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# On Schur–Convex Integrals of Non-Negative Functions

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In the article we give some answers to a question of Albert Marshal and Ingram Olkin [1]. Namely, given a Schur-convex non-negative function $\varphi : \mathbf{R}^n \to \mathbf{R}$ and the integral $\psi(\theta) = \int_{R^n} K(\theta, x)\varphi(x)dx$ with a kernel $K : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, find conditions on the kernel $K$ which guarantee that the function $\psi$ is Shur-convex.

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## 1. Introduction

Inequalities have been used in almost all branches of Mathematics.In their monograph : 'Inequalities: Theory of Majorization and Its Applications' [1], Albert Marshal and Ingram Olkin have raised the problem: *Under what conditions about the function $K$ and the measure $\mu$, the function $\psi$ is Schur-convex. Here we have put* $\psi(\theta) = \int_{R^n} K(\theta, x)\varphi(x)d\mu(x)$, *where* $\theta, x \in R^n$, $K(\theta, x) : R^n \times R^n \to R$, $\varphi : R^n \to R$ *and* $\varphi$ *is Schur-convex.*

In [1], the authors have proved a theorem, when $K(\theta, x)$ is of the form $K(\theta, x) = g(\theta - x)$. F. Proshan, J. Sethuraman [2], K.W. Chen [3] and N. Elezovic, J. Pecaric [4] have proved a series of theorems about this problem. All proved assertions are considered in the sense of the Lebesgue integral. In [5]there are given results when :

1. If $K(\theta, x) = g(\theta . x)$ is convex, then $\psi$ is Schur-convex.
2. $K(\theta, x) = g(\theta \times x)$.

We formulate these theorems in our paper, in the secton Related results, as Theorem 1 and Theorem 2. The conclusion in Theorem 1 is very important, because we integrate a convex function and the result is Schur-convex function.

After Theorem 1, we show an Example 1.

The next two theorems, which we present here, in the section Main results, are natural extensions of [5]. Their results are very important in this sense.

The conditions in Theorem 3 and Theorem 4 are:

3. $K(\theta, x) = g(x^{\theta})$. 4. $K(\theta, x) = g(\theta^{x})$.

In Theorem 2, Theorem 3 and Theorem 4 $g$ is a Schur-convex function, and the conclusion is that $\psi$ is a Schur-concave function. In all four conditions, $\varphi$ is non-negative. After Theorem 3 and Theorem 4 we show three examples. In Example 4 after Theorem 4, we use the function of Laplace

$$Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp(-t^2/2) dt.$$

The results here could be apply in Probability theory. All the results in theorems and examples can be used as a generator for S-convex and S-concave functions.

## 2. Preliminaries.

Everywhere in the paper we rely essentialy on Lemma 1. Therefore all theorems are formulated for $n$, but in the proofs, we consider only the case $n = 2$.

We note $\mathbf{R}_+^n = [0, \infty)^n$, and $\mathbf{R}_{++}^n = (0, \infty)^n$.

**Definition 1.** Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbf{R}^n$. We say $x \prec y$, if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k x_{[i]} y_{[i]}$, $k = 1, 2, ..., n-1$; $\sum_{i=1}^n x_{[i]} x_{[i]} = \sum_{i=1}^n x_{[i]} y_{[i]}$. Here we note $x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}$, as a rearrangement of the coordinates of $x = (x_1, x_2, ..., x_n)$. Analogically $y_{[1]} \geq y_{[2]} \geq ... y_{[n]}$.

**Definition 2.** Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbf{R}^n$. We say $x \prec_w y$, if $\sum_{i=1}^k x_{[i]} x_{[i]} \leq \sum_{i=1}^k x_{[i]} y_{[i]}$, $k = 1, 2, ..., n$. Here we note $x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}$, as a rearrangement of the coordinates of $x = (x_1, x_2, ..., x_n)$. Analogically $y_{[1]} \geq y_{[2]} \geq ... y_{[n]}$.

**Definition 3.** The real function $\varphi$, defined on the set $\mathbf{A}$, $\mathbf{A} \subseteq \mathbf{R}^n$, is Schur-convex if: ($x \prec y$ on the set $\mathbf{A}$) implies $\varphi(x) \leq \varphi(y)$. Analogically, the real function $\varphi$, defined on the set $\mathbf{A}$, $\mathbf{A} \subseteq \mathbf{R}^n$, is Schur-concave if: ( $x \prec y$ on the set $\mathbf{A}$) implies $\varphi(x) \geq \varphi(y)$.

**Lemma 1.([1])** *If* $x \prec y$, *x can be obtained by y after sequential applying of finite number T-transformations.(The matrix of the T-transformations is* $\mathbf{T} = \lambda\mathbf{E} + (\mathbf{1} - \lambda)\mathbf{Q}$, *where* $\lambda \in [0, 1]$ *and* $\mathbf{Q}$ *is obtained by unit matrix, after changing the places of two columns.)*

**Lemma 2.([1])** *The real function* $\varphi$, *defined on the set* $\mathbf{A}$, $\mathbf{A} \subseteq \mathbf{R}^n$, *satisfied the condition: with* $x \prec_w y$ *above* $\mathbf{A}$ *implies* $\varphi(x) \leq \varphi(y)$; *if and only if* $\varphi$ *is monotone, non-decreasing and Schur-convex above* $\mathbf{A}$.

**Lemma 3.** *The function* $f(x) = x(c - x)$, $x \in [0, c/2]$, $c > 0$, *is non-decreasing.*

**Lemma 4.** *The function* $f(\alpha) = x_1^\alpha - x_2^\alpha$, $x_1 \geq x_2 \geq 0$, $\alpha \geq 0$, *is non-decreasing.*

**Lemma 5.** *The function* $f(x) = \ln x \ln(c - x)$, $x \in [1, c/2]$, $c > 2$, *is non-decreasing.*

**Lemma 6.([1])** *Let the function* $f(x)$ *be defined on the interval* $I \subset \mathbf{R}$. *Then the function* $\varphi(x) = \prod_{i=1}^n f(x_i), x \in \mathbf{R}^n$, *is Shur-convex if and only if* $f(x)$ *is convex on* $I$.

Let $K(\theta, x) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $\varphi : \mathbf{R}^n \to \mathbf{R}$ and $\psi(\theta) = \int_{R^n} K(\theta, x)\varphi(x)dx$, always when the integral exists.

## 3. Related results.

Here we present the results, which are formulated and proved in [5].

**Theorem 1.** *Let* $K(\theta, x) = g(\theta.x)$ *(Here* $\theta.x = \sum_{i=1}^n \theta_i x_i$*) ,where* $g : \mathbf{R} \to \mathbf{R}$. *If* $\varphi$ *is non-negative and symmetric, and g is convex, then* $\psi$ *is Schur-convex.*

Example 1. If in the conditions of Theorem1 we put $g(t) = \exp(-2t)$, $\varphi(x) = \pi^{-n/2} \exp(-\sum_{i=1}^n x_i^2)$ we obtain, that the function $\psi(\theta) = \exp(\sum_{i=1}^n \theta_i^2)$ is Schur-convex.

**Theorem 2.** *Let* $K(\theta, x) : \mathbf{R}^n_{++} \times \mathbf{R}^n_+ \to \mathbf{R}$, *and* $K(\theta, x) = g(\theta \times x)$, *where* $\theta \times x = (\theta_1 x_1, \theta_2 x_2, ..., \theta_n x_n)$ *and* $g : \mathbf{R}^n_+ \to \mathbf{R}$. *If* $\varphi$ *and g are Shur-convex and non-decreasing, and* $\varphi$ *is non-negative, then* $\psi$ *is Shur-concave.*

## 4. Main results.

Here we present new results, which are natural extension of the previous section.

**Theorem 3.** *Let* $\mathbf{A} = [\mathbf{1}, \infty)^n$, $K(\theta, x) : \mathbf{R}^n_{++} \times \mathbf{A} \to \mathbf{R}$, *and* $K(\theta, x) = g(x^\theta)$, *where* $x^\theta = (x_1^{\theta_1}, x_2^{\theta_2}, ..., x_n^{\theta_n})$ *and* $g : \mathbf{A} \to \mathbf{R}$. *If* $\varphi$ *and* $g$ *are Shur-convex and non-decreasing, and moreover* $\varphi$ *is non-negative, then* $\psi$ *is Shur-concave.*

P r o o f. We note

$$\Delta = \psi(\theta) - \psi(\xi) = \int\int_{\mathbf{A}} \varphi(x_1, x_2)[g(x_1^{\theta_1}, x_2^{\theta_2}) - g(x_1^{\xi_1}, x_2^{\xi_2})]dx_1 dx_2$$

by $\xi \prec \theta$ and $0 < \theta_1 \le \xi_1 \le \xi_2 \le \theta_2$. Let us substitute $x_1 = x_1^{\theta_1}$, $x_2 = x_2^{\xi_2}$.

Then $\delta = \frac{\Delta}{\theta_1 \xi_2} = \frac{\psi(\theta) - \psi(\xi)}{\theta_1 \xi_2} =$

$$= \int\int_{\mathbf{A}} [g(x_1, x_2^{\theta_2/\xi_2}) - g(x_1^{\xi_1/\theta_1}, x_2)] \times$$

$$\varphi(x_1^{1/\theta_1}, x_2^{1/\xi_2}) x_1^{(1-\theta_1)/\theta_1} x_2^{(1-\xi_2)/\xi_2} dx_1 dx_2$$

and

$$\int\int_{1<\mathbf{x_1}\le\mathbf{x_2}} [\, g(x_1, x_2^{\theta_2/\xi_2}) - g(x_1^{\xi_1/\theta_1}, x_2)] \times$$

$$\varphi(x_1^{1/\theta_1}, x_2^{1/\xi_2}) x_1^{(1-\theta_1)/\theta_1} x_2^{(1-\xi_2)/\xi_2} dx_1 dx_2$$

$$= \int\int_{\mathbf{x_1}\ge\mathbf{x_2}>1} [\, g(x_2, x_1^{\theta_2/\xi_2}) - g(x_2^{\xi_1/\theta_1}, x_1)] \times$$

$$\varphi(x_1^{1/\xi_2}, x_2^{1/\theta_1}) x_2^{(1-\theta_1)/\theta_1} x_1^{(1-\xi_2)/\xi_2} dx_1 dx_2,$$

since $\varphi$ is Shur-convex, $\varphi$ is symmetric. Let us replace $a = \theta_2/\xi_2$, $b = \xi_1/\theta_1$.

According to Lemma 3 $\theta_1\theta_2 \le \xi_1\xi_2$, hence $1 \le a \le b$. Then

$$-\delta = \int\int_{\mathbf{x_1}\ge\mathbf{x_2}>1} [\, g(x_1^b, x_2) - g(x_1, x_2^a)] \times$$

$$\varphi(x_1^{1/\theta_1}, x_2^{1/\xi_2}) x_1^{(1-\theta_1)/\theta_1} x_2^{(1-\xi_2)/\xi_2} dx_1 dx_2 -$$

$$- \int\int_{\mathbf{x_1}\ge\mathbf{x_2}>1} [\, g(x_1^a, x_2) - g(x_1, x_2^b)] \times$$

$$\varphi(x_1^{1/\xi_2}, x_2^{1/\theta_1}) x_2^{(1-\theta_1)/\theta_1} x_1^{(1-\xi_2)/\xi_2} dx_1 dx_2.$$

Obviously $(x_1^a, x_2) \prec_w (x_1^b, x_2)$, $(x_1, x_2^a) \prec_w (x_1, x_2^b)$.
Applying Lemma 2, we obtain

$$g(x_1^a, x_2) \le g(x_1^b, x_2),\ g(x_1, x_2^a) \le g(x_1, x_2^b).$$

According to Lemma 4, we get

$$x_1 - x_2 \le x_1^a - x_2^a \le x_1^b - x_2^a, \text{ i.e. } x_1 + x_2^a \le x_1^b + x_2$$

or

$$(x_1,\ x_2^a) \prec_w (x_1^b, x_2), \text{ i.e. } g(x_1^b, x_2) - g(x_1,\ x_2^a) \ge 0.$$

Again using Lemma 4 we obtain $x_1^{1/\xi_2} - x_2^{1/\xi_2} \le x_1^{1/\theta_1} - x_2^{1/\theta_1}$. Since then $x_1^{1/\xi_2} + x_2^{1/\theta_1}) \le x_1^{1/\theta_1} + x_2^{1/\xi_2}$, i.e. $(x_1^{1/\xi_2}, x_2^{1/\theta_1}) \prec_w (x_1^{1/\theta_1}, x_2^{1/\xi_2})$
and then

$$\varphi(x_1^{1/\xi_2}, x_2^{1/\theta_1}) \le \varphi(x_1^{1/\theta_1}, x_2^{1/\xi_2}).$$

Moreover

$$x_2^{1/\theta_1 - 1/\xi_2} \le x_1^{1/\theta_1 - 1/\xi_2}, \text{ i.e. } x_1^{1/\xi_2} x_2^{1/\theta_1} \le x_1^{1/\theta_1} x_2^{1/\xi_2}.$$

Therefore

$$-\delta \ge \int\int_{\mathbf{A}} [\, g(x_1^b, x_2) - g(x_1, x_2^a) - g(x_1^a, x_2) + g(x_1, x_2^b)] C dx_1 dx_2 \ge 0,$$

i.e.

$$\delta \le 0 \quad (C = \varphi(x_1^{1/\xi_2}, x_2^{1/\theta_1}) x_1^{1/\xi_2}, x_2^{1/\theta_1}).$$

Therefore $\psi$ is Schur-concave. ■

Example 2. If in the conditons of Theorem 3 we put $g(t) = \sum_{i=1}^n t_i$, $\varphi(x) = 1$, where $x \in \mathbf{A} = [\mathbf{1}, \mathbf{c}]^n$, $\theta, t, \in \mathbf{R}_{++}^n$, $1 < c \in \mathbf{R}$, then the function $\chi(\theta) \sum_{i=1}^n \frac{c^{\theta_i+1}-1}{\theta_i+1}$ is Schur-concave.

Proof. We have $\psi(\theta) = \int_{\mathbf{A}} K(\theta, x)\varphi(x)dx$, where $K(\theta, x) = \sum_{i=1}^n x_i^{\theta_i}$, according to the condition. Then we obtain $\psi(\theta) = \sum_{i=1}^n \int_{\mathbf{A}} x_i^{\theta_i} dx$. We see that the functions $g(t)$ and $\varphi(x)$ satisfy the conditions of Theorem 3, i.e. they are Shur-convex, non-decreasing, and moreover $\varphi$ is non-negative.

If we put $D = (c-1)^{n-1}$ then we get $\psi(\theta) = D\sum_{i=1}^n \frac{c^{\theta_i+1}-1}{\theta_i+1}$. If we substitute $\chi(\theta) = \psi(\theta)/D$, then our conclusion is that the function $\chi(\theta) = \sum_{i=1}^n \frac{c^{\theta_i+1}-1}{\theta_i+1}$ is Schur-concave.

**Theorem 4.** *Let* $\mathbf{A} = [\mathbf{1}, \infty)^n$, $\mathbf{B} = (\mathbf{1}, \infty)^n$, $K(\theta, x) : \mathbf{B} \times \mathbf{R}_+^n \to \mathbf{R}$, *and* $K(\theta, x) = g(\theta^x)$, *where* $\theta^x = (\theta_1^{x_1}, \theta_2^{x_2}, \dots, \theta_n^{x_n})$ *and* $g : \mathbf{A} \to \mathbf{R}$. *If* $\varphi$ *and* $g$ *are Shur-convex and non-decreasing, and furthermore* $\varphi$ *is non-negative, then* $\psi$ *is Shur-concave.*

Proof. We note

$\Delta = \psi(\theta) - \psi(\xi) = \int\int_{\mathbf{R}_+^2} \varphi(x_1, x_2)[g(\theta_1^{x_1}, \theta_2^{x_2}) - g(\xi_1^{x_1}, \xi_2^{x_2})]dx_1dx_2$, as $\xi \prec \theta$ and $1 < \theta_1 \le \xi_1 \le \xi_2 \le \theta_2$.

Let us substitute $x_1 = \ln x_1,\ x_2 = \ln x_2$. Then

$$\Delta = \int\int_{\mathbf{A}} \frac{[g(x_1^{\ln\theta_1}, x_2^{\ln\theta_2}) - g(x_1^{\ln\xi_1}, x_2^{\ln\xi_2})]\varphi(\ln x_1, \ln x_2)}{x_1x_2}dx_1dx_2.$$

Now again, we substitute $x_1 = x_1^{1/\ln\theta_1},\ x_2 = x_2^{1/\ln\xi_2}$. Then, if we put $a = \ln\theta_2/\ln\xi_2,\ b = \ln\xi_1/\ln\theta_1$, we obtain

$$\delta = \ln\theta_1\ln\xi_2\Delta = \int\int_{\mathbf{A}} \frac{[g(x_1, x_2^a) - g(x_1^b, x_2)]\varphi(\ln x_1/\ln\theta_1, \ln x_2/\ln\xi_2)}{x_1x_2}dx_1dx_2.$$

Applying Lemma 1 we get $1 \le a \le b$ (because $\theta_i\xi_i \in (1,\infty)$ ).Besides

$$\ln x_1(1/\ln\theta_1 - 1/\ln\xi_2) \ge \ln x_2(1/\ln\theta_1 - 1/\ln\xi_2)$$

i.e.

$$\ln x_1/\ln\xi_2 \ln x_2/\ln\theta_1 \le \ln x_1/\ln\theta_1 \ln x_2/\ln\xi_2.$$

Therefore

$$(\ln x_1/\ln\xi_2, \ln x_2/\ln\theta_1) \prec_w (\ln x_1/\ln\theta_1, \ln x_2/\ln\xi_2).$$

Further, the proof does not differ essentially from the proof of Theorem 3.

Here we apply Lemma 5 and then we show, that $\delta \le 0$, i.e. $\psi$ is Shur-concave. ■

Example 3. If in the conditons of Theorem 4 we put $\varphi(x) = \sum_{i=1}^n x_i$, $g(t) = \sum_{i=1}^n t_i$, . where $x \in \mathbf{A} = [\mathbf{0}, \mathbf{c}]^n$, $\theta \in \mathbf{B} = (\mathbf{1}, \infty)^n, t \in [\mathbf{1}, \infty)^n$ $c \in \mathbf{R}_{++}$, then the function $\chi(\theta) = \sum_{i=1}^n \frac{[(n+1)c\ln\theta_i - 2]\theta_i^c - (n-1)c\ln\theta_i + 2}{\ln^2\theta_i}$ is Schur-concave.

Proof. We have $\psi(\theta) = \int_{\mathbf{A}} K(\theta, x)\varphi(x)dx$, where $K(\theta, x) = \sum_{i=1}^n \theta_i^{x_i}$, according to the condition. Then we obtain $\psi(\theta) = \sum_{i=1}^n \int_{\mathbf{A}} \theta_i^{x_i}(\sum_{i=1}^n x_i)dx$. We see that the functions $g(t)$ and $\varphi(x)$satisfy the conditions of Theorem 3, i.e. they are Shur-convex and non-decreasing, and moreover $\varphi$ is non-negative.

It is obvious $\psi(\theta) = I_1 + (n-1)I_2$, where $I_1 = c^{n-1}\sum_{i=1}^n \int_0^c t\theta_i^t dt$,

$$I_2 = c^{n-2}\sum_{i=1}^n \int_0^c \theta_i^t dt \int_0^c t dt = \frac{c^n}{2}\sum_{i=1}^n \int_0^c \theta_i^t dt = \frac{c^n}{2}\sum_{i=1}^n \frac{\theta_i^c - 1}{\ln \theta_i}$$

$$I_1 = c^{n-1}[\sum_{i=1}^n \frac{t\exp(t\ln\theta_i)}{\ln\theta_i} |_0^c - \sum_{i=1}^n \frac{1}{\ln\theta_i}\int_0^c \exp(t\ln\theta_i)dt]$$

$$= c^{n-1}\sum_{i=1}^n \frac{\theta_i^c c\ln\theta_i - \theta_i^c + 1}{\ln^2\theta_i}.$$

Then we obtain $\psi(\theta) = \frac{c^{n-1}}{2}\sum_{i=1}^n \frac{[(n+1)c\ln\theta_i - 2]\theta_i^c - (n-1)c\ln\theta_i + 2}{\ln^2\theta_i}$. If we put

$\chi(\theta) = \frac{2}{c^{n-1}}\psi(\theta)$, then we get $\chi(\theta) = \sum_{i=1}^n \frac{[(n+1)c\ln\theta_i - 2]\theta_i^c - (n-1)c\ln\theta_i + 2}{\ln^2\theta_i}$

is Schur-concave.

Example 4. If in the conditons of Theorem 4 we put $g(t) = \sum_{i=1}^n t_i$, where $x \in \mathbf{A} = [\mathbf{0}, \mathbf{c}]^n$, $\theta \in \mathbf{B} = (\mathbf{1}, \infty)^n, t \in [\mathbf{1}, \infty)^n$, $c \in \mathbf{R}_{++}$, and $\varphi(x) = \Pi_{i=1}^n [1 - \exp(-x_i^2/2)]$. Then the function

$$\chi(\theta) = \sum_{i=1}^n \frac{\theta_i^c - \sqrt{\pi/2}\exp(\ln^2\frac{\theta_i}{2})[\Phi(c - \ln\theta_i) - \Phi(\ln\theta_i)] - 1}{\ln\theta_i}$$

is Schur-concave. Here we put $\Phi(x) = \sqrt{\frac{2}{\pi}} {}_0\int^x \exp(-t^2/2)dt$ the function of Laplace.

Proof. Let us put $f(t) = 1 - \exp(-t^2/2)$, $t > 0$. Then we calculate $d(\ln f)/dt = t\exp(-t^2/2)/[1 - \exp(-t^2/2)] > 0$. Besides

$$d^2(\ln f)/dt^2 = d[\frac{\exp(-t^2/2)}{1 - \exp(-t^2/2)}t]/dt$$

$$= [d(\frac{\exp(-t^2/2)}{1 - \exp(-t^2/2)})dt]t + \frac{\exp(-t^2/2)}{1 - \exp(-t^2/2)} > 0.$$

Then according to Lemma 6 $\varphi(x) = \Pi_{i=1}^n [1 - \exp(-x_i^2/2)]$ is Shur-convex.

We have $\psi(\theta) = \int_{\mathbf{A}} K(\theta,x)\varphi(x)dx$, where $K(\theta,x) = \sum_{i=1}^{n}\theta_i^{x_i}$, according to the condition. Then we obtain $\psi(\theta) = \sum_{i=1}^{n}\int_{\mathbf{A}}\theta_i^{x_i}\Pi_{i=1}^{n}[1-\exp(-x_i^2/2)]dx$. We see that the functions $g(t)$ and $\varphi(x)$ satisfy the conditions of Theorem 3, i.e. they are Shur-convex and non-decreasing, and moreover $\varphi$ is non-negative.

Let us put

$$D = \int_{[\mathbf{0},\mathbf{c}]^{n-1}} \Pi_{i=1}^{n-1}[1-\exp(-x_i^2/2)]dx, \ x \in \mathbf{R}^{n-1}.$$

Then we obtain

$$\psi(\theta) = D\sum\nolimits_{i=1}^{n}\int_0^c \theta_i^t[1-\exp(-t^2/2)]dt$$

$$= D\sum\nolimits_{i=1}^{n}[\int_0^c \theta_i^t dt - \int_0^c \theta_i^t \exp(-t^2/2)dt] =$$

$$= D\sum\nolimits_{i=1}^{n}[\frac{\theta_i^c - 1}{\ln\theta_i} - \frac{1}{\ln\theta_i}e^{\frac{\ln^2\theta_i}{2}}\int_0^c e^{-\frac{(t-\ln\theta_i)^2}{2}}dt]$$

$$= D\sum\nolimits_{i=1}^{n}\frac{\theta_i^c - \sqrt{\pi/2}\exp(\ln^2\frac{\theta_i}{2})[\Phi(c-\ln\theta_i)-\Phi(\ln\theta_i)]-1}{\ln\theta_i}.$$

If we put $\chi(\theta) = \psi(\theta)/D$, then we obtain

$$\chi(\theta) = \sum\nolimits_{i=1}^{n}\frac{\theta_i^c - \sqrt{\pi/2}\exp(\ln^2\frac{\theta_i}{2})[\Phi(c-\ln\theta_i)-\Phi(\ln\theta_i)]-1}{\ln\theta_i}$$

is Schur-concave.

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