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A Weighted Prime Geodesic Theorem [1]

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We consider a weighted form of the prime geodesic theorem for a compact Riemann surface as a ψ_1 – level analogue of the classical von Koch theorem. A result in that direction is obtained for higher dimensional real hyperbolic manifolds with cusps.

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1 Introduction

In 1901, von Koch proved that the Riemann hypothesis is equivalent to the best possible bound in the prime number theorem. The precise version of von Koch's result says that the Riemann hypothesis is equivalent to the following asymptotic equation

$$\pi(x) = \mathrm{li}\,(x) + E(x) \tag{1}$$

as $x \to +\infty$, where $E(x) = O\left(x^{\frac{1}{2}} \log x\right)$ and $\pi(x)$ denotes the number of prime numbers not larger than x. The best estimate of the error term known up to now is $E(x) = O\left(x \exp\left(-c\,(\log x)^{\frac{3}{5}}\right)(\log\log x)^{-\frac{1}{5}}\right)$, obtained by Vinogradov in 1958.

The set of lengths of prime geodesics over a compact locally symmetric space is related to the Selberg zeta function in a manner which is evocative to the relationship between prime numbers and the Riemann zeta function. Let R be a d-dimensional compact Riemannian manifold whose simply connected Riemannian covering manifold $\mathcal{H}$ is a symmetric space of noncompact type and of rank 1. Then, $\mathcal{H}$ can be represented as G/K where G is a noncompact

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connected simple Lie group of rank 1 with finite center and K is a maximal compact subgroup of G. As a consequence, R can be represented as $\Gamma\backslash G/K$, where Γ is a discrete subgroup of G acting freely on $\mathcal{H}$ such that $\Gamma\backslash G$ is compact.

Let us recall that a prime geodesic C_γ over R corresponds to the conjugacy class of a primitive hyperbolic element $\gamma \in \Gamma$. As usual, let $\pi_\Gamma(x)$ be the number of prime geodesics C_γ of length $l(C_\gamma) \le \log x$. Then the prime geodesic theorem states

$$\pi_\Gamma(x) \sim \frac{x^{d-1}}{d-1} (\log x)^{-1} . \tag{2}$$

Relation (2) was independently proved by Gangolli [6] and DeGeorge [4] (see also Wakayama [17]) when R is compact and extended by Gangolli and Warner [7] to the noncompact finite volume case. The search for the optimal error bound in the prime geodesic theorem is widely open. However, one should hardly expect to get a more informative general result without entering into details of compact locally symmetric spaces case by case.

2 Classification

Let R be as above. Being symmetric and of rank 1, the covering manifold $\mathcal{H}$ is known to be a real, a complex or a quaternionic hyperbolic space or the hyperbolic Cayley plane (see e.g., [3])

2.1 Real hyperbolic space $H\mathbb{R}^d$

We may represent real hyperbolic space $\mathcal{H} = H\mathbb{R}^d$ as $\mathcal{H} = SO_0(d,1)/SO(d)$, $d \ge 3$. Here, $\dim(\mathcal{H}) = d$ according to [10]. Also, due to [5], the critical strip of the Selberg zeta function associated to R in this case is given by $0 \le \mathrm{Re}(s) \le d-1$.

2.2 Complex hyperbolic space $H\mathbb{C}^m$

We may write $\mathcal{H} = G/K$, where $G = SU(1,m)/\mathbb{Z}_k$, $K = S(U(1) \times U(m))/\mathbb{Z}_k$, $m \ge 2$ for any divisor k of $m+1$. In this case, $\dim(\mathcal{H}) = 2m$ and the critical strip is given by $0 \le \mathrm{Re}(s) \le 2m$.

2.3 Quaternionic hyperbolic space $H\mathbb{H}^m$

A possible representation is $\mathcal{H} = Sp(1,m)/Sp(1) \times Sp(m)$, $m \ge 2$. Now, $\dim(\mathcal{H}) = 4m$ and the critical strip is given by $0 \le \mathrm{Re}(s) \le 4m+2$.

2.4 Hyperbolic Cayley plane $H\mathbb{C}a^2$

The unique representation of the hyperbolic Cayley plane $\mathcal{H} = H\mathbb{C}a^2$ is $\mathcal{H} = F_4^{-20}/Spin(9)$. Here, $\dim(\mathcal{H}) = 16$ and the critical strip in this exceptional case is given by $0 \leq \mathrm{Re}(s) \leq 22$.

One could expect that obtaining a more precise version of the prime geodesic theorem would depend on $\dim(R)$ and an appropriate choice of a zeta function. As explained by Gangolli and Warner in [7, Section 5], the Selberg zeta function is not sufficient for this purpose in the case of dimension d higher than 3, since it provides us information only on the error terms related to the poles in the strip $(d-2, d-1]$.

3 Compact Riemann surfaces case

Let $R = \Gamma\backslash\mathcal{H}$ be a compact Riemann surface of genus $g \geq 2$, where Γ is a strictly hyperbolic discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathcal{H}$ is the upper half-plane. Denote by Γ_{h} resp. $\mathrm{P}\Gamma_{\mathrm{h}}$ the set of the $\Gamma-$conjugacy classes of hyperbolic resp. primitive hyperbolic elements in Γ. We set $\Lambda(\gamma) = l(C_{\gamma_0})$ for $\gamma \in \Gamma_{\mathrm{h}}$, where $\gamma = \gamma_0^{j(\gamma)}$, $\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}$, $j(\gamma) \in \mathbb{N}$. Let $N(\gamma_0) = e^{l(C_{\gamma_0})}$. As usual, we introduce $\psi_0(x) = \sum_{\gamma\in\Gamma_{\mathrm{h}}, N(\gamma)\leq x} \Lambda(\gamma)$ and define $\psi_n(x)$ recursively by $\psi_n(x) = \int\limits_0^x \psi_{n-1}(t)dt$ for $n \in \mathbb{N}$. One has (see, e.g., [14, p. 98], [15, p. 245])

$$\psi_n(x) = \frac{1}{n!} \sum_{\gamma\in\Gamma_{\mathrm{h}}, N(\gamma)\leq x} \Lambda(\gamma)\,(x - N(\gamma))^n . \tag{3}$$

The Selberg zeta function for the group Γ is defined by

$$Z_\Gamma(s) = \prod_{\gamma_0\in\mathrm{P}\Gamma_{\mathrm{h}}} \prod_{k=0}^{+\infty} \left(1 - N(\gamma_0)^{-s-k}\right)^{-1}, \quad \mathrm{Re}(s) > 1,$$

and meromorphically continued to the whole complex plane. Nontrivial zeros of Z_Γ are located at $s_n = \frac{1}{2} + ir_n$ and $\tilde{s}_n = \frac{1}{2} - ir_n$ where $r_n = \sqrt{\lambda_n - \frac{1}{4}}$ for $\lambda_n \geq \frac{1}{4}$ and $r_n = -i\sqrt{-\lambda_n + \frac{1}{4}}$ for $\lambda_n < \frac{1}{4}$. Here, λ_n ranges through the sequence of eigenvalues of the Laplace-Beltrami operator on $\Gamma\backslash\mathcal{H}$.

Randol [15] proved the prime geodesic theorem with the error term (see also [11], [12], [9], [2])

$$\pi_\Gamma(x) = \sum_{\frac{3}{4}<s_n\leq 1} \mathrm{li}\,(x^{s_n}) + O\left(x^{\frac{3}{4}}(\log x)^{-1}\right), \tag{4}$$

where $\lambda_n = s_n(1-s_n)$ is a small eigenvalue in $\left[0, \frac{3}{16}\right]$ of the Laplacian Δ_0 acting on $L^2(R)$.

In view of the remarkable fact that the analogue of the Riemann hypothesis holds for the Selberg zeta in this case, the error term in (4) is far from the expected $O\left(x^{\frac{1}{2}+\varepsilon}\right)$. The biggest obstacle is the higher density of nontrivial zeros of the Selberg zeta function when compared to the Riemann zeta case. This gives us a motivation to take a look at a weighted prime geodesic theorem in the following form.

Theorem 1 *For every compact Riemann surface of genus $g \geq 2$, the asymptotic equation*

$$\sum_{\gamma\in\Gamma_h, N(\gamma)\leq x} \Lambda(\gamma)\left(1-\frac{N(\gamma)}{x}\right) = \sum_{n=0}^{M} \frac{x^{s_n}}{s_n(s_n+1)} + O\left(x^{\frac{1}{2}}\log x\right) \tag{5}$$

holds true as $x \to +\infty$. The implied constant depends solely on Γ. Here, M is determined by the condition $\lambda_n \in \left[0, \frac{1}{4}\right)$ if and only if $0 \leq n \leq M$.

Proof. It is enough to prove the following relation

$$\frac{\psi_1(x)}{x} = \sum_{n=0}^{M} \frac{x^{s_n}}{s_n(s_n+1)} + O\left(x^{\frac{1}{2}}\log x\right), \quad x \to +\infty.$$

For $c > 1$ we get (see [8, Th. 40.])

$$\psi_n(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{R'_\Gamma(s)}{R_\Gamma(s)} \frac{x^{s+n}}{s(s+1)\ldots(s+n)}\,ds,$$

where R_Γ is the Ruelle zeta function defined by

$$R_\Gamma(s) = \prod_{\gamma_0\in \mathrm{P}\Gamma_\mathrm{h}} \left(1-N(\gamma_0)^{-s}\right)^{-1}, \quad \mathrm{Re}\,(s) > 1,$$

and meromorphically continued to the whole complex plane. Let $\varepsilon > 0$ be a number such that real zeros s_0, s_1,...,s_M of Z_Γ belong to the segment $\left(\frac{1}{2}+2\varepsilon, 1\right]$.

Applying the Cauchy residue theorem to the rectangle given by vertices $c - iT$, $c + iT$, $\frac{1}{2} + 2\varepsilon + iT$, $\frac{1}{2} + 2\varepsilon - iT$ and letting $T \to +\infty$, we obtain

$$\psi_1(x) = \sum_{n=0}^{M} \frac{x^{s_n+1}}{s_n(s_n+1)} - \frac{1}{2\pi i} \int\limits_{\frac{1}{2}+2\varepsilon-i\infty}^{\frac{1}{2}+2\varepsilon+i\infty} \frac{R'_\Gamma(s)}{R_\Gamma(s)} \frac{x^{s+1}}{s(s+1)} ds. \tag{6}$$

Reasoning as in [2, p. 1839], we easily derive

$$\frac{R'_\Gamma(s)}{R_\Gamma(s)} = O\left(\frac{1}{\varepsilon}\left(\frac{T}{\log T}\right)^{2\max(0,1+\varepsilon-\sigma)}\right) \tag{7}$$

for $s = \sigma + iT$, $\sigma \geq \frac{1}{2} + \varepsilon$, $T \geq 1000$, $T \neq$ all r_n. Hence, (6) and (7) imply

$$\psi_1(x) = \sum_{n=0}^{M} \frac{x^{s_n+1}}{s_n(s_n+1)} - \frac{1}{2\pi i} \int\limits_{\frac{1}{2}+2\varepsilon-iT}^{\frac{1}{2}+2\varepsilon+iT} \frac{R'_\Gamma(s)}{R_\Gamma(s)} \frac{x^{s+1}}{s(s+1)} ds \tag{8}$$

$$+ O\left(\frac{x^{\frac{3}{2}+2\varepsilon}}{\varepsilon^2 \log T}\right).$$

The integral on the right hand side can be estimated in the same way as in [9]. We put $A = N + \frac{1}{2}$ for some $N \in \mathbb{N}$ and apply the Cauchy residue theorem over the rectangle with vertices $\frac{1}{2} + 2\varepsilon - iT$, $\frac{1}{2} + 2\varepsilon + iT$, $-A + iT$, $-A - iT$. Having in mind that (see, e.g., [16])

$$R_\Gamma(s) = \frac{Z_\Gamma(s+1)}{Z_\Gamma(s)},$$

we deduce

$$-\frac{1}{2\pi i} \int\limits_{\frac{1}{2}+2\varepsilon-iT}^{\frac{1}{2}+2\varepsilon+iT} \frac{R'_\Gamma(s)}{R_\Gamma(s)} \frac{x^{s+1}}{s(s+1)} ds = \alpha x + \beta x \log x + \gamma + \delta \log x \tag{9}$$

$$+ \sum_{n=1}^{M} \frac{x^{\widetilde{s}_n+1}}{\widetilde{s}_n(\widetilde{s}_n+1)} + \sum_{0 \leq r_n \leq T} \left(\frac{x^{s_n+1}}{s_n(s_n+1)} + \frac{x^{\widetilde{s}_n+1}}{\widetilde{s}_n(\widetilde{s}_n+1)}\right)$$

$$-\sum_{n=1}^{M}\frac{x^{s_n}}{(s_n-1)\,s_n}-\sum_{n=1}^{M}\frac{x^{\widetilde{s}_n}}{(\widetilde{s}_n-1)\,\widetilde{s}_n}-\sum_{0\le r_n\le T}\left(\frac{x^{s_n}}{(s_n-1)\,s_n}+\frac{x^{\widetilde{s}_n}}{(\widetilde{s}_n-1)\,\widetilde{s}_n}\right)$$

$$+(4g-4)\sum_{k=2}^{+\infty}\frac{1}{k(k-1)}x^{1-k}+O\left(\frac{x^{\frac{3}{2}+2\varepsilon}}{\varepsilon T}\right)+O\left(x^{1-A}\right)$$

where α, β, γ and δ are constants that depend solely on Γ. Passing to the limit $A\to+\infty$ in (9) and taking into account (8), we obtain

$$\psi_1(x)=\alpha x+\beta x\log x+\gamma+\delta\log x \tag{10}$$

$$+\sum_{n=0}^{M}\frac{x^{s_n+1}}{s_n\,(s_n+1)}+\sum_{n=1}^{M}\frac{x^{\widetilde{s}_n+1}}{\widetilde{s}_n\,(\widetilde{s}_n+1)}+\sum_{0\le r_n\le T}\left(\frac{x^{s_n+1}}{s_n\,(s_n+1)}+\frac{x^{\widetilde{s}_n+1}}{\widetilde{s}_n\,(\widetilde{s}_n+1)}\right)$$

$$-\sum_{n=1}^{M}\frac{x^{s_n}}{(s_n-1)\,s_n}-\sum_{n=1}^{M}\frac{x^{\widetilde{s}_n}}{(\widetilde{s}_n-1)\,\widetilde{s}_n}-\sum_{0\le r_n\le T}\left(\frac{x^{s_n}}{(s_n-1)\,s_n}+\frac{x^{\widetilde{s}_n}}{(\widetilde{s}_n-1)\,\widetilde{s}_n}\right)$$

$$+(4g-4)\sum_{k=2}^{+\infty}\frac{1}{k(k-1)}x^{1-k}+O\left(\frac{x^{\frac{3}{2}+2\varepsilon}}{\varepsilon^2\log T}\right).$$

Define $N(t)$ to be the number of $s_n=\frac{1}{2}+ir_n$ (or $\tilde{s}_n=\frac{1}{2}-ir_n$) such that $r_n=\sqrt{\lambda_n-\frac{1}{4}}$, $\lambda_n\ge\frac{1}{4}$ and $r_n\le t$. It is known that $N(t)=O\left(t^2\right)$. Hence,

$$\sum_{0\le r_n\le T}\left(\frac{x^{s_n+1}}{s_n\,(s_n+1)}+\frac{x^{\widetilde{s}_n+1}}{\widetilde{s}_n\,(\widetilde{s}_n+1)}\right)=O\left(x^{\frac{3}{2}}\log T\right),$$

$$\sum_{0\le r_n\le T}\left(\frac{x^{s_n}}{(s_n-1)\,s_n}+\frac{x^{\widetilde{s}_n}}{(\widetilde{s}_n-1)\,\widetilde{s}_n}\right)=O\left(x^{\frac{1}{2}}\log T\right).$$

Now, equation (10) becomes

$$\psi_1(x)=O\left(x\log x\right)+\sum_{n=0}^{M}\frac{x^{s_n+1}}{s_n\,(s_n+1)}+O\left(x^{\frac{3}{2}}\right)+O\left(x^{\frac{3}{2}}\log T\right) \tag{11}$$

$$+O\left(x\right)+O\left(x^{\frac{1}{2}}\right)+O\left(x^{\frac{1}{2}}\log T\right)+O\left(\frac{1}{x}\right)+O\left(\frac{x^{\frac{3}{2}+2\varepsilon}}{\varepsilon^2\log T}\right)$$

$$=\sum_{n=0}^{M}\frac{x^{s_n+1}}{s_n\,(s_n+1)}+O\left(x^{\frac{3}{2}}\log T\right)+O\left(\frac{x^{\frac{3}{2}+2\varepsilon}}{\varepsilon^2\log T}\right).$$

The optimal size of the error term in (11) is given when $T \sim x$ and $\varepsilon \sim \frac{1}{\log x}$. We finally obtain

$$\psi_1(x) = \sum_{n=0}^{M} \frac{x^{s_n+1}}{s_n\,(s_n+1)} + \left(x^{\frac{3}{2}} \log x\right).$$

This completes the proof of theorem. ∎

4 Real hyperbolic manifolds with cusps case

We first recall our refinement [1] of Park's prime geodesic theorem for higher dimensional real hyperbolic manifolds with cusps [14]. The obtained estimate coincides with the best known result (4) in the Riemann surfaces case.

Let Γ be a discrete co-finite torsion free subgroup of $G = \mathrm{SO}_0(d,1)$ satisfying the condition $\Gamma \cap P = \Gamma \cap N(P)$ for $P \in \mathfrak{P}_\Gamma$, where $\mathfrak{P}_\Gamma$ is the set of $\Gamma-$conjugacy classes of $\Gamma-$cuspidal parabolic subgroups in G and $N(P)$ is the unipotent part of P. Denote by K a maximal compact subgroup of G. The manifold $R = \Gamma\backslash G/K$ is a d-dimensional real hyperbolic manifold with cusps. In [1] the authors proved the theorem in the following form

Theorem 2 *Let R be a d-dimensional real hyperbolic manifold with cusps, $d \geq 3$. Then*

$$\pi_\Gamma(x) = \sum_{\frac{3}{2}d_0 < s_n(k) \leq 2d_0} (-1)^k \,\mathrm{li}\left(x^{s_n(k)}\right) + O\left(x^{\frac{3}{2}d_0}\,(\log x)^{-1}\right)$$

as $x \to +\infty$, where $d_0 = \frac{d-1}{2}$, $(s_n(k)-k)\,(2d_0-k-s_n(k))$ is a small eigenvalue in $\left[0, \frac{3}{4}d_0^2\right]$ of Δ_k on $\pi_{\sigma_k,\lambda_n(k)}$ with $s_n(k) = d_0 + i\lambda_n(k)$ or $s_n(k) = d_0 - i\lambda_n(k)$ in $\left(\frac{3}{2}d_0, 2d_0\right]$, Δ_k is the Laplacian acting on the space of $k-$forms over R and $\pi_{\sigma_k,\lambda_n(k)}$ is the principal series representation.

Concerning a weighted prime geodesic theorem in this setting, we prove the next result.

Theorem 3 *Let R be a d-dimensional real hyperbolic manifold with cusps, $d \geq 3$. Then*

$$\sum_{\gamma\in\Gamma_h, N(\gamma)\leq x} \Lambda(\gamma)\left(1 - \frac{N(\gamma)}{x}\right) =$$

$$\sum_{s_n(k)\in(d_0,2d_0]} (-1)^k \frac{x^{s_n(k)}}{s_n(k)\,(s_n(k)+1)} + O\left(x^{\frac{3}{2}d_0-\frac{1}{4}}\right)$$

as $x \to +\infty$.

Proof. By [1, relation (6)]

$$\psi_d(x) = \sum_{s_n(k)\in(d_0,2d_0]} (-1)^k \frac{x^{s_n(k)+d}}{s_n(k)\,(s_n(k)+1)\dots(s_n(k)+d)} + \tag{12}$$

$$\sum_{s_n(0)=d_0\pm i\lambda_n(0)} \frac{x^{s_n(0)+d}}{s_n(0)\,(s_n(0)+1)\dots(s_n(0)+d)}.$$

Reasoning as in [1, pages 4–5], we introduce the function

$$\Delta_n^+ f(x) = \int\limits_{x}^{x+h} \int\limits_{x_{n-1}}^{x_{n-1}+h} \dots \int\limits_{x_1}^{x_1+h} f^{(n)}(x_0)dx_0\dots dx_{n-1}$$

and obtain

$$h^{-(d-1)}\Delta_{d-1}^+ \frac{x^{s_n(k)+d}}{s_n(k)(s_n(k)+1)\dots(s_n(k)+d)} = \tag{13}$$

$$\frac{x^{s_n(k)+1}}{s_n(k)\,(s_n(k)+1)} + O\left(h^{s_n(k)+1}\right),$$

$$h^{-(d-1)}\Delta_{d-1}^+ \frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\dots(s_n(0)+d)} = \tag{14}$$

$$O\left(\min\left(x^{d_0+1}\left|s_n(0)\right|^{-2}, h^{-(d-1)}\left|s_n(0)\right|^{-(d+1)} x^{d+d_0}\right)\right).$$

Following [12, pp. 463-464] and using (14), we deduce

$$h^{-(d-1)}\Delta_{d-1}^+ \sum_{s_n(0)=d_0\pm i\lambda_n(0)} \frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\dots(s_n(0)+d)} \tag{15}$$

$$= O\left(x^{d_0+1}\int\limits_{d_0}^{M} t^{-2}dN(t)\right) + O\left(h^{-(d-1)}x^{d+d_0}\int\limits_{M}^{+\infty} t^{-(d+1)}dN(t)\right)$$

$$= O\left(x^{d_0+1}M^{d-2}\right) + O\left(h^{-(d-1)}x^{d+d_0}M^{-1}\right)$$

for some $M > 2d_0$, where $N(t) = O\left(t^d\right)$ denotes the counting function of $s_n(0) = d_0 + i\lambda_n(0)$. Thus, (12), (13) and (15) imply

$$h^{-(d-1)}\Delta^{+}_{d-1}\psi_d(x) = \sum_{s_n(k)\in(d_0,2d_0]} \frac{(-1)^k x^{s_n(k)+1}}{s_n(k)\left(s_n(k)+1\right)} + O\left(h^{2d_0+1}\right) \tag{16}$$

$$+O\left(x^{d_0+1}M^{d-2}\right) + O\left(h^{-(d-1)}x^{d+d_0}M^{-1}\right).$$

Substituting $M \sim x^{\frac{1}{4}}$, $h \sim x^{\frac{3}{4}}$ into (16), we get

$$\psi_1(x) \leq h^{-(d-1)}\Delta^{+}_{d-1}\psi_d(x) = \tag{17}$$

$$\sum_{s_n(k)\in(d_0,2d_0]} (-1)^k \frac{x^{s_n(k)+1}}{s_n(k)\left(s_n(k)+1\right)} + O\left(x^{\frac{3}{2}d_0+\frac{3}{4}}\right).$$

Reasoning in an analogous way, one proves

$$\sum_{s_n(k)\in(d_0,2d_0]} (-1)^k \frac{x^{s_n(k)+1}}{s_n(k)\left(s_n(k)+1\right)} + O\left(x^{\frac{3}{2}d_0+\frac{3}{4}}\right) \leq \psi_1(x). \tag{18}$$

Combining (17) and (18), we obtain

$$\psi_1(x) = \sum_{s_n(k)\in(d_0,2d_0]} (-1)^k \frac{x^{s_n(k)+1}}{s_n(k)\left(s_n(k)+1\right)} + O\left(x^{\frac{3}{2}d_0+\frac{3}{4}}\right). \tag{19}$$

Now, the theorem follows from (19) and the fact that

$$\sum_{\gamma\in\Gamma_h, N(\gamma)\leq x} \Lambda(\gamma)\left(1 - \frac{N(\gamma)}{x}\right) = \frac{\psi_1(x)}{x}.$$

■

5 Concluding remarks

1. Weighted prime number theorems that we investigate are to be compared to the following classical theorem for the Riemann zeta function.

Theorem A. [13, Th. 30., p. 83f] *Let θ be the upper bound of the real parts of the zeros of $\zeta(s)$. Then*

(a) $\psi_1(x) = \frac{x^2}{2} + O\left(x^{\theta+1}\right)$,

(b) $\psi(x) = x + O\left(x^{\theta} \log^2 x\right)$,

(c) $\pi(x) = \mathrm{li}\, x + O\left(x^{\theta} \log x\right)$.

The facts that $\sum_{0<\gamma\leq T} \frac{1}{\gamma} = O\left(\log^2 T\right)$ and $\sum_{\gamma>T} \frac{1}{\gamma^2} = O\left(\frac{\log T}{T}\right)$ (see, [13, Th. 25b., p. 70]) for the nontrivial zeros $\frac{1}{2} + i\gamma$ of $\zeta(s)$ allow us flexibility in moving from one level to another that is not present in the Selberg zeta case because of $\sum_{0<r_n\leq T} \frac{1}{r_n} = O(T)$ (see, [9, (6.14), p. 113]) and $\sum_{0<r_n\leq T} \frac{1}{r_n^2} = O(\log T)$ (see, [9, (5.10), p. 91]) for the Riemann surface and analogously for higher dimensional manifolds.

2. Let us notice that the proof of Theorem 3. yields another proof of Theorem 1., after taking into account that the first integral on the right hand side of (15) is $O(\log M)$ in the compact Riemann surface case. Indeed, with an appropriate caution related to the difference in the definitions of our $\Lambda(\gamma)$ and Hejhal's $\Lambda(\gamma) = l\left(C_{\gamma_0}\right)\left(1 - N(\gamma)^{-1}\right)^{-1}$, the result could be deduced from [9, Th. 6.16, p. 110]. We found our first proof worth presenting because of the term $O\left(\frac{x^{\frac{3}{2}+2\varepsilon}}{\varepsilon^2 \log T}\right)$ in (8) as opposed to $O\left(\frac{x^2 \log x}{T}\right)$ (see, [9, Remark. 6.17., p. 110]). The former might be useful for other applications as well.

3. We could summarize our results in the following form.

Theorem 4 *With the notation as in Theorem 3., let*

$$G_m(x) = \frac{1}{m!} \sum_{\gamma \in \Gamma_h, N(\gamma) \leq x} \Lambda(\gamma) \left(1 - \frac{N(\gamma)}{x}\right)^m -$$

$$\sum_{s_n(k) \in (d_0, 2d_0]} (-1)^k \frac{x^{s_n(k)}}{s_n(k)\,(s_n(k)+1)\ldots(s_n(k)+m)}.$$

Then,

(a) $G_m(x) = O\left(x^{d_0}\right)$ *if* $m \geq d$,

(b) $G_m(x) = O\left(x^{d_0} \log x\right)$ *if* $m = d - 1 = 2d_0$,

(c) $G_m(x) = O\left(x^{\frac{3}{4}(d-m-1)}\right)$ *if* $0 \leq m < 2d_0$.

Indeed, integrating the relation (12) $m - d$ times and having in mind that the series on the right hand side of (12) is absolutely convergent, we obtain (a). If we apply the operator $h^{-1}\Delta_1^+$ in the way we applied $h^{-(d-1)}\Delta_{d-1}^+$ in (13) and (14), we shall get (b). By the same reasoning the application of $h^{-(d-m)}\Delta_{d-m}^+$, $0 \leq m \leq d-2$, will give us (c). Actually, the assertion (b) coincides with Theorem 1. from the Riemann surface case. Theorem 2. is derived from (c) with $m = 0$.

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