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A Study of Derivations in Prime Near-Rings[1]

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Let N be a prime near-ring. We show two main results for N to be a commutative ring: (1) If there exist $k, l \in \mathbb{N}$ such that N admits a derivation d satisfying either $d([x, y]) = x^k[x, y]x^l$ for all $x, y \in N$ or $d([x, y]) = -x^k[x, y]x^l$ for all $x, y \in N$, then N is a commutative ring. (2) If there exist $k, l \in \mathbb{N}$ such that N admits a derivation d satisfying either $d(x \circ y) = x^k(x \circ y)x^l$ for all $x, y \in N$ or $d(x \circ y) = -x^k(x \circ y)x^l$ for all $x, y \in N$, then N is a commutative ring.

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1. Introduction

Throughout this paper N stands for a zero-symmetric right near-ring. N is called zero-symmetric if $x0 = 0$ for all $x \in N$ (recall that right distributivity yields $0x = 0$). According to [6], a near-ring N is said to be prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$. An additive endomorphism d of N is called a derivation on N if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as noted in [5], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. Let $Z(N)$ be the multiplicative center of N, that is, $Z(N) = \{x \in N | xy = yx \text{ for all } y \in N\}$. Note that $Z(N) \neq \emptyset$ since $0 \in Z(N)$. For any $x, y \in N$, we denote $[x, y] = xy - yx$ and $x \circ y = xy + yx$ the well-known Lie and Jordan products, respectively.

Over the past few years, many authors have investigated commutativity of prime and semi-prime rings admitting suitably constrained derivations

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[3,8,9,10,12,14]. Some comparable results on near-rings have also been derived, see e.g. [1,2,4,7,11,13]. In [8] the authors showed that a prime ring R must be commutative if it admits a derivation d such that either $d([x,y]) = [x,y]$ for all $x, y \in K$ or $d([x,y]) = -[x,y]$ for all $x, y \in K$, where K is a nonzero ideal of R. The aim of this paper is to go a further step in this direction, and extend some results on prime rings admitting a special type of generalized derivation to prime near-rings.

The rest of the paper is organized as follows. In Section 2, we present our main results. In Section 3, we draw a conclusion.

2. The results

Let $\mathbb{N}$ be the non-negative integers including 0. The proof of our main results will rely on essentially the following lemma.

Lemma 2.1. *([5]) Let N be a prime near-ring. If N admits a nonzero derivation d for which $d(N) \subset Z(N)$, then N is a commutative ring.*

Theorem 2.2. *Let N be a prime near-ring. If there exist $k, l \in \mathbb{N}$ such that N admits a nonzero derivation d satisfying either*

(i) $d([x,y]) = x^k[x,y]x^l$ for all $x, y \in N$, or

(ii) $d([x,y]) = -x^k[x,y]x^l$ for all $x, y \in N$,

then N is a commutative ring.

Proof. We first assume that (i) holds. Since $[x, yx] = [x,y]x$, replacing y by yx in (i), we obtain

$$d([x,y]x) = d([x,yx]) = x^k[x,yx]x^l = x^k[x,y]x^{l+1} \tag{0.1}$$

for all $x, y \in N$. By definition, we have $d([x,y]x) = d([x,y])x + [x,y]d(x)$. Using (0.1) and assumption (i) we have

$$x^k[x,y]x^{l+1} = x^k[x,y]x^{l+1} + [x,y]d(x), \tag{0.2}$$

and thus $[x,y]d(x) = 0$. Replacing y by zy, we have

$$[x,zy]d(x) = [x,z]yd(x) = 0 \tag{0.3}$$

for all $x, y, z \in N$, which implies

$$[x,z]Nd(x) = 0 \tag{0.4}$$

for all $x, z \in N$. In view of the primeness of N, (0.4) yields that for each $x \in N$,

$$d(x) = 0 \quad \text{or} \quad x \in Z(N). \tag{0.5}$$

By a one-line calculation we know that if $x \in Z(N)$ then $d(x) \in Z(N)$. Hence, (0.5) forces that for all $x \in N$, $d(x) \in Z(N)$, i.e., $d(N) \subset Z(N)$. It then follows from Lemma 2.1 that N is a commutative ring.

Next, we assume that (ii) holds. Substituting yx for y in (ii), we can similarly obtain

$$d([x, y]x) = d([x, yx]) = -x^k[x, yx]x^l = -x^k[x, y]x^{l+1} \tag{0.6}$$

for all $x, y \in N$. By definition, we have $d([x, y]x) = d([x, y])x + [x, y]d(x)$. Using (0.6) and assumption (ii) we have

$$-x^k[x, y]x^{l+1} = -x^k[x, y]x^{l+1} + [x, y]d(x), \tag{0.7}$$

and thus $[x, y]d(x) = 0$. The rest of the proof is the same as before. ■

The following simple example demonstrates that the primeness hypothesis in Theorem 2.2 is necessary even in the case of arbitrary rings.

Example 2.3. Let R be a nontrivial commutative ring and $N = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \middle| \, x, y \in R \right\}$. If we define $d : N \to N$ by $d \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, then it is straightforward to check that d is a nonzero derivation on N. If $A = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, where $x \neq 0$, then $ANA = 0$ which proves that N is not prime. Furthermore, d satisfies the condition $d([A, B]) = [A, B]$ for all $A, B \in N$, but N is a noncommutative ring.

Theorem 2.4. *Let N be a prime near-ring. If there exist $k, l \in \mathbb{N}$ such that N admits a nonzero derivation d satisfying either*

(iii) $d(x \circ y) = x^k(x \circ y)x^l$ for all $x, y \in N$, or

(iv) $d(x \circ y) = -x^k(x \circ y)x^l$ for all $x, y \in N$,

then N is a commutative ring.

Proof. We first assume that (iii) holds. Since $x \circ (yx) = (x \circ y)x$, replacing y by yx in (iii), we obtain

$$d((x \circ y)x) = d(x \circ (yx)) = x^k(x \circ (yx))x^l = x^k(x \circ y)x^{l+1} \tag{0.8}$$

for all $x, y \in N$. By definition, we have $d((x \circ y)x) = d(x \circ y)x + (x \circ y)d(x)$. Then using assumption (iii) we have

$$d((x \circ y)x) = x^k(x \circ y)x^{l+1} + (x \circ y)d(x). \tag{0.9}$$

Combining (0.8) and (0.9), we conclude that $(x \circ y)d(x) = 0$ so that

$$xyd(x) = -yxd(x) \tag{0.10}$$

for all $x, y \in N$. Replacing y by zy in (0.10) and using (0.10), we obtain

$$xzyd(x) = -zyxd(x) = (-z)(-xyd(x)) = (-z)(-x)yd(x) \tag{0.11}$$

and thus $(xz - (-z)(-x))yd(x) = 0$ for all $x, y, z \in N$. Replacing x by $-x$ gives

$$(-xz + zx)yd(-x) = 0, \tag{0.12}$$

and hence

$$-[z, x]yd(x) = [z, x]yd(-x) = 0 \tag{0.13}$$

for all $x, y, z \in N$. By virtue of the primeness of N, we have that for each $x \in N$,

$$d(x) = 0 \quad \text{or} \quad x \in Z(N). \tag{0.14}$$

Since (0.14) is the same as (0.5), arguing as in the proof of Theorem 2.2 we obtain that N is a commutative ring.

Next, we assume that (iv) holds. Substituting yx for y in (iv), we can similarly obtain

$$d((x \circ y)x) = d(x \circ (yx)) = -x^k(x \circ (yx))x^l = -x^k(x \circ y)x^{l+1} \tag{0.15}$$

for all $x, y \in N$. By definition, we have $d((x \circ y)x) = d(x \circ y)x + (x \circ y)d(x)$. Then using assumption (iv) we have

$$d((x \circ y)x) = -x^k(x \circ y)x^{l+1} + (x \circ y)d(x). \tag{0.16}$$

Combining (0.15) and (0.16), we conclude that $(x \circ y)d(x) = 0$. The rest of the proof is the same as before. ■

The following simple example demonstrates that the primeness hypothesis in Theorem 2.4 is necessary even in the case of arbitrary rings.

Example 2.5. Let S be a ring and $N = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & z \\ y & 0 & 0 \end{pmatrix} \middle| x, y, z \in S \right\}$ and define a map $d : N \to N$ such that $d \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & z \\ y & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$. Clearly, the endomorphism d is a derivation on N. If $A = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $x \neq 0$, then $ANA = 0$ which proves that N is not prime. Furthermore, d satisfies the condition $d(A \circ B) = A \circ B$ for all $A, B \in N$, but N is a noncommutative ring.

3. Conclusion

In this paper, we study the prime near-rings with derivations. We prove that a prime near-ring which admits a nonzero derivation satisfying certain differential identities is a commutative ring. For future research, more general constraints on the derivation would be interesting. In addition, can the hypotheses of Theorem 2.2 and Theorem 2.4 be weakened such that the identities hold in some nonzero semigroup ideal I of N ?

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