Mathematica
Balkanica

New Series Vol. 25, 2011, Fasc. 5

On Tetravalent Near-Bipartite Arc-Transitive Circulants

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A graph Γ is said to be *near-bipartite* if there exists a subset of a vertex set of Γ with no adjacent vertices, such that its complement induces a bipartite graph. A *circulant* is a Cayley graph on a cyclic group. A graph is *arc-transitive* if its automorphism group acts transitively on the set of its arcs.

In this paper a question which tetravalent arc-transitive circulants are near-bipartite is considered. In particular, it is shown that if the order of a tetravalent arc-transitive circulant has a prime divisor $p > 13$ such that $2p = (2s+1)^2 + 1$ then it is near-bipartite. It is also shown that any tetravalent arc-transitive circulant of even order is near-bipartite.

MSC2010: 05C15, 05C25

Key Words: circulant, arc-transitive, near-bipartite graph.

1 Introduction

In this paper we consider near-bipartite graphs which are a natural generalization of bipartite graphs. In particular, a *near-bipartite* graph, is a graph in which there exists an independent set $I \subset V(\Gamma)$ of vertices, such that the induced graph $\Gamma[V(\Gamma) \setminus I]$ is bipartite. A *chromatic number* of a graph is the minimum number of colors needed to color the vertices of the graph in such a way that adjacent vertices have different colors. If the graph is near-bipartite, than we can color the vertices of this graph with three colors, since we can color the vertices in the independent set with one color, and for the remaining vertices two colors suffice. Therefore, near-bipartite graphs have chromatic number at most 3. Conversely, if a graph has chromatic number 3, then it is near-bipartite, since we can choose the vertices of one color to be the required independent set, and the remaining vertices are colored with 2 colors, which means that the remaining graph is bipartite.

[1]Supported in part by "Agencija za raziskovalno dejavnost Republike Slovenije", research program P1-0285 and "Mladi raziskovalec" research program.

In this paper, we deal with the question which tetravalent arc-transitive circulants, that is, tetravalent Cayley graphs on cyclic groups with the automorphism groups acting transitively on the set of ordered pairs of adjacent vertices, are near-bipartite. In the background of this question, is the following general problem : *Which combinatorial properties of a graph are implied by its symmetry?* In particular, is there any relationship between the symmetry and vertex-colorability of tetravalent graphs? As a partial answer to this question the following theorem is proven in this paper.

Theorem 1.1 *Let Γ be a tetravalent arc-transitive circulant. Then:*

(i) $\Gamma \cong K_5$; *or*

(ii) Γ *is near-bipartite; or*

(iii) Γ *is a normal circulant of odd order* n, $(n,5)=1$, *without a prime divisor* $p > 13$, *such that* $2p = (2s+1)^2 + 1$, *and, moreover, with respect to any* $\mathrm{Aut}(\Gamma)$*-invariant partition* $\mathcal{B}$ *either the quotient graph* $\Gamma_{\mathcal{B}}$ *is a tetravalent arc-transitive normal non-near-bipartite circulant or* $\Gamma_{\mathcal{B}} \cong K_5$.

The chromatic number of the complete graph K_5 is 4, implying that this graph is not near bipartite. An example of a graph from Theorem 1.1(iii) is the circulant $Cay(\mathbb{Z}_{13}, \{\pm 1, \pm 5\})$ which in fact is also not near-bipartite.

The paper is organized as follows. In Section 2 we recall some facts regarding arc-transitive circulants. In Section 3 we prove our main result concerning near-bipartite tetravalent arc-transitive circulants.

2 Preliminaries

Throughout this paper graphs are assumed to be finite, simple, connected and undirected, and groups are finite. Given a graph Γ, let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ be the set of its vertices, edges, arcs and the automorphism group of Γ, respectively.

Let U and W be disjoint subsets of $V(\Gamma)$. The subgraph of Γ induced by U will be denoted by $\Gamma[U]$. Similarly, let $\Gamma[U, W]$ denote the bipartite subgraph of Γ induced by the edges having one endvertex in U and the other endvertex in W.

A graph Γ is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if its automorphism group acts transitively on $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, respectively. For $I \subseteq V(\Gamma)$ the induced subgraph $\Gamma[I]$ is defined as the subgraph of Γ with vertex set I and edges of $E(\Gamma)$ with both endvertices in I. A subset $I \subset V(\Gamma)$ is called an *independent set* of Γ if no two vertices of I are adjacent in Γ. A graph Γ is said to be *near-bipartite* if there exists an independent set $I \subset V(\Gamma)$, such that the induced graph $\Gamma[V(\Gamma) \setminus I]$ is bipartite.

A *chromatic number* of a graph is the minimum number of colors needed to color the vertices of the graph in such a way that adjacent vertices have different colors. A *chromatic index* of a graph is the minimum number of colors needed to color the edges of the graph in such a way that adjacent edges have different colors.

Let G be a finite group with identity element 1, and let $S \subset G\backslash\{1\}$ be such that $S^{-1} = S$. We define the *Cayley graph* $Cay(G, S)$ on the group G with respect to the connection set S, to be the graph with vertex set G, in which two vertices $x, y \in G$ are adjacent if and only if $x^{-1}y \in S$. A *circulant* of order n is a Cayley graph on a cyclic group of order n.

To state the classification of connected arc-transitive circulants, which has been obtained independently by Kovács [1] and Li [2], and will be needed in Section 3, we need to recall certain graph products and the concept of normal Cayley graphs.

The *wreath (lexicographic) product* $\Sigma[\Gamma]$ of a graph Γ by a graph Σ is the graph with vertex set $V(\Sigma) \times V(\Gamma)$ such that $\{(u_1, u_2), (v_1, v_2)\}$ is an edge if and only if either $\{u_1, v_1\} \in E(\Sigma)$, or $u_1 = v_1$ and $\{u_2, v_2\} \in E(\Gamma)$. For a positive integer b and a graph Σ, denote by $b\Sigma$ the graph consisting of b vertex-disjoint copies of the graph Σ. The graph $\Sigma[\overline{K_b}] - b\Sigma$ is called the *deleted wreath (deleted lexicographic) product* of Σ and $\overline{K_b}$, where $\overline{K_b} = bK_1$.

Let $\Gamma = Cay(G, S)$ be a Cayley graph on a group G. Denote by $\mathrm{Aut}(G, S)$ the set of all automorphisms of G which fix S setwise, that is,

$$\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) | S^\sigma = S\}.$$

It is easy to check that $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\Gamma)$ and that it is contained in the stabilizer of the identity element $1 \in G$. It follows from the definition of Cayley graphs that the left regular representation G_L of G induces a regular subgroup of $\mathrm{Aut}(\Gamma)$. Following Xu [3], $\Gamma = Cay(G, S)$ is called a *normal Cayley graph* if G_L is normal in $\mathrm{Aut}(\Gamma)$, that is, if $\mathrm{Aut}(G, S)$ coincides with the vertex stabilizer $1 \in G$. Moreover, if Γ is a normal Cayley graph, then $\mathrm{Aut}(\Gamma) = G_L \rtimes \mathrm{Aut}(G, S)$.

Proposition 2.1 [1, 2] *Let Γ be a connected arc-transitive circulant of order n. Then one of the following holds:*

(i) $\Gamma \cong K_n$;

(ii) $\Gamma = \Sigma[\overline{K_d}]$, *where* $n = md$, $m, d > 1$ *and* Σ *is a connected arc-transitive circulant of order* m;

(iii) $\Gamma = \Sigma[\overline{K_d}] - d\Sigma$, *where* $n = md$, $d > 3$, $gcd(d, m) = 1$ *and* Σ *is a connected arc-transitive circulant of order* m;

(iv) Γ *is a normal circulant.*

Given a transitive group G acting on a set V, we say that a partition $\mathcal{B}$ of V is *G-invariant* if the elements of G permute the parts, that is, *blocks* of $\mathcal{B}$, setwise. If the trivial partitions $\{V\}$ and $\{\{v\} : v \in V\}$ are the only G-invariant partitions of V, then G is said to be *primitive*, and is said to be *imprimitive* otherwise. Clearly, the orbits of any proper normal subgroup of G form a non-trivial G-invariant partition. If the set V above is the vertex set of a vertex-transitive graph Γ, and $\mathcal{B}$ is a G-invariant partition, then clearly any two blocks $B, B' \in \mathcal{B}$ induce isomorphic vertex-transitive subgraphs.

Given a graph Γ and a partition $\mathcal{B}$ of its vertex set let the *quotient graph corresponding to $\mathcal{B}$* be the graph $\Gamma_{\mathcal{B}}$ whose vertex set equals $\mathcal{B}$ with $A, B \in \mathcal{B}$ adjacent if there exist vertices $a \in A$ and $b \in B$, such that a is adjacent to b in Γ. In case $\mathcal{B}$ is an $\mathrm{Aut}(\Gamma)$-invariant partition of an arc-transitive graph arising from the set of orbits of a normal subgroup of $\mathrm{Aut}(\Gamma)$ we clearly have that $X_{\mathcal{B}}$ is arc-transitive, $\Gamma[B]$ is empty graph and $\Gamma[B, B']$ is a regular graph for any two blocks $B, B' \in \mathcal{B}$.

3 Tetravalent near-bipartite arc-transitive circulants

In this section we will deal with the question which tetravalent arc-transitive circulants are near-bipartite.

Lemma 3.1 *Let Γ be an arc-transitive tetravalent circulant which is not near-bipartite. Then Γ is isomorphic to K_5 or Γ is a normal circulant.*

Proof. Let Γ be an arc-transitive tetravalent circulant. Then, by Proposition 2.1, Γ is a complete graph, a wreath product, a deleted wreath product or it is a normal circulant.

If Γ is a complete graph, then since it is tetravalent it is in fact isomorphic to the complete graph K_5.

If Γ is a deleted wreath product $\Gamma = \Sigma[\overline{K}_d] - d\Sigma$, where $d > 3$, then $val(\Gamma) = val(\Sigma)\cdot(d-1)$. Hence the only case when $val(\Gamma) = 4$ is that $val(\Sigma) = 1$, and therefore $\Sigma \cong K_2$ and $d = 5$, but this graph is bipartite, and therefore it also near-bipartite.

If, however, Γ is a wreath product $\Gamma = \Sigma[\overline{K}_d]$, then we have $val(\Gamma) = d \cdot val(\Sigma)$, and since $d > 1$, either $d = 2$ and $\Gamma = C_n[2K_1]$, or $d = 4$ and $\Gamma = K_2[4K_1]$. In the first case, since we can clearly color the vertices of C_n with three colors (C_n is near-bipartite), we can, with the use of vertex coloring of the C_n, color the graph $\Gamma = C_n[2K_1]$ in such a way, that if a vertex v of C_n is colored with i then we color the two vertices corresponding to this vertex (the two vertices in the same $2K_1$ corresponding to v) with the color i. This gives us a good 3-vertex coloring of Γ, and therefore Γ is near bipartite. In the latter case, $\Gamma = K_2[4K_1]$ is a bipartite graph, and thus also near-bipartite. ■

Lemma 3.2 *A tetravalent arc-transitive circulant of order p, where p is a prime such that $2p = (2s+1)^2 + 1$, $p > 13$, is near-bipartite.*

Proof. Let $\Gamma = Cay(\mathbb{Z}_p, S)$ be a tetravalent arc-transitive circulant on $\mathbb{Z}_p$. By Lemma 3.1 we may assume that Γ is normal. Therefore $\mathrm{Aut}(\Gamma)_0 = \mathrm{Aut}(\mathbb{Z}_p, S) \leq \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$ is cyclic, and so, without lose of generality, we may assume that $\Gamma = Cay(\mathbb{Z}_p, \{\pm 1, \pm m\})$, where $m = 2s+1$, and m and s are positive integers.

Let us define a set $I = \{0, 2, \ldots, 2s, \ldots, p-m-2\}$. Then $|I| = s^2$. The lemma follows by the following two claims.

Claim 1: I is an independent set.

Let $x > y$ be two different elements from I. Then $x - y$ is an even number, and $2 \leq x - y \leq p - m - 2$, so x and y cannot be adjacent. Namely, if that was the case then their difference would have to be an odd number, or equal to $p-1$, or $p - m$, which is impossible.

Claim 2: Graph $\Sigma = \Gamma[\mathbb{Z}_p \backslash I]$ is bipartite.

We will show that there exists no cycle of odd length in Σ. For $j \in \{1, 3, \ldots, p - m - 3\}$ it is obvious that $j+1$ and $j-1$ are both elements of I. We also claim that $j + m$ or $j - m$ is element of I. For $k \in \{1, 3, \ldots, p - 2m - 2\}$ we have $k + m \leq p - m - 2$ and since $p > 13$, we also have $s \geq 3$. Therefore $p - 2m - 2$ is a positive integer. If $k \in \{p - 2m, \ldots, p - m - 3\}$, then $k - m \in I$, since $k - m$ is an even number, and $p - 3m \leq k - m \leq p - 2m - 3$ which is equivalent to $2s^2 - 4s - 2 \leq k - m \leq 2s^2 - 2s - 4$ and since $s \geq 3$, it follows that $k - m \in I$.

We can conclude that all elements of $\{1, 3, \ldots, p - m - 3\}$ have at least 3 neighbors in I, so they have valency at most 1 in Σ, and they cannot lie on any cycle.

Thus if there exists a cycle of odd length in Σ, then it must be made of vertices from $J = \{-1, -2, \ldots, -m, -m-1\}$. If $x \neq -1$ and $x \neq -m-1, x \in J$ then x has only two neighbors in J, namely $x - 1$ and $x + 1$. So, the only possible cycle on vertices from J is the one which contains all the vertices from J. But this cycle is of even length $|J| = m + 1 = 2s + 2$. ∎

Proof of the Theorem 1.1: Let $\Gamma = Cay(\mathbb{Z}_n, S)$ be a tetravalent arc-transitive circulant. Assume that Γ is not near-bipartite. By Lemma 3.1, Γ is a normal circulant or $\Gamma \cong K_5$. Assume that $\Gamma \ncong K_5$. Then $\mathrm{Aut}(\Gamma) = \mathbb{Z}_n \rtimes \mathrm{Aut}(\mathbb{Z}_n, S)$. Let H be a subgroup of $\mathbb{Z}_n$. Since $\mathbb{Z}_n$ is cyclic, H is characteristic in $\mathbb{Z}_n$, and since $\mathbb{Z}_n$ is normal in $\mathrm{Aut}(\Gamma)$ it follows that H is normal in $\mathrm{Aut}(\Gamma)$. If $H \neq \mathbb{Z}_n$ then H is intransitive on $V(\Gamma)$, and the orbits of H form an $\mathrm{Aut}(\Gamma)$-invariant partition $\mathcal{B}$ of $V(\Gamma)$.

If Γ is of even order then there exists a subgroup $H < \mathbb{Z}_n$ of index 2 which gives an Aut(Γ)-invariant partition $\mathcal{B}$ of $V(\Gamma)$ consisting of two blocks, and thus Γ is bipartite. Since Γ is assumed to be non-near-bipartite it follows that it is of odd order.

With respect to any Aut(Γ)-invariant partition $\mathcal{B}$ we can make a quotient graph $\Gamma_{\mathcal{B}}$, which is again arc-transitive circulant either of valency 2 or 4. If the quotient graph is near-bipartite, then we can color the vertices of $\Gamma_{\mathcal{B}}$ with three colors, and then if the vertex v of $\Gamma_{\mathcal{B}}$ is colored with color i, then we color all the vertices in Γ which are in the orbit of H corresponding to v with color i, and this gives us a good 3-vertex coloring of Γ since there are no edges inside the blocks. If the quotient graph has valency 2, then it is clearly near-bipartite.

Since Γ is not near-bipartite Lemma 3.1 implies that either $\Gamma_{\mathcal{B}}$ is a tetravalent normal circulant or $\Gamma_{\mathcal{B}} \cong K_5$. In particular, we can reduce Γ to a smaller tetravalent normal circulant $X = Cay(\mathbb{Z}_p, S')$ for any prime divisor p of n.

Now we can assume that $S' = \{\pm 1, \pm s\}$. Since $\mathrm{Aut}(\mathbb{Z}_p, S')$ is cyclic, there exists some element in $\mathrm{Aut}(\mathbb{Z}_p, S')$ which cyclically rotates elements from S', therefore, we can assume that it acts as a multiplication by s. From this we obtain

$$s^2 \equiv -1 \pmod p.$$

This gives us $s^2 + 1 = mp$. If $m = 2$ then we use Lemma 3.2 to obtain the result. ■

Corollary 3.3 *A tetravalent arc-transitive circulant of even order is near-bipartite.*

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