$ is the initial capital and $c > 0$ is the premium income rate.

As a measure of risk one usually takes the probability of ruin $\psi(u) = P(R(t) < 0 \text{ for some } t > 0 \mid R(0) = u)$. In few cases $\psi(u)$ can be calculated explicitly. In most cases, however, one solves the problem either giving upper and lower bounds for $\psi(u)$ or approximating the risk process $R(t)$.

It is transparent that the uncertainty of the risk process is borne by the accumulated claim whose distribution function (df) has the form

$$\mathbf{P}\{S(t) < x\} = \sum_{k=0}^{\infty} \mathbf{P}\{S(t) < x \mid N(t) = n\}\mathbf{P}\{N(t) = n\}$$

$$(1.1) \qquad = \sum_{k=0}^{\infty} (F_{X_1} * \ldots * F_{X_n})(x)\mathbf{P}\{N(t) = n\}.$$

In view of (1.1), in fact an approximation of $S(t)$ is pursued.

The main goal of our survey is to offer a unified approach to the approximation of the risk process. As a "back testing" we check our approach on the well known diffusion approximation (Section 3) and the α-stable approximation (Section 4). Then, in Section 5, we pay attention to a new approximation using a Sato process. It seems that for first time it was introduced by I. Mitov in his Ph.D.Thesis [6] (see also Mitov et al.[7]).

2. Preliminaries

Let us make use of the basic idea to change time and space. We apply continuous and strictly increasing in both coordinates mappings $\zeta_n(t, x) = (\tau_n(t), u_n(x))$, $n \geq 1$ in a way that the claim sizes $X_{nk} = u_n^{-1}(X_k)$ become smaller but their number $N_n(t)$ increases properly. Here $N_n(t) = N(\tau_n(t)) = \max\{k : T_{nk} = \tau_n^{-1}(T_k) \leq t\}$. In this way we are supplied with a sequence of point processes $\mathcal{N}_n = \{(T_{nk}, X_{nk}) : k = 1, 2, \ldots\}$, $n \geq 1$, and associated random processes $N_n(t)$, $S_n(t) = \sum_{k=1}^{N_n(t)} X_{nk}$ and $R_n(t) = u_n + c_n t - S_n(t)$. If we succeed in showing that $R_n \Rightarrow R_0$ in $\mathcal{D}$, then we might consider R_0 as a weak approximation of the initial risk process R.

This approach consists of three steps:

Step 1 (accompanying point process $\mathcal{N}_n^{(a)}$).

The classical limit theory for sums of independent rv's is related to point processes with deterministic time points $\mathcal{N}_n^{(a)} = \{(t_{nk}, X_{nk}) : k = 1, 2, \ldots\}$. Let X_{nk} be the same normalized claim sizes as in the point process $\mathcal{N}_n$. The time points t_{nk} are chosen so that the corresponding counting function $k_n(t) = \max\{k : t_{nk} \leq t\}$ is finite for every fixed n and t, tends to ∞ as $n \to \infty$, and (under certain conditions on X_{nk}) the weak convergence

$$(2.1) \qquad Z_n(.) := \sum_{k=1}^{k_n(.)} X_{nk} \Rightarrow Z(.) \quad \text{in} \quad \mathcal{D}$$

holds. Note that $k_n(t)$ is not uniquely determined by (2.1) and depends on the tails $1 - \mathbf{P}\{X_{nk} < x\}$.

We denote the class of all nondecreasing cadlag functions $y : (0, \infty) \to (0, \infty)$ equipped with the topology of the weak convergence by $\mathcal{M}(0, \infty)$, $\mathcal{M} \subset \mathcal{D}$. It is a Polish space. Denote by $\mathcal{P}$ the set of all probability measures on $\mathcal{M}$.

The limit process Z has independent increments and sample paths in $\mathcal{M}$. For such processes it is known (cf. Whitt [9]) that:

- the finite dimensional distributions are determined by the univariate marginal distributions;
- any sequence of nondecreasing processes is tight;
- the set $\mathcal{P}$ is closed with respect to the weak topology.

Therefore it is sufficient to prove the convergence $Z_n(t) \xrightarrow{d} Z(t)$ for all t in a dense subset of $(0, \infty)$ in order to state the weak convergence $Z_n \Rightarrow Z$ in $\mathcal{D}$. Moreover, if the limit process is stochastically continuous, then the weak convergence (2) holds under the Skorohod's J_1-topology.

Step 2 (random time change).

Here we call *random time change* any mapping $\theta : (0, \infty) \to (0, \infty)$, $\theta(0) = 0$ and $\theta(s) \to \infty$ for $s \to \infty$, which is stochastically continuous and has sample paths in $\mathcal{M}(0, \infty)$. Given both counting processes N_n (of the point process with random time points) and k_n (of the accompanying point process with deterministic time points) there exists a random time change $\theta_n(t)$ (cf. Pancheva, Kolkovska and Jordanova [8]) such that

$$(2.2) \qquad N_n(t) \stackrel{d}{=} k_n(\theta_n(t)).$$

On $\mathcal{M}(0, \infty)$ a convergence in the Skorohod's M_1-topology coincides with a pointwise convergence on a dense subset of $(0, \infty)$, which itself coincides with a convergence for all continuity points of the limit function. Hence, it is sufficient to assume that

$$(2.3) \qquad \theta_n(t) \xrightarrow{d} \theta(t), \; n \to \infty$$

for all t in a dense subset of $(0, \infty)$. Then $\theta_n \Rightarrow \theta$ in the M_1-topology and the limit time process θ has sample paths in $\mathcal{M}$.

Step 3 (continuity of the composition).

Now the accumulated claim associated with the point process $\mathcal{N}_n = \{(T_{nk}, X_{nk}) : k = 1, 2, ...\}$ can be expressed as

$$S_n(t) = \sum_{k=1}^{N_n(t)} X_{nk} = \sum_{k=1}^{k_n(\theta_n(t))} X_{nk} = Z_n \circ \theta_n(t).$$

The composition $Z_n \circ \theta_n$ maps $\mathcal{M}(0,\infty) \times \mathcal{M}(0,\infty)$ into $\mathcal{M}(0,\infty)$. The independence of T_{nk} and X_{nk} implies the independence of θ_n and Z_n. Both convergences (2.1) and (2.3) then mean that

$$(Z_n, \theta_n) \Rightarrow (Z, \theta) \in \mathcal{M} \times \mathcal{M}.$$

Unfortunately, the composition map is in general not continuous at (Z, θ) in the M_1-topology. Whitt ([9], Theorem 13.2.4) gives conditions for the M-continuity of the composition. These conditions are equivalent to the statement that both processes Z and θ do not jump simultaneously. Finally, under this condition, one may claim that

$$S_n = Z_n \circ \theta_n \Rightarrow Z \circ \theta =: S \tag{2.4}$$

in the M_1-topology. In a case when both Z and θ are stochastically continuous the weak convergence (2.4) holds also in the stronger J_1-topology on $\mathcal{M}$.

3. Diffusion approximation

In this section we specify the initial model $\mathcal{I}$ as follows. Define the interarrival times by $Y_k := T_k - T_{k-1},\ k \geq 1,\ T_0 = 0$ and assume that

i) $\{Y_k\}$ are independent and identically distributed random variables (iid rv's) with finite variance σ_Y^2 and expected value $EY = \frac{1}{\lambda},\ \lambda > 0$;

ii) $\{X_k\}$ are iid rv's with $\sigma_X^2 < \infty$ and $EX = \mu$.

Iglehart [4] investigated this model and suggested a Brownian motion approximation of the accumulated claim process. Indeed, by Donsker Invariance Principle,

$$Z_n(.) := \sum_{k=1}^{[n.]} \frac{X_k - \mu}{\sigma_x \sqrt{n}} \Rightarrow B(.) \quad \text{in} \quad \mathcal{D}(0,\infty). \tag{3.1}$$

The sample paths of the Brownian motion $B(t)$ are a.s. continuous. On the space C (of all continuous functions), $C \subset \mathcal{D}$, the Skorohod's J_1-topology coincides with the uniform topology.

From (3.1) we already know what kind of time-space transforms to choose, namely $\zeta_n(t,x) = (nt, \frac{x}{\sigma_X \sqrt{n}})$. Hence, in the new coordinates, we have

$$\mathcal{N}_n = \left\{ \left(T_{nk} = \frac{T_k}{n}, X_{nk} = \frac{X_k}{\sigma_X \sqrt{n}} \right) : k \geq 1 \right\}$$

and

$$N_n(t) = \sum_k \mathbf{1}_{\left\{ \frac{T_k}{n} \leq t \right\}} = N(nt).$$

Under assumption i) the Law of Large Numbers (LLN) claims that

$$\theta_n(t) := \frac{N(nt)}{n} \xrightarrow{P} \lambda t, \quad n \to \infty.$$

Further, in view of (2.2) and (3.1), we observe that $k_n(t) = [nt]$ and

$$N_n(t) = N(nt) = [n\theta_n(t)] = k_n(\theta_n(t)).$$

Under assumption ii) the accumulated claim process $S_n(t)$, associated with $\mathcal{N}_n$, can be expressed as

$$\begin{aligned} S_n(t) &= \sum_{k=1}^{N_n(t)} X_{nk} = \sum_{k=1}^{k_n(\theta_n(t))} \frac{X_k}{\sigma_X \sqrt{n}} \\ &= Z_n(\theta_n(t)) + \frac{\mu}{\sigma_X} (\frac{N(nt) - \lambda nt}{\sqrt{n}}) + \frac{\lambda \mu n t}{\sigma_X \sqrt{n}}. \end{aligned}$$

Thus, the risk process in the new coordinates is defined as $R_n(t) := \frac{R(nt)}{\sigma_X \sqrt{n}} =$

$$= \frac{u}{\sigma_X \sqrt{n}} + \frac{\lambda \mu}{\sigma_X} (\frac{c}{\lambda \mu} - 1) t \sqrt{n} - Z_n(\theta_n(t)) - \frac{\mu}{\sigma_X} (\frac{N(nt) - \lambda nt}{\sqrt{n}}).$$

In order to guarantee the weak convergence of R_n we assume additionally that a second order LLN holds, namely

iii) $\frac{N(nt) - \lambda nt}{\sqrt{n}} \xrightarrow{P} 0, \quad n \to \infty.$

The risk process in the new coordinates is connected with an increasing number of customers in $[0,t]$. Thus it is reasonable to assume that the initial capital $u = u(n)$ increases with n whereas the safety loading $\rho = (\frac{c}{\lambda\mu} - 1) = \rho(n)$ decreases in n in such a way that

iv) $\frac{u(n)}{\sigma_X\sqrt{n}} \to u_0, \quad \frac{\lambda\mu}{\sigma_X}\rho(n)\sqrt{n} \to \rho_0, \quad n \to \infty.$

Then

$$R_n(t) \Rightarrow R_0(t) = u_0 + \rho_0 t - \lambda^{1/2}B(t) \quad \text{in} \quad \mathcal{D},$$

where we have used the selfsimilarity $B(\lambda t) \stackrel{d}{=} \lambda^{1/2}B(t)$ of the Brownian motion.

4. α-stable Levy motion approximation

In this section we look at the results of Meerschaert and Scheffler [5] through the three-steps approach to the risk process approximation performed in Section 2. The initial model $\mathcal{I}$ is specified here by assuming that

i) the claim sizes $\{X_k\}$ are iii rv's whose df belongs to the domain of attraction of an α-stable law (briefly $X \in DA(Z_\alpha)$) with $\alpha \in (0,1)$;

ii) the interarrival times $\{Y_k\}$ are iid rv's, $Y \in DA(D_\beta)$, $\beta \in (0,1)$.

Under these conditions the stable Functional Central Limit Theorem (FCLT) claims that there exist normalizing sequences $B(n) > 0$ and $b(n) > 0$ such that for all t in a dense subset in $(0, \infty)$

$$Z_n(t) := \sum_{k=1}^{[nt]} \frac{X_k}{B(n)} \stackrel{d}{\to} Z(t), \quad Z(1) = Z_\alpha \tag{4.1}$$

and

$$\frac{T_{[nt]}}{b(n)} = \sum_{k=1}^{[nt]} \frac{Y_k}{b(n)} \stackrel{d}{\to} D(t), \quad D(1) = D_\beta. \tag{4.2}$$

The limit process $Z(t)$ is an α-stable one-sided Levy motion and hence $\frac{1}{\alpha}$-selfsimilar and stochastically continuous. Its sample paths belong to $\mathcal{M}(0,\infty)$. Similarly, the process $D(t)$ is one-sided β-stable Levy motion, stochastically continuous with sample paths in $\mathcal{M}(0,\infty)$. Consequently,

$$Z_n \Rightarrow Z \quad \text{and} \quad \frac{T_{[n.]}}{b(n)} \Rightarrow D(.) \quad \text{in} \quad \mathcal{M}$$

with respect to the J_1-topology.

Define the hitting time process of $D(.)$ by $E(t) := \inf\{x : D(x) > t\}$. It is β-selfsimilar, hence stochastically continuous, but not any more a Levy process. Its sample paths belong to $\mathcal{M}(0,\infty)$. Take a sequence $\tilde{b}(n)$ asymptotically inverse to $b(n)$ in the sense that $b(\tilde{b}(n)) \sim n$. Then convergence (4.2) implies the weak convergence of the random time changes

$$\theta_n(.) := \frac{N(n.)}{\tilde{b}(n)} \Rightarrow E(.),\ n \to \infty,$$

in $\mathcal{M}(0, \infty)$ with respect to the J_1-topology (cf. Theorem 3 in Bingham [1]).

Now take convergence (4.1) along the subsequence $\{n' = \tilde{b}(n)\}$. We get

$$Z'_n(.) := \sum_{k=1}^{[\tilde{b}(n).]} \frac{X_k}{B(\tilde{b}(n))} \Rightarrow Z(.) \quad \text{in} \quad \mathcal{M}. \tag{4.3}$$

Convergences (4.2) and (4.3) suggest the choice of the proper time-space changes, namely $\zeta_n(t, x) = (nt, \frac{x}{B(\tilde{b}(n))})$. Then we have

$$\mathcal{N}_n = \left\{ \left(T_{nk} = \frac{T_k}{n}, X_{nk} = \frac{X_k}{B(\tilde{b}(n))} \right) : k \geq 1 \right\}.$$

Consequently,

$$N_n(t) = N(nt) = \tilde{b}(n)\theta_n(t) = k_n(\theta_n(t))$$

and

$$S_n(t) = \sum_{k=1}^{N_n(t)} X_{nk} = \sum_{k=1}^{\tilde{b}(n)\theta_n(t)} \frac{X_k}{B(\tilde{b}(n))} = Z'_n \circ \theta_n(t).$$

Recall that from Section 2, in order to use the continuity property of the composition

$$Z'_n \circ \theta_n \Rightarrow Z \circ E$$

in $\mathcal{M}$ with respect to the Skorohod M_1-topology, we need one more assumption, namely

iii) both limit random processes do not jump together with probability 1.

Now, the risk process $R_n(t)$, associated with $\mathcal{N}_n$, can be expressed as

$$R_n(t) := \frac{R(nt)}{B(\tilde{b}(n))} = \frac{u(n)}{B(\tilde{b}(n))} + \frac{c(n)nt}{B(\tilde{b}(n))} - Z'_n(\theta_n(t)).$$

In addition assume that the initial capital and the income rate increase with n in a way that

iv) $\frac{u(n)}{B(b(n))} \to u_0, \quad \frac{c(n)n}{B(b(n))} \to c_0, \quad n \to \infty.$

Finally, we may claim that

$$R_n(t) \Rightarrow R_0(t) = u_0 + c_0 t - Z(E(t)) \quad \text{in} \quad \mathcal{D}.$$

Here the random time-changed α-stable Levy motion $Z \circ E$ is not anymore a Levy process but a β/α -selfsimilar process whose increments are neither independent nor stationary.

To this section also belongs the risk approximation studied by Furrer et al. [3]. In their model the claim sizes X_k are iid rv's with $EX = \mu$ and $X \in DA(Z_\alpha)$, $\alpha \in (1,2)$. The interarrival times Y_k are iid rv's such that the counting process $N_n(t)$ is a renewal process satisfying the first and second order LLN:

$$\frac{N_n(t)}{n} \to \lambda t \quad \text{and} \quad \frac{N_n(t) - n\lambda t}{B(n)} \to 0, \quad n \to \infty.$$

Then, under the usual assumptions, the limiting risk process has the form $R_0(t) = u_0 + c_0 t - \lambda^{1/\alpha} Z_\alpha(t)$, where Z_α is the α-stable Levy motion.

5. Approximation by a subordinated Sato process

Definition. A selfsimilar random process with independent but not necessarily stationary additive increments is referred to as a Sato process (cf. Embrechts and Maejima [2]).

In this section we drop the assumption of identically distributed claim sizes and specify the initial insurance model $\mathcal{I}$ by the assumptions:

i) The claim sizes X_k are independent Pareto-distributed rv's with

$$P(X_k > x) = \left(\frac{Ck^\delta}{x}\right)^\alpha \quad \text{for} \quad x > Ck^\delta, \quad \alpha \in (0,1);$$

ii) the interarrival times Y_k are iid rv's whose df G has a regularly varying tail $1 - G(x) \sim x^{-\beta} L(x)$ with $\beta \in (0,1)$ and $L(x)$ - slowly varying function.

In this very heavy-tailed case the stable FCLT applies and we get that there exist normalizing sequences $C(n) > 0$ and $b(n) > 0$ such that the weak convergences

$$S_n(.) := \sum_{k=1}^{[n.]} \frac{X_k}{C(n)} \Rightarrow \mathbf{S}(.) \tag{5.1}$$

and

$$(5.2) \qquad \frac{T_{[n.]}}{b(n)} = \sum_{k=1}^{[n.]} \frac{Y_k}{b(n)} \Rightarrow D(.)$$

hold in $\mathcal{M}$. The process $\mathbf{S}(t)$ in (5.1) is a Sato process with stochastically continuous sample paths in $\mathcal{M}(0,\infty)$. Its selfsimilarity parameter is $H = \delta + 1/\alpha$. The limit process $D(t)$ in (5.2) is the β-stable Levy motion. As before we denote its hitting time process by $E(t)$ and obtain from (5.2) that

$$\theta_n(.) := \frac{N(b(n).)}{n} \Rightarrow E(.), \; n \to \infty.$$

Now, it is clear that the time-space changed initial point process has the form:

$$\left\{ \left(T_{nk} = \frac{T_k}{b(n)}, X_{nk} = \frac{X_k}{C(n)} \right) : k \geq 1 \right\}.$$

The associated random processes are:

- the counting process $N_n(t) = N(b(n)t) = n\theta_n(t) = k_n(\theta_n(t))$,
- the accumulated claim $\tilde{S}_n(t) := \sum_{k=1}^{N_n(t)} X_{nk} = \sum_{k=1}^{n\theta_n(t)} \frac{X_k}{C(n)} = S_n \circ \theta_n(t)$,
- the risk process $R_n(t) := \frac{R(b(n)t)}{C(n)} = \frac{u(n)}{C(n)} + \frac{c(n)b(n)t}{C(n)} - S_n \circ \theta_n(t)$.

Let us assume condition iii) from the previous section and

iv) $\frac{u(n)}{C(n)} \to u_0, \quad \frac{c(n)b(n)}{C(n)} \to c_0, \quad n \to \infty.$

Then we observe that

$$R_n(t) \Rightarrow R_0(t) = u_0 + c_0 t - \mathbf{S} \circ E(t) \quad \text{in} \quad \mathcal{D}.$$

Moreover, using the selfsimilarity of $\mathbf{S}$ and E, since $E(1) \stackrel{d}{=} D^{-\beta}(1)$ one gets

$$\mathbf{S} \circ E(t) \stackrel{d}{=} \left(\frac{D(1)}{t} \right)^{-\beta H} \mathbf{S}(1).$$

The last simple formula appears to be very useful in simulating the subordinated Sato process (cf. Mitov et al. [7]).

References

[1] N.H. Bingham. Limit theorems for occupation times of Markov processes, *Z. Wahrsch. verw. Geb.*, **17**, 1971, 1-22.

[2] P. Embrechts, M. Maejima. *Selfsimilar processes*, Princeton Uni. Press, Princeton, 2002.

[3] H. Furrer, Zb. Michna, A. Weron. Stable Levy motion approximation in collective risk theory, *Insurance: Mathematics and Economics*, **20**, 1997, 97-114.

[4] D.L. Iglehart. Diffusion approximation in collective risk Theory, *J. Appl. Probab.*, **6**, 1969, 285-292.

[5] M.M. Meerschaert, H.-P. Scheffler. Limit theorems for continuous time random walks with infinite mean waiting times, *J. Appl. Probab.*, **41**, 2004, 623-638.

[6] I.K. Mitov. *Processes with independent increments with application to finance, Ph.D. Thesis*, Sofia University, Sofia, 2009.

[7] I.K. Mitov, S.T. Rachev, F.J. Fabozzi. Approximation of aggregate and extremal losses within the very heavy tails framework, *J. of Quantitative Finance*, **10**, 2010, DOI: 10.1080/14697681003718414.

[8] E. Pancheva, E. Kolkovska, P. Jordanova. Random time changed extremal processes, *Probab. Theory Appl.*, **51**, No 4, 2006, 752-772.

[9] W. Whitt. *Stochastic-process limits*, Springer-Verlag, New York, 2002.

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
8 Acad. G. Bonchev str.
1113 Sofia, BULGARIA
E-MAIL: epancheva@math.bas.bg

FinAnalytica Inc.
21A Srebarna Str., 5th floor
1407 Sofia, BULGARIA
E-MAIL: ivan.mitov@finanalytica.com