Mathematica
Balkanica

New Series Vol. 25, 2011, Fasc. 4

Strong Insertion of a Continuous Finction between Two Comprable α-Continuous ((C)Continuous) Functions [1]

Majid Mirmiran

Presented by V. Kiryakova

Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a continuous function between two comparable real-valued functions.

2010 Math. Subject Class.: Primary 54C08, 54C10, 54C50;Secondary 26A15, 54C30.

Key Words: Strong insertion, Strong binary relation, $C-$open set, Semi-preopen set, $\alpha-$open set, Lower cut set.

1. Introduction

The concept of a $C-$open set in a topological space was introduced by E. Hatir, T. Noiri and S. Yksel in 1996 [5]. The authors define a set S to be a $C-$*open* set if $S = U \cap A$, where U is open and A is semi-preclosed. A set S is a $C-$*closed* set if its complement is $C-$open set or equivalently if $S = U \cup A$, where U is closed and A is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an $\alpha-$open set and a $C-$open set. This enable them to provide the following decomposition of continuity: a function is continuous if and only if it is $\alpha-$continuous and $C-$continuous.

Recall that a subset A of a topological space (X, τ) is called $\alpha-$*open* if A is the difference of an open and a nowhere dense subset of X. A set A is called $\alpha-$*closed* if its complement is $\alpha-$open or equivalently if A is union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace

[1]This work was supported by University of Isfahan and Centre of Excellence for Mathematics (University of Isfahan).

are called *semi-preopen or $\beta-$open.* A set is *semi-preclosed or $\beta-$closed* if its complement is semi-preopen or $\beta-$open.

We have that a set A is $\beta-$open if and only if $A \subseteq Cl(Int(Cl(A)))$ [3].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [10] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subset of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [2, 4].

Hence, a real-valued function f defined on a topological space X is called $C-$continuous (resp. $\alpha-$continuous) if the preimage of every open subset of $\mathbb{R}$ is $C-$open (resp. $\alpha-$open) subset of X.

Results of Katětov [6, 7] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [1], are used in order to give necessary and sufficient conditions for the strong insertion of a continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [8].

A property P defined relative to a real-valued function on a topological space is a *$c-$property* provided that any constant function has property P and provided that the sum of a function with property P and any continuous function also has property P. If P_1 and P_2 are $c-$property, the following terminology is used:(i) A space X has the *weak $c-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a continuous function h such that $g \leq h \leq f$.(ii) A space X has the *strong $c-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, is given a sufficient condition for the weak $c-$insertion property. Also for a space with the weak $c-$insertion property, we give necessary and sufficient conditions for the space to have the strong $c-$insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a continuous function, the necessary definitions and terminology are stated.

The abbreviations c , Cc and αc are used for continuous , $C-$continuous and $\alpha-$continuous, respectively.

Let (X,τ) be a topological space, the family of all $\alpha-$open, $\alpha-$closed, $C-$open and $C-$closed will be denoted by $\alpha O(X,\tau)$, $\alpha C(X,\tau)$, $CO(X,\tau)$ and $CC(X,\tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X,τ). Respectively, we define the *$\alpha-$closure, $\alpha-$interior, $C-$closure* and *$C-$interior* of a set A, denoted by $\alpha Cl(A), \alpha Int(A), CCl(A)$ and $CInt(A)$ as follows:
$\alpha Cl(A) = \cap\{F : F \supseteq A, F \in \alpha C(X,\tau)\}$,
$\alpha Int(A) = \cup\{O : O \subseteq A, O \in \alpha O(X,\tau)\}$,
$CCl(A) = \cap\{F : F \supseteq A, F \in CC(X,\tau)\}$ and
$CInt(A) = \cup\{O : O \subseteq A, O \in CO(X,\tau)\}$.

Respectively, we have $\alpha Cl(A), CCl(A)$ are $\alpha-$closed, semi-preclosed and $\alpha Int(A), CInt(A)$ are $\alpha-$open, semi-preopen.
The following first two definitions are modifications of conditions considered in [6, 7].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ \nu$ implies $x\ \rho\ \nu$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:
1) If $A_i\ \rho\ B_j$ for any $i \in \{1,\ldots,m\}$ and for any $j \in \{1,\ldots,n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1,\ldots,m\}$ and any $j \in \{1,\ldots,n\}$.
2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.
3) If $A\ \rho\ B$, then $Cl(A) \subseteq B$ and $A \subseteq Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [1] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f,\ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then

$A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. *Let g and f be real-valued functions on a topological space X with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$, then there exists a continuous function h defined on X such that $g \leq h \leq f$.*

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [7] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = Int(H(t_2)) \setminus Cl(H(t_1))$. Hence $h^{-1}(t_1, t_2)$ is an open subset of X, i.e., h is a continuous function on X. ■

The above proof used the technique of proof of Theorem 1 of [6].

If a space has the strong $c-$insertion property for (P_1, P_2), then it has the weak $c-$insertion property for (P_1, P_2).The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak $c-$insertion property to satisfy the strong $c-$insertion property.

Theorem 2.2. *Let P_1 and P_2 be $c-$property and X be a space that satisfies the weak $c-$insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g \leq f, g$ has property P_1 and f has property P_2. The space X has the strong $c-$insertion property for (P_1, P_2) if and only if there* *exist lower cut sets $A(f-g,2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that (i) for each n, F_n and $A(f-g,2^{-n})$ are completely separated*

Proof. Theorem 3.1, of [9]. ■

Theorem 2.3. *Let P_1 and P_2 be $c-$properties and assume that the space X satisfied the weak $c-$insertion property for (P_1, P_2). The space X satisfies the strong $c-$insertion property for (P_1, P_2) if and only if X satisfies the strong $c-$insertion property for (P_1, c) and for (c, P_2).*

Proof. Theorem 3.2, of [9]. ■

3. Applications

Corollary 3.1. *If for each pair of disjoint $\alpha-$closed (resp. $C-$closed) sets F_1, F_2 of X , there exist open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then X has the weak $c-$insertion property for $(\alpha c, \alpha c)$ (resp. (Cc, Cc)).*

Proof. Let g and f be real-valued functions defined on the X, such that f and g are αc (resp. Cc), and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $\alpha Cl(A) \subseteq \alpha Int(B)$ (resp. $CCl(A) \subseteq CInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an $\alpha-$closed (resp. $C-$closed) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$open (resp. $C-$open) set, it follows that $\alpha Cl(A(f,t_1)) \subseteq \alpha Int(A(g,t_2))$ (resp. $CCl(A(f,t_1)) \subseteq CInt(A(g,t_2))$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. *If for each pair of disjoint $\alpha-$closed (resp. $C-$closed) sets F_1, F_2, there exist open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then every $\alpha-$continuous (resp. $C-$continuous) function is continuous.*

Proof. Let f be a real-valued $\alpha-$continuous (resp. $C-$continuous) function defined on the X. Set $g = f$, then by Corollary 3.1, there exists a continuous function h such that $g = h = f$. ■

Corollary 3.3. *If for each pair of disjoint $\alpha-$closed (resp. $C-$closed) sets F_1, F_2 of X , there exist open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then X has the strong $c-$insertion property for $(\alpha c, \alpha c)$ (resp. (Cc, Cc)).*

proof Let g and f be real-valued functions defined on the X, such that f and g are αc (resp. Cc), and $g \leq f$. Set $h = (f+g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are continuous functions hence h is a continuous function. ■

Corollary 3.4. *If for each pair of disjoint subsets F_1, F_2 of X , such that F_1 is $\alpha-$closed and F_2 is $C-$closed, there exist open subsets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then X have the weak $c-$insertion property for $(\alpha c, Cc)$ and $(Cc, \alpha c)$.*

Proof. Let g and f be real-valued functions defined on the X, such that g is αc (resp. Cc) and f is Cc (resp. αc), with $g \leq f$.If a binary relation ρ is defined by $A \; \rho \; B$ in case $CCl(A) \subseteq \alpha Int(B)$ (resp. $\alpha Cl(A) \subseteq CInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $C-$closed (resp. $\alpha-$closed) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$open (resp. $C-$open) set, it follows that $CCl(A(f, t_1)) \subseteq \alpha Int(A(g, t_2))$ (resp. $\alpha Cl(A(f, t_1)) \subseteq CInt(A(g, t_2)))$. Hence $t_1 < t_2$ implies that $A(f, t_1) \; \rho \; A(g, t_2)$. The proof follows from Theorem 2.1. ■

Before stating the consequences of Theorems 2.2, and 2.3, we state and prove the necessary lemmas.

Lemma 3.1. *The following conditions on the space X are equivalent:*

(i) For each pair of disjoint subsets F_1, F_2 of X, such that F_1 is $\alpha-$closed and F_2 is $C-$closed, there exist open subsets G_1, G_2 of X such that $F_1 \subseteq G_1, F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$.

(ii) If F is a $C-$closed (resp. $\alpha-$closed) subset of X which is contained in an $\alpha-$open (resp. $C-$open) subset G of X, then there exists an open subset H of X such that $F \subseteq H \subseteq Cl(H) \subseteq G$.

Proof. (i) ⇒ (ii) Suppose that $F \subseteq G$, where F and G are $C-$closed (resp. $\alpha-$closed) and $\alpha-$open (resp. $C-$open) subsets of X, respectively. Hence, G^c is an $\alpha-$closed (resp. $C-$closed) and $F \cap G^c = \varnothing$.

By (i) there exists two disjoint open subsets G_1, G_2 of X s.t., $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \varnothing \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since G_2^c is a closed set containing G_1 we conclude that $Cl(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq Cl(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that F_1, F_2 are two disjoint subsets of X, such that F_1 is $\alpha-$closed and F_2 is $C-$closed.

This implies that $F_2 \subseteq F_1^c$ and F_1^c is an $\alpha-$open subset of X. Hence by (ii) there exists an open set H s.t., $F_2 \subseteq H \subseteq Cl(H) \subseteq F_1^c$.
But

$$H \subseteq Cl(H) \Rightarrow H \cap (Cl(H))^c = \varnothing$$

and

$$Cl(H) \subseteq F_1^c \Rightarrow F_1 \subseteq (Cl(H))^c.$$

Furthermore, $(Cl(H))^c$ is an open set of X. Hence $F_2 \subseteq H, F_1 \subseteq (Cl(H))^c$ and $H \cap (Cl(H))^c = \varnothing$. This means that condition (i) holds. ∎

Lemma 3.2. *Suppose that X is a topological space. If each pair of disjoint subsets F_1, F_2 of X, where F_1 is $\alpha-$closed and F_2 is $C-$closed, can separate by open subsets of X then there exists a continuous function $h : X \to [0,1]$ s.t., $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

Proof. Suppose F_1 and F_2 are two disjoint subsets of X, where F_1 is $\alpha-$closed and F_2 is $C-$closed. Since $F_1 \cap F_2 = \varnothing$, hence $F_2 \subseteq F_1^c$. In particular, since F_1^c is an $\alpha-$open subset of X containing $C-$closed subset F_2 of X, by Lemma 3.1, there exists an open subset $H_{1/2}$ of X s.t.,

$$F_2 \subseteq H_{1/2} \subseteq Cl(H_{1/2}) \subseteq F_1^c.$$

Note that $H_{1/2}$ is also an $\alpha-$open subset of X and contains F_2, and F_1^c is an $\alpha-$open subset of X and contains a $C-$closed subset $Cl(H_{1/2})$ of X. Hence, by Lemma 3.1, there exists open subsets $H_{1/4}$ and $H_{3/4}$ s.t.,

$$F_2 \subseteq H_{1/4} \subseteq Cl(H_{1/4}) \subseteq H_{1/2} \subseteq Cl(H_{1/2}) \subseteq H_{3/4} \subseteq Cl(H_{3/4}) \subseteq F_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain open subsets H_t of X with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_1$ and $h(x) = 1$ for $x \in F_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, F_2 \subseteq H_t$; hence $h(F_2) = \{0\}$. Furthermore, by definition, $h(F_1) = \{1\}$. It remains only to prove that h is a continuous function on X. For every $\beta \in \mathbb{R}$, we have if $\beta \leq 0$ then $\{x \in X : h(x) < \beta\} = \varnothing$ and if $0 < \beta$ then $\{x \in X : h(x) < \beta\} = \cup\{H_t : t < \beta\}$, hence, they are open subsets of X. Similarly, if $\beta < 0$ then $\{x \in X : h(x) > \beta\} = X$ and if $0 \leq \beta$ then $\{x \in X : h(x) > \beta\} = \cup\{(Cl(H_t))^c : t > \beta\}$ hence, every of them is an open subset of X. Consequently h is a continuous function. ■

Lemma 3.3. *Suppose that X is a topological space. If each pair of disjoint subsets F_1, F_2 of X, where F_1 is $\alpha-$closed and F_2 is $C-$closed, can separate by open subsets of X, and F_1 (resp. F_2) is a countable intersection of open subsets of X, then there exists a continuous function $h : X \to [0,1]$ s.t., $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h(F_2) = \{1\}$ (resp. $h(F_1) = \{1\}$).*

Proof. Suppose that $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), where G_n is an open subset of X. We can suppose that $G_n \cap F_2 = \varnothing$ (resp. $G_n \cap F_1 = \varnothing$), otherwise we can substitute G_n by $G_n \setminus F_2$ (resp. $G_n \setminus F_1$). By Lemma 3.2, for every $n \in \mathbb{N}$, there exists a continuous function $h_n : X \to [0,1]$ s.t., $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$) and $h_n(X \setminus G_n) = \{1\}$. We set $h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that h is a continuous function from X to $[0,1]$. Since for every $n \in \mathbb{N}, F_2 \subseteq X \setminus G_n$ (resp. $F_1 \subseteq X \setminus G_n$), therefore $h_n(F_2) = \{1\}$ (resp. $h_n(F_1) = \{1\}$) and consequently $h(F_2) = \{1\}$ (resp. $h(F_1) = \{1\}$). Since $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$), hence $h(F_1) = \{0\}$ (resp. $h(F_2) = \{0\}$). It suffices to show that if $x \notin F_1$ (resp. $x \notin F_2$), then $h(x) \neq 0$.

Now if $x \notin F_1$ (resp. $x \notin F_2$), since $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), therefore there exists $n_0 \in \mathbb{N}$ s.t., $x \notin G_{n_0}$, hence $h_{n_0}(x) = 1$, i.e., $h(x) > 0$. Therefore $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$). ■

Lemma 3.4. *Suppose that X is a topological space such that every two disjoint $C-$closed and $\alpha-$closed subsets of X can be separated by open subsets of X. The following conditions are equivalent:*

(i) For every two disjoint subsets F_1 and F_2 of X, where F_1 is $\alpha-$closed and F_2 is $C-$closed, there exists a continuous function $h : X \to [0,1]$ s.t., $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h^{-1}(1) = F_2$ (resp. $h^{-1}(1) = F_1$).

(ii) Every $\alpha-$closed (resp. $C-$closed) subset of X is a countable intersection of open subsets of X.

(iii) Every $\alpha-$open (resp. $C-$open) subset of X is a countable union of closed subsets of X.

Proof. (i) ⇒ (ii) Suppose that F is an $\alpha-$closed (resp. $C-$closed) subset of X. Since $\varnothing$ is a $C-$closed (resp. $\alpha-$closed) subset of X, by (i) there exists a continuous function $h : X \to [0,1]$ s.t., $h^{-1}(0) = F$. Set $G_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, G_n is an open subset of X and $\bigcap_{n=1}^{\infty} G_n = \{x \in X : h(x) = 0\} = F$.

(ii) ⇒ (i) Suppose that F_1 and F_2 are two disjoint subsets of X, where F_1 is $\alpha-$closed and F_2 is $C-$closed. By Lemma 3.3, there exists a continuous function $f : X \to [0,1]$ s.t., $f^{-1}(0) = F_1$ and $f(F_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two closed subsets of X and $(G \cup F) \cap F_2 = \varnothing$. By Lemma 3.3, there exists a continuous function $g : X \to [\frac{1}{2}, 1]$ s.t., $g^{-1}(1) = F_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define h by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then h is well-defined and a continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence h defined on X and maps to $[0,1]$. Also, we have $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) ⇔ (iii) By De Morgan law and noting that the complement of every open subset of X is a closed subset of X and complement of every closed subset of X is an open subset of X, the equivalence is hold. ■

Corollary 3.5. *If for every two disjoint subsets F_1 and F_2 of X, where F_1 is $\alpha-$closed (resp. $C-$closed) and F_2 is $C-$closed (resp. $\alpha-$closed), there exists a continuous function $h : X \to [0,1]$ s.t., $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$ then X has the strong $c-$insertion property for $(\alpha c, Cc)$ (resp. $(Cc, \alpha c)$).*

Proof. Since for every two disjoint subsets F_1 and F_2 of X, where F_1 is $\alpha-$closed (resp. $C-$closed) and F_2 is $C-$closed (resp. $\alpha-$closed), there exists a continuous function $h : X \to [0,1]$ s.t., $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$, define $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then G_1 and G_2 are two disjoint open subsets of X that contain F_1 and F_2, respectively. Hence by Corollary 3.4, X has the weak $c-$insertion property for $(\alpha c, Cc)$ and $(Cc, \alpha c)$. Now, assume that g and f are functions on X such that $g \leq f$, g is αc (resp. Cc) and f is c. Since $f - g$ is αc (resp. Cc), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is an $\alpha-$closed (resp. $C-$closed) subset of X. By Lemma 3.4, we can choose a sequence $\{F_n\}$ of closed subsets of X s.t., $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$ and for every $n \in \mathbb{N}, F_n$ and $A(f - g, 2^{-n})$ are disjoint subsets of X. By Lemma 3.2, F_n and $A(f - g, 2^{-n})$ can be completely separated by continuous functions. Hence by Theorem 2.2, X has the strong $c-$insertion property for $(\alpha c, c)$ (resp. (Cc, c)).

By an analogous argument, we can prove that X has the strong $c-$insertion property for (c, Cc) (resp. $(c, \alpha c)$). Hence, by Theorem 2.3, X has the strong $c-$insertion property for $(\alpha c, Cc)$ (resp. $(Cc, \alpha c)$). ■

Acknowledgement

This research was partially supported by Centre of Excellence for Mathematics(University of Isfahan).

References

[1] F. Brooks, Indefinite cut sets for real functions, *Amer. Math. Monthly,* **78**, 1971, 1007-1010.

[2] J. Dontchev, The characterization of some peculiar topological space via $\alpha-$ and $\beta-$sets, *Acta Math. Hungar.,* **69**, 1-2, 1995, 67-71.

[3] J. Dontchev, Between $\alpha-$ and $\beta-$sets, *Math. Balkanica (N.S)*, **12**, 1998, 295-302.

[4] M. Ganster and I. Reilly, A decomposition of continuity, *Acta Math. Hungar.,* **56**, 3-4, 1990, 299-301.

[5] E. Hatir, T. Noiri and S. Yksel, A decomposition of continuity, *Acta Math. Hungar.,* **70**, 1-2, 1996, 145-150.

[6] M. Katětov, On real-valued functions in topological spaces, *Fund. Math.,* **38**, 1951, 85-91.

[7] M. Katětov, Correction to, "On real-valued functions in topological spaces", *Fund. Math.,* **40**, 1953, 203-205.

[8] E. Lane, Insertion of a continuous function, *Pacific J. Math.,* **66**, 1976, 181-190.

[9] M. Mirmiran, Insertion of a function belonging to a certain subclass of $\mathbb{R}^X$, *Bull. Iran. Math. Soc.,*, **28**, No. 2, 2002, 19-27.

[10] M. Przemski, A decomposition of continuity and $\alpha-$continuity, *Acta Math. Hungar.,* **61**, 1-2, 1993, 93-98.

Department of Mathematics
University of Isfahan
Isfahan 81746-73441, IRAN

Received 29.05.2010

E-mail: mirmir@sci.ui.ac.ir