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A Characterization of n-Seminorm

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The notion of n-norm is introduced in [6] as a generalization of the notion of 2-norm introduced by Gahler in [1]. The notion of n-seminorm is introduced in [5] as a generalization of the notion of 2-seminorm. An equivalent definition of 2-norm is given in [2]. Using the technique in [2], some classes of subsets of X^n are introduced in [3]. Using them, in this paper, it is given a characterization of n-seminorm in the sense of Minkowski.

1. Introduction

Definition 1.1 Let X be a vector space over the field $\mathbf{\Phi}$ and let $p : X^n \to \mathbb{R}$ be a mapping which satisfies the conditions:

- a) $p(x_1, x_2, ..., x_n) = 0$, for every linearly dependant set $\{x_1, ..., x_n\}$;
- b) $p(x_1, ..., x_n) = p(x_{\pi(1)}, ..., x_{\pi(n)})$, for every permutation $\pi : \{1, ..., n\} \to \{1, ..., n\}$;
- c) $p(\alpha x_1, x_2, ..., x_n) = |\alpha|\, p(x_1, x_2, ..., x_n)$, for any scalar α and arbitrary $x_1, ..., x_n \in X$;
- d) $p\left(x_1 + x_1', x_2, ..., x_n\right) \leq p(x_1, x_2, ..., x_n) + p\left(x_1', x_2, ..., x_n\right)$, for arbitrary $x_1, x_1', x_2, ..., x_n \in X$.

The function $p : X^n \to \mathbb{R}$ which satisfies the conditions $a) - d)$ is called n-seminorm, and the ordered pair $(X,\ p)$ is called real n-seminormed space.

Let X be a vector space over a field $\mathbf{\Phi}$. (The field $\mathbf{\Phi}$ is either $\mathbb{R}$ or $\mathbb{C}$) The notions of n-absorbing, n-balanced and n-convex set in X^n are introduced in [3], as a generalization of the notions of absorbing, balanced and convex set in X, respectively. In [3] are, also, introduced two new notions i.e. n-invariant set and the set $\triangle_n$ and some of the basic properties of these subsets of X^n are given. Since we will use these subsets in our subsequent work, first we will give their definitions and some of their properties.

Let $M_n(\mathbf{\Phi})$ denote the set of quadratic matrixes of order n over the field $\mathbf{\Phi}$.

For arbitrary $(x_1, ..., x_n)$ and for any matrix $A \in M_n(\mathbf{\Phi})$, we define operation $\cdot : M_n(\mathbf{\Phi}) \times X^n \to X^n$, by $A \cdot (x_1, ..., x_n) = A\,(x_1, ..., x_n)^T$, where the right side of the equation is multiplication of a matrix with a vector.

For example, in the case $n = 2$, for $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\mathbf{\Phi})$ and $(x_1, x_2) \in X^2$, we get $A \cdot (x_1, x_2) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot (x_1, x_2) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} (x_1, x_2)^T$ $= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2)$.

Definition 1.2. The set $V \subseteq X^n$ is n-balanced if $AV^T \subseteq V$, for arbitrary matrix $A \in M_n(\mathbf{\Phi})$ with $|\det A| \leq 1$, where $V^T = \{(x_1,...,x_n)^T | (x_1,...,x_n) \in V\}$.

Definition 1.3. The set $S \subseteq X^n$ is n-convex if $(x_1,..., x_{i-1}, tx_i + (1-t)\,x_i',..., x_n) \in S$, for arbitrary $(x_1,..., x_{i-1}, x_i,..., x_n)\,,(x_1,...,x_{i-1}, x_i',...,x_n) \in S$ and for any $t \in [0, 1]$.

Definition 1.4. The set $W \subseteq X^n$ is n-absorbing if for every $(x_1,...,x_n) \in X^n$ there exists $A_{x_1,..,x_n} \in M_n(\mathbf{\Phi})$ with $\det A_{x_1,..,x_n} > 0$ such that

$$(x_1, ..., x_n) \in A_{x_1,...,x_n} W^T, \text{ i.e. } (A_{x_1,...,x_n})^{-1} (x_1, ..., x_n)^T \in W.$$

Definition 1.5. The set $P \subseteq X^n$ is n-invariant if $AP^T \subseteq P$, for every $A \in M_n(\mathbf{\Phi})$ with $\det A = 1$, where $P^T = \{(x_1, ..., x_n)^T |\, (x_1, ..., x_n) \in P\}$.

Definition 1.6. The set $\Delta_n \subseteq X^n$ consists of all elements $(x_1,...,x_n) \in X^n$ for which the set $\{x_1,..., x_n\}$ is linearly dependent.

The following lemma is important for the characterization of n-seminorms.

Lemma 1.7. *If $W \subseteq X^n$ is n-absorbing and n-invariant set, then $\Delta_n \subseteq W$.*

The proof of Lemma 1 is given in [3].

Theorem 1.8. *Let $p : X^n \to \mathbb{R}$ be n-seminorm. The set $F = \{(x_1, ..., x_n)\ |p\,(x_1, ..., x_n\,) < 1\}$ is n-convex, n-absorbing and n-balanced set and $\Delta_n \subseteq F$.*

The proof of Theorem 1 is given in [3].

2. Main results concerning the characterization of n-seminorms as n-functionals of Minkowski

Let the set $U \subseteq X^n$ be n-absorbing set. For $(x_1, x_2, ..., x_n) \in X^n$, we define the sets $H_U(x_1, x_2, .., x_n) \subseteq M_n(\mathbf{\Phi})$ and $M_U(x_1, x_2, ..., x_n) \subseteq \mathbb{R}$, to be the following sets:

$$H_U(x_1, x_2, ..., x_n) = \left\{A \,|\, \det A > 0, (x_1, x_2, ..., x_n) \in AU^T\right\} \text{ and}$$
$$M_U(x_1, x_2, ..., x_n) = \{\det A \,|\, A \in H_U(x_1, x_2, ..., x_n)\}.$$

For the n-absorbing set $U \subseteq X^n$, we define $\mu_U : X^n \to \mathbb{R}$ to be the function $\mu_U(x_1, x_2, ..., x_n) = \inf\{t \,|\, t \in M_U(x_1, x_2, ..., x_n)\}$.

We call this function **n-functional of Minkowski.**

Lemma 2.1. *Let the set $U \subseteq X^n$ be n-absorbing, n-invariant and n-convex set and $(x_1, x_2, ..., x_n) \in X^n$. Then the set $M_U(x_1, x_2, ..., x_n)$ is a semiline with end point $\mu_U(x_1, x_2, ..., x_n)$.*

Proof. Let $U \subseteq X^n$ be n-absorbing, n-invariant and n-convex set and $(x_1, x_2,..., x_n) \in X^n$. Let $H_U(x_1, x_2,..., x_n) \subseteq M_n(\mathbf{\Phi})$ and $M_U(x_1, x_2,..., x_n) \subseteq \mathbb{R}$ be the sets defined as above. We will prove that the set $M_U(x_1, x_2,..., x_n)$ is a semiline with end point $\mu_U(x_1, ..., x_n)$. If $t \in M_U(x_1, x_2, ..., x_n)$ is an arbitrary element, then there exists $A \in M_n(\mathbf{\Phi}),\ A \in H_U(x_1, x_2, ..., x_n)$ with $\det A = t$ and $A^{-1}(x_1, x_2, ..., x_n)^T \in U$. Since $\det A = t$ and $\det A^{-1} = \dfrac{1}{\det A} = \dfrac{1}{t}$, and because U is n-invariant, we get that $\left(\dfrac{1}{t}x_1, x_2, ..., x_n\right) \in U$. Using the fact that the set U is n-absorbing and n-invariant, according to Lemma 1.7, we get that $(0, x_2, ..., x_n) \in U$. Because the set U is n-convex, for arbitrary $s > t,\ 0 < \dfrac{t}{s} < 1$, we have that

$$\frac{t}{s}\left(\frac{1}{t}x_1, x_2, ..., x_n\right) + \left(1 - \frac{t}{s}\right)(0, x_2, ..., x_n) \in U$$

i.e.

$$\left(\frac{1}{s}x_1, x_2, ..., x_n\right) = \left(\frac{t}{s}\frac{1}{t}x_1 + \left(1 - \frac{t}{s}\right)0, x_2, ..., x_n\right) \in U.$$

Using this and the fact that U is n-invariant set, we get that, for $B \in M_n(\mathbb{R})$ with $\det B = s,\ B^{-1}(x_1, x_2, ..., x_n)^T \in U$.

Since $s>t$ is arbitrary, we get that $M_U(x_1,x_2,...,x_n)$ is a semiline with end point $\mu_U(x_1,...,x_n)$. ■

Lemma 2.2. *Let $U\subseteq X^n$ be n-convex, n-absorbing and n-invariant set and let $(x_1,x_2,...,x_n)\in X^n$. Then $\mu_U(x_1,x_2,...,x_n)$ is subadditive with respect to every variable, i.e.*

$$\mu_U(x_1,x_2,...,x_i+x_i',...,x_n)\leq\mu_U(x_1,x_2,...,x_i,...,x_n)+\mu_U(x_1,x_2,...,x_i',...,x_n),$$

where $x_i',x_1,x_2,...,x_n\in X$ are arbitrary elements.

Proof. Let $U\subseteq X^n$ be n-convex, n-absorbing and n-invariant set and let $(x_1,x_2,...,x_n)\in X^n$. It is enough to prove that the function $\mu_U(x_1,x_2,...,x_n)$ is subadditive with respect to the first variable. For arbitrary chosen elements $x_1,x_1',x_2,...,x_n$, let $s>\mu_U(x_1,x_2,...,x_n)$ and $t>\mu_U(x_1',x_2,...,x_n)$ be arbitrary chosen. Using the definitions and the properties of $M_U(x_1,x_2,...,x_n)$ and $M_U(x_1',x_2,...,x_n)$, we get that there exist $A\in M_U(x_1,x_2,...,x_n)$ with $\det A=s$ and $B\in M_U(x_1',x_2,...,x_n)$ with $\det B=t$ such that

$$A^{-1}(x_1,x_2,...,x_n)^T\in U,\ B^{-1}(x_1',x_2,...,x_n)^T\in U.$$

Since U is n-invariant, we have that $(\frac{1}{t}x_1,x_2,...,x_n)\in U$ and $(\frac{1}{s}x_1',x_2,...,x_n)\in U$.

Because the set U is n-convex, for $u=t+s$ and $\alpha=\frac{t}{u}$, $\beta=\frac{s}{u}$ $(\alpha+\beta=1)$, we get that

$$\left(\frac{1}{u}\left(x_1+x_1'\right),x_2,...,x_n\right)=\left(\frac{1}{u}x_1+\frac{1}{u}x_1',x_2,...,x_n\right)=$$
$$=\left(\frac{t}{u}\left(\frac{1}{t}x_1\right)+\frac{s}{u}\left(\frac{1}{s}x_1'\right),x_2,...,x_n\right)=\left(\alpha\left(\frac{1}{t}x_1\right)+\beta\left(\frac{1}{s}x_1'\right),x_2,...,x_n\right)\in U$$

Therefore, for every matrix $C\in M_n(\mathbf{\Phi})$ with $\det C=u$, it holds

$$C^{-1}(x_1+x_1',x_2,...,x_n)^T\in U,$$

i.e. $u\in M_U(x_1+x_1',x_2,...,x_n)$. According to the definition of the functional μ_U, we have

$$\mu_U(x_1+x_1',x_2,...,x_n)\leq u=s+t,$$

where s and t are arbitrary elements.

So, we proved that $\mu_U(x_1+x_1^{'},x_2,...,x_n)\le\mu_U(x_1,x_2,...,x_n)+\mu_U(x_1^{'},x_2,...,x_n)$, i.e. that the funcional μ_U is subadditive. ■

Lemma 2.3. *Let $U\subseteq X^n$ be n-convex, n-absorbing and n-invariant set and let $(x_1,x_2,...,x_n)\in X^n$. Then $\mu_U(x_1,x_2,...,x_n)$ is homogenous for every positive scalar, with respect to every variable, i.e. $\mu_U(x_1,x_2,...,tx_i,...,x_n) = t\mu_U(x_1,x_2,...,x_i,...,x_n)$, for arbitrary $t>0$ and for every $(x_1,x_2,...,x_n)\in X^n$.*

Proof. Let $U\subseteq X^n$ be n-convex, n-absorbing and n-invariant set and let $(x_1,x_2,...,x_n)\in X^n$ and $t>0$ be arbitrary chosen. For arbitrary $s\in M_U(tx_1,x_2,...,x_n)$ there exist a matrix A with $\det A=s>0$ and $(tx_1,x_2,...,x_n)\in AU^T$, i.e. $A^{-1}(tx_1,x_2,...,x_n)^T\in U$. Using that U is n-invariant set, we get that $\left(\frac{t}{s}x_1,x_2,...,x_n\right)\in U$. Since $t,s>0$, there exists B with $\det B=(\frac{t}{s})^{-1}$ such that $(x_1,x_2,...,x_n)\in BU^T$. Therefore $\frac{s}{t}\in M_U(x_1,x_2,...,x_n)$, and, according to the definition of the functional μ_U, we have that $\mu_U(x_1,x_2,...,x_n)\le\frac{s}{t}$ i.e. $t\mu_U(x_1,x_2,...,x_n)\le s$. Since s is arbitrary, we get that

$$t\mu_U(x_1,x_2,...,x_n)\le\mu_U(tx_1,x_2,...,x_n)\,. \tag{1}$$

Let $s\in M_U(x_1,x_2,...,x_n)$ be arbitrary chosen. Then there exist $A\in H_U(x_1,x_2,...,x_n)$ with $\det A=s$ and $(x_1,x_2,...,x_n)\in AU^T$ i.e. $A^{-1}(x_1,x_2,...,x_n)^T\in U$. Using that U is n-invariant set and $\det A^{-1}=\frac{1}{s}$, we get that $\left(\frac{1}{s}x_1,x_2,...,x_n\right)\in U$, i.e. $\left(\frac{1}{ts}(tx_1),x_2,...,x_n\right)\in U$. Since U is n-invariant set, for each matrix C with $\det C=st>0$, we have that $(tx_1,x_2,...,x_n)\in CU^T$ i.e. $C^{-1}(tx_1,x_2,...,x_n)^T\in U$. Therefore $st\in M_U(tx_1,x_2,...,x_n)$, and, according to the definition of the functional μ_U, we have that

$$\mu_U(tx_1,x_2,...,x_n)\le ts.$$

Since $s\in M_U(x_1,x_2,...,x_n)$ is arbitrary, we get that

$$\mu_U(tx_1,x_2,...,x_n)\le t\mu_U(x_1,x_2,...,x_n) \tag{2}$$

From (1) and (2), we get $\mu_U(tx_1,x_2,...,x_n)=t\mu_U(x_1,x_2,...,x_n)$, for arbitrary $t>0$ and for every $(x_1,x_2,...,x_n)\in X^n$.

In the case $t=0$, we have $(0x_1,x_2,...,x_n)=(\mathrm{o},x_2,...,x_n)\in\Delta_n\subseteq U$. So, for every $A\in M_n(\mathbf{\Phi})$ with $\det A>0$, $A^{-1}(\mathrm{o},x_2,...,x_n)^T\in\Delta_n\subseteq U$. Therefore $M_U(\mathrm{o},x_2,...,x_n)=(0,+\infty)$, i.e. $\mu_U(0x_1,x_2,...,x_n)=0=0\mu_U(\mathrm{o},x_2,...,x_n)$. ■

Lemma 2.4. *Let $U \subseteq X^n$ be n-convex, n-absorbing and n-balanced set and let $(x_1, x_2, ..., x_n) \in X^n$. Then $\mu_U(x_1, x_2, ..., x_n)$ is homogenous for every scalar, with respect to every variable, i.e. $\mu_U(x_1, x_2,..., tx_i,..., x_n) = t\mu_U(x_1,x_2,...,x_i,...,x_n)$, for arbitrary scalar t and for every $(x_1, x_2,..., x_n) \in X^n$.*

P r o o f. Let $U \subseteq X^n$ be n-convex, n-absorbing and n-balanced set and let $\alpha \in \mathbf{\Phi}$ with $|\alpha| = 1$ and $(x_1, x_2, ..., x_n) \in X^n$ be arbitrary chosen. Let

$$M_U(\alpha x_1, x_2, ..., x_n) = \{\det A | A \in H_U(\alpha x_1, x_2, ..., x_n)\}$$

and

$$M_U(x_1, x_2, ..., x_n) = \{\det A | A \in H_U(x_1, x_2, ..., x_n)\}.$$

For arbitrary $s \in M_U(x_1, x_2, ..., x_n)$ there exists $A \in M_n(\mathbf{\Phi})$ with $\det A = s$ such that $A^{-1}(x_1, x_2, ..., x_n)^T \in U$. For the matrixes

$$B = \begin{bmatrix} \overline{\alpha} & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix} \text{ and } \overline{B} = \begin{bmatrix} \alpha & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

we have that $B\overline{B} = E$ and $A^{-1} = A^{-1}E = A^{-1}B\overline{B}$. Since $A^{-1}(x_1, x_2, ..., x_n)^T \in U$, using the last equation, we get $\left(A^{-1}B\overline{B}\right)(x_1, x_2, ..., x_n)^T \in U$. Therefore $(A^{-1}B)\left(\overline{B}(x_1, x_2, ..., x_n)^T\right)^T \in U$, i.e. $(A^{-1}B)(\alpha x_1, x_2, ..., x_n)^T \in U$.

Since U is n-balanced set and $|\det B| = |\alpha| = 1$, we get that $\overline{B}\left((A^{-1}B)(\alpha x_1, x_2, ..., x_n)^T\right)^T \in U$, i.e. $\left(\overline{B}(A^{-1}B)\right)(\alpha x_1, x_2, ..., x_n)^T \in U$.

Because of $\det\left(\overline{B}A^{-1}B\right) = \det\overline{B}\det A^{-1}\det B = \alpha\det A^{-1}\overline{\alpha} = \det A^{-1} = \frac{1}{s} > 0$ and $(\alpha x_1, x_2, ..., x_n) \in \left(\overline{B}A^{-1}B\right)^{-1}U^T$, we get that $B^{-1}A\overline{B}^{-1} = \overline{B}AB \in H_U(\alpha x_1, x_2, ..., x_n)$ i.e. $s \in M_U(\alpha x_1, x_2, ..., x_n)$. Since s is an arbitrary element which belongs to $M_U(x_1, x_2, ..., x_n)$, we conclude that

$$M_U(x_1, x_2, ..., x_n) \subseteq M_U(\alpha x_1, x_2, ..., x_n). \tag{3}$$

Let us now suppose $t \in M_U(\alpha x_1, x_2,..., x_n)$ is arbitrary chosen. Because of the definition of $M_U(\alpha x_1, x_2,..., x_n)$, there exists $D \in M_n(\mathbf{\Phi})$ such that $\det D = t$ and $D^{-1}(\alpha x_1, x_2,..., x_n)^T \in U$. Therefore $D^{-1}\left(\overline{B}(x_1, x_2,..., x_n)^T\right)^T \in U$ i.e. $(D^{-1}\overline{B})(x_1, x_2,..., x_n)^T \in U$.

Since U is n-balanced set and $|\det B| = |\overline{\alpha}| = 1$, we have that $BU^T \subseteq U$. Therefore $(BD^{-1}\overline{B})\,(x_1, x_2..., x_n)^T \in U$. Using that $\det BD^{-1}\overline{B} = \overline{\alpha}\det D^{-1}\alpha = \frac{1}{t} > 0$, we get that $\left(\overline{B}D^{-1}B\right)^{-1} \in H_U\,(x_1, x_2, ..., x_n)$, i.e. $t \in M_U\,(x_1, x_2, ..., x_n)$. Since t is an arbitrary element which belongs to $M_U\,(\alpha x_1, x_2, ..., x_n)$, we conclude that

(4) $$M_U\,(\alpha x_1, x_2, ..., x_n) \subseteq M_U\,(x_1, x_2, ..., x_n)$$

From (3) and (4), we get that

(5) $$M_U\,(\alpha x_1, x_2, ..., x_n) = M_U\,(x_1, x_2, ..., x_n)$$

for any scalar α with $|\alpha| = 1$.

Using (5) and the definition of the n-functional of Minkowski, μ_U, we get

(6) $$\mu_U\,(\alpha x_1, x_2, ..., x_n) = \mu_U\,(x_1, x_2, ..., x_n)\,,$$

for any scalar α with $|\alpha| = 1$.

Now, let α be an arbitrary scalar and $(x_1, x_2, .., x_n) \in X^n$ be an arbitrary chosen element. Then, using that $\left|\frac{\alpha}{|\alpha|}\right| = 1$, $|\alpha| > 0$ and (6), we get

$$\mu_U\,(\alpha x_1, x_2, ..., x_n) = \mu_U\left(|\,\alpha\,|\,\frac{\alpha}{|\alpha|}x_1, x_2, ..., x_n\right) =$$
$$= |\,\alpha\,|\,\mu_U\left(\frac{\alpha}{|\,\alpha\,|}x_1, x_2, ..., x_n\right) = |\alpha|\,\mu_U\,(x_1, x_2, ..., x_n)$$

Therefore $\mu_U\,(\alpha x_1, x_2, ..., x_n) = |\alpha|\,\mu_U\,(x_1, x_2, ..., x_n)$, for any scalar $\alpha \in \mathbf{\Phi}$ and for an arbitrary $(x_1, x_2, ..., x_n) \in X^n$. ■

Lemma 2.5. *Let $U \subseteq X^n$ be n-convex, n-absorbing and n-balanced set and let $(x_1, x_2, ..., x_n) \in X^n$. Then $\mu_U(x_1, x_2, ..., x_n)$ does not change the value under any permutation of its variables, i.e. $\mu_U\,(x_1, ..., x_n) = \mu_U\left(x_{\pi(1)}, ..., x_{\pi(n)}\right)$, for any permutation $\pi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$.*

Proof. Let $U \subseteq X^n$ be n-convex, n-absorbing and n-balanced set. Let $(x_1, x_2, ..., x_n) \in X^n$ be arbitrary chosen and let $\pi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be an arbitrary permutation. Each permutation matrix has determinant equals to 1 or -1. Let $\pi\,(k) = i_k$ and let us denote the permutation matrix with $B_{i_1 i_2 \dots i_n}$ and $C = (B_{i_1 i_2 \dots i_n})^{-1}$. Then $\det C = \frac{1}{\det B_{i_1 i_2 \dots i_n}} = \frac{1}{\pm 1}$, i.e. $|\det C\,| = 1$.

Let $t \in M_U\left(x_{\pi(1)}, ..., x_{\pi(n)}\right)$ be arbitrary chosen. For $A \in H_U\left(x_{\pi(1)}, ..., x_{\pi(n)}\right)$ with $\det A = t$, $A^{-1}\left(x_{\pi(1)}, ..., x_{\pi(n)}\right)^T \in U$. Using that U is n-balanced set, for the matrix C, we have $C\left(A^{-1}\left(x_{\pi(1)},...,x_{\pi(n)}\right)^T\right)^T \in U$ i.e. $(CA^{-1}B_{i_1 i_2 \dots i_n})$ $(x_1,...,x_n)^T \in U$. Therefore $\left((B_{i_1 i_2 \dots i_n})^{-1} AC^{-1}\right)^{-1}(x_1,...,x_n)^T \in U$. Since $\det\left((B_{i_1 i_2 \dots i_n})^{-1} AC^{-1}\right) = \det(B_{i_1 i_2 \dots i_n})^{-1}\det A \det C^{-1} = t$, we get that $t \in M_U(x_1,...,x_n)$.

We proved that $M_U\left(x_{\pi(1)},...,x_{\pi(n)}\right) \subseteq M_U(x_1,...,x_n)$.

Let $t \in M_U(x_1,...,x_n)$ be arbitrary chosen. For $B \in H_U(x_1,...,x_n)$ with $\det B = t$, $B^{-1}(x_1,...,x_n)^T \in U$. Then $(B^{-1}C)\left(x_{\pi(1)},....,x_{\pi(n)}\right)^T \in U$. Since U is n-balanced set and $|B_{i_1 i_2 \dots i_n}| = 1$, we have

$$(B_{i_1 i_2 \dots i_n}B^{-1}C)\left(x_{\pi(1)},...,x_{\pi(n)}\right)^T \in U$$

i.e.

$$\left(C^{-1}B\left(B_{i_1 i_2 \dots i_n}\right)^{-1}\right)^{-1}\left(x_{\pi(1)},...,x_{\pi(n)}\right)^T \in U.$$

Because of $\det\left(C^{-1}B\left(B_{i_1 i_2 \dots i_n}\right)^{-1}\right) = \det C^{-1}\det B\det\left(B_{i_1 i_2 \dots i_n}\right)^{-1} = t$ we get that $t \in M_U\left(x_{\pi(1)},...,x_{\pi(n)}\right)$. Therefore $M_U(x_1,...,x_n) \subseteq M_U\left(x_{\pi(1)},...,x_{\pi(n)}\right)$.

So, we proved that $M_U(x_1,...,x_n) = M_U\left(x_{\pi(1)},...,x_{\pi(n)}\right)$.

Using the last equation and the definition of the n-functional of Minkowski, μ_U, we get $\mu_U(x_1,...,x_n) = \mu_U\left(x_{\pi(1)},...,x_{\pi(n)}\right)$. ■

Lemma 2.6. *Let $U \subseteq X^n$ be n-convex, n-absorbing and n-invariant set and let the set $\{x_1, x_2, ..., x_n\}$ be linearly dependant. Then $\mu_U(x_1, x_2, ..., x_n) = 0$.*

Proof. Let $U \subseteq X^n$ be n-convex, n-absorbing and n-invariant set and let the set $\{x_1, x_2,..., x_n\}$ be linearly dependant i.e. $(x_1, x_2,..., x_n) \in \triangle_n$. For any matrix $A = [a_{ij}]_{n\times n}$ with $\det A > 0$, the set of vectors $\left\{\sum\limits_{j=1}^{n} a_{1j}x_j, \sum\limits_{j=1}^{n} a_{2j}x_j,..., \sum\limits_{j=1}^{n} a_{nj}x_j\right\}$ is linearly dependent. Therefore, for any matrix $A = [a_{ij}]_{n\times n}$ with $\det A > 0$, $A^{-1}(x_1, x_2, ..., x_n)^T \in \Delta_n \subseteq U$. So, $M_U(x_1, x_2, ..., x_n) = (0, +\infty)$ i.e. $\mu_U(x_1, x_2, ..., x_n) = 0$. ■

The following theorem 2.7 is a direct consequence of the previous Lemmas and the fact that every n-balanced set is n-invariant set.

Theorem 2.7. *Let $U \subseteq X^n$ be n-convex, n-absorbing and n-balanced set. Then the n-functional of Minkowski, μ_U, is a seminorm on X^n.*

Theorem 2.8. *Let $U \subseteq X^n$ be n-convex, n-absorbing and n-balanced set and let $\mu_U : X^n \to \mathbf{R}$ be the n-funcional of Minkowski. If*
$V = \{ (x_1, x_2, ..., x_n) \mid \mu_U (x_1, x_2, ..., x_n) < 1 \}$ *and*
$W = \{ (x_1, x_2, ..., x_n) \mid \mu_U (x_1, x_2, ..., x_n) \leq 1 \}$
then $V \subset U \subset W$ and $\mu_U = \mu_V = \mu_W$.

Proof. It is obvios that $V \subseteq W$. If $(x_1, x_2, ..., x_n) \in V$ then $\mu_U (x_1, x_2, ..., x_n) < 1$. Because of the definition of μ_U, we have that $1 \in M_U (x_1, x_2, ..., x_n)$ i.e. $E_n \in H_U (x_1, x_2, ..., x_n)$, where E_n is the unit matrix of order n. Therefore $(x_1, x_2, ..., x_n) \in U$. So, we proved that $V \subseteq U$.

If $(x_1, ..., x_n) \in U$ then $E \in H_U (x_1, x_2, ..., x_n)$, so $\mu_U (x_1, x_2, ..., x_n) \leq 1$ i.e. $(x_1, x_2, ..., x_n) \in W$. So, we proved that $U \subseteq W$.

Therefore $V \subset U \subset W$. Because of this, we have

$$H_V (x_1, x_2, ..., x_n) \subseteq H_U (x_1, x_2, ..., x_n) \subseteq H_W (x_1, x_2, ..., x_n)$$

i.e.

$$M_V (x_1, x_2, ..., x_n) \subseteq M_U (x_1, x_2, ..., x_n) \subseteq M_W (x_1, x_2, ..., x_n)$$

Using the definition of μ_U, we get that $\mu_W \leq \mu_U \leq \mu_V$.

If s and t are real numbers such that $\mu_W (x_1, x_2, ..., x_n) < s < t$, then, according to the definition of μ_W, we get that

$$\mu_W \left(\frac{1}{s} x_1, x_2, ..., x_n\right) = \frac{1}{s} \mu_W (x_1, x_2, ..., x_n) < 1 \text{ i.e. } \left(\frac{1}{s} x_1, x_2, ..., x_n\right) \in W.$$

Therefore $\mu_U \left(\frac{1}{s} x_1, x_2, ..., x_n\right) \leq 1$ i.e. $\mu_U (x_1, x_2, ..., x_n) \leq s$, so

$$\mu_U \left(\frac{1}{t} x_1, x_2, ..., x_n\right) \leq \frac{s}{t} < 1 \text{ i.e. } \left(\frac{1}{t} x_1, x_2, ..., x_n\right) \in V.$$

Using the definition of μ_V, we get $\mu_V (x_1, x_2, ..., x_n) \leq t$. Since $t > \mu_W (x_1, x_2, ..., x_n)$ is an arbitrary element, we get that
$\mu_W (x_1, x_2, ..., x_n) \geq \mu_V (x_1, x_2, ..., x_n)$. Therefore $\mu_V \leq \mu_W$.
We proved that $\mu_W = \mu_U = \mu_V$. ■

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