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On the H-Invariants in the Selberg Class

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We give a survey of the presently known results on H-invariants in the Selberg class, including the recent solution of a problem posed by Perelli.

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1. Introduction

Analogies between the Riemann zeta function and other zeta or L-functions have been observed over the years. While these functions are seemingly independent of each other, there is a growing evidence that they are all somehow connected. The most important question about L-functions is related to the position of their zeros yielding various generalizations of the Riemann hypothesis.

This was motivation for mathematicians to try to understand, or at least classify, all of the objects which are believed to satisfy the Riemann hypothesis. There were different attempts in this direction. The Langlands program is an attempt to understand all L-functions and to relate them to automorphic forms.

In 1991, Selberg gave a set of precise axioms which are believed to characterize the L-functions for which Riemann hypothesis holds. Elements of the mentioned class, called the Selberg class, are Dirichlet series with an Euler product representation, meromorphic continuation and a functional equation of the proper shape. Although the exact nature of the class is conjectural, the hope is that the definition of the class will lead to the classification of its elements and give an insight into their relationship to automorphic forms and the Riemann hypothesis.

There are numerous problems concerning the Selberg class appropriate for the investigation. Roughly speaking, these problems are of two types: classical and structural.

By the classical problems we mean the extension of the problems about the classical L-functions to the Selberg class. Of course, the most important one is the Generalized Riemman hypothesis.

Structural problems are problems concerning the structure of Selberg class or its subclasses. It happens quite often that a property of a single L-function can be proved by considering it as a member of a class of L-functions.

Hence, the study of such classes is quite important. Some problems of this type are the classification of the functions from the Selbarg class, independence properties, study of invariants, countability and rigidity conjectures for the Selberg class.

In this paper we are focused on study of invariants. The notion of invariants comes from the fact that the factor in the functional equation is not unique. We will give definition of an invariant and list some important invariants and results concerning them. A special attention is devoted to the set of H-invariants, defined below.

2. Classes $\mathcal{S}$ and $\mathcal{S}^\natural$

The Selberg class of functions $\mathcal{S}$, introduced by A. Selberg in [11], is a general class of functions F satisfying the following properties:

(i) (Dirichlet series) F possesses a Dirichlet series representation

$$F(s)=\sum_{n=1}^{\infty}\frac{a_F(n)}{n^s},$$

that converges absolutely for $\sigma>1$, $s=\sigma+it$.

(ii) (Analytic continuation) There exists an integer $m\geq 0$ such that $(s-1)^mF(s)$ is an entire function of a finite order.

(iii) (Functional equation) The function F satisfies the functional equation $\Phi_F(s)=\omega\overline{\Phi_F(1-\overline{s})}$, where

$$\Phi_F(s)=F(s)Q_F^s\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j)=F(s)\gamma(s),$$

with $Q_F>0$, $r\geq 0$, $|\omega|=1$, $\lambda_j>0$, $\operatorname{Re}\mu_j\geq 0$, $j=1,\dots,r$.

(iv) (Ramanujan conjecture) For every $\epsilon>0$, $a_F(n)\ll n^{\epsilon}$.

(v) (Euler product)

$$\log F(s)=\sum_{n=1}^{\infty}\frac{b_F(n)}{n^s},$$

where $b_F(n)=0$, for all $n\neq p^m$ with $m\geq 1$ and p a prime, and $b_F(n)\ll n^{\theta}$, for some $\theta<\frac{1}{2}$.

The smallest number m from the axiom (ii) is denoted by m_F and called the polar order of F. The factor $\gamma(s)$ from the axiom (iii) is called the γ-factor, while the factors therein $\Gamma(\lambda_j s+\mu_j)$ are called Γ-factors.

Zeros of the function F located at the poles of γ-factor, i.e. $\rho = -\frac{\mu_j+k}{\lambda_j}$, $k = 0, 1, 2, \ldots$, $j = 1, \ldots, r$, are called the trivial zeros, and these are the only zeros in the half-plane $\sigma < 0$. The trivial zeros of F coincide with the zeros of $s^{-m_F}G(s)$, where $G^{-1}(s) = Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$. The other zeros are the non-trivial zeros, and they lie in the critical strip $0 \leq \sigma \leq 1$. Let us notice that the case $\rho = 0$, if present, usually requires a special attention, since it may arise as both a trivial and a non-trivial zero [7].

An extended Selberg class $\mathcal{S}^\natural$ is a class of functions satisfying conditions (i), (ii) and (iii).

It is believed that the class $\mathcal{S}^\natural$ contains all L functions of the interest in the number theory. Most of them are in the Selberg class. Here are some examples: the Riemann zeta function $\zeta(s)$, the Dirichlet L-function $L(s, \chi)$ and its shifts $L(s + i\theta, \chi)$, $\theta \in \mathbb{R}$, the Dedekind zeta function $\zeta_K(s)$ of an algebraic number field K, the Hecke L-function $L_K(s, \chi)$, the L-function associated with a holomorphic newform of a congruence subgroup of $SL2(\mathbb{Z})$. Moreover, the class $\mathcal{S}$ contains some L-functions provided certain classical conjectures hold, for example Artin L-function belongs to $\mathcal{S}$ if Artin conjecture holds, automorphic L-functions are in $\mathcal{S}$ provided the Ramanujan conjecture holds true.

It is important to mention that the class $\mathcal{S}^\natural$ is not suitable for formulating the Generalized Riemann hypothesis, since some functions from this class may have infinitely many nontrivial zeros in the half-plane $\sigma > 1$. Axioms (iv) and (v), are of key importance for the Generalized Riemann hypothesis.

Very nice survey papers concerning Selberg and extended Selberg class are [3, 4, 9, 10].

Let us also mention here that the Selberg class is an important subclass of the fundamental class of functions introduced by Jorgenson and Lang [2]. This gives us a possibility to derive and use an explicit formula for the Selberg class, as it is shown in [1, 7, 12].

3. Invariants

From the definition of the γ-factor there directly follows that the form of it is not unique. It is easy to see that different shapes can be obtained using the Legendre duplication formulae and factorial formulae for gamma function. Neither, the data $(Q_F, \lambda, \mu, \omega)$ of the functional equation of F, where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$, are uniquely determined by F.

This gives rise to the notion of an invariant, introduced in the series of papers written by J. Kaczorowski and A. Perelli. Hence, an invariant (a numerical invariant) of a function $F \in \mathcal{S}^\natural$ is an expression (a number) defined in terms of the data of F which is uniquely determined by F itself.

A more formal definition of an invariant can be given using the notion of a parameter, as in [5]. An expression depending on the data $(Q, \lambda, \mu, \omega)$ is called a parameter, usually denoted by $I(Q, \lambda, \mu, \omega)$. A parameter $I(Q, \lambda, \mu, \omega)$ is an invariant if $I(Q, \lambda, \mu, \omega) = I(Q', \lambda', \mu', \omega')$ for any pair of data $(Q, \lambda, \mu, \omega)$,

$(Q', \lambda', \mu', \omega')$ of the functional equation of F , for every $F \in \mathcal{S}^\natural$. Invariants are also denoted by $I(Q, \lambda, \mu, \omega)$ and I_F when referred to a function F . A generic invariant I is called numerical if $I_F \in \mathbb{C}$ for every $F \in \mathcal{S}^\natural$.

In order to give a characterization of invariants introduced in [5] we need to introduce two following terms. A parameter is stable by multiplication formula if $I((Q, \lambda, \mu, \omega) = I(Q', \lambda', \mu', \omega')$, where $(Q', \lambda', \mu', \omega')$ are the new data obtained by application of the multiplication formula to a Γ-factor.

The definition of a parameter stable by factorial formula is completely analogous to the previous one.

The above mentioned characterization is given by the following theorem.

Theorem 3.1. *[5] A parameter is an invariant if and only if it is stable by the multiplication and factorial formula.*

Application of the above theorem enables us to prove that some interesting parameters are actually invariants [5]. Here are some examples.

Let a, b be positive real numbers. We say that a and b are $\mathbb{Q}$-equivalent if $a/b \in \mathbb{Q}$, and we denote by h_F, the γ-class number, the number of $\mathbb{Q}$-equivalence classes arising from the numbers $\lambda_1, \ldots. \lambda_r$ of gamma factor of $F \in \mathcal{S}^\natural$. It is shown that h_F is an invariant.

Important examples of (numerical) invariants are the numbers $H_F(n)$, for an integer $n \geq 0$, defined by

$$H_F(n) = 2 \sum_{j=1}^{r} \frac{B_n(\mu_j)}{\lambda_j^{n-1}},$$

where $B_n(x)$ is the n-th Bernoulli polynomial. The numbers $H_F(n)$ are called the H-invariants.

Special cases

$$H_F(0) = 2 \sum_{j=1}^{r} \lambda = d_F$$

and

$$H_F(1) = 2 \sum_{j=1}^{r} \left(\mu_j - \frac{1}{2} \right) = \xi_F = \eta_F + i\theta_F$$

are particularly important. They are called the degree, and the ξ-invariant, respectively. A real and imaginary part of the ξ-invariant are called, respectively, the parity and the shift of $F \in \mathcal{S}^\natural$, and they are invariants, too.

Other important invariants for the functions $F \in \mathcal{S}^\natural$ are the conductor or modulus of F , denoted by q_F, and the root number ω_F^*. They are defined as follows:

$$q_F = (2\pi)^{d_F} Q_F^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

(1)

$$\omega_F^* = \omega\, e^{-i\frac{\pi}{2}(\eta_F+1)} \left(\frac{q_F}{(2\pi)^{d_F}} \right)^{i\theta_F/d_F} \prod_{j=1}^{r} \lambda_j^{-2i \operatorname{Im} \mu_j}.$$

There are several problems concerning the functional equations and invariants. The problem of characterizing the shape of admissible functional equations in $\mathcal{S}$ is raised by Selberg, amongst others. However, very little is known unconditionally about it, although several conjectures are proposed [5]. Another problem is to construct a universal set of basic invariants. Note that a set $\{I_j\}_{j\in J}$ of numerical invariants is called a set of basic invariants if $I_j(F_1) = I_j(F_2)$ for all $j \in J$ implies that $F_1(s)$ and $F_2(s)$ satisfy the same functional equation, for any F_1 , $F_2 \in \mathcal{S}^\natural$. In other words, a set of basic invariants characterizes the functional equation of every $F \in \mathcal{S}^\natural$. More precisely, such set is called a global set of basic invariants, contrary to a local set of basic invariants, characterizing the functional equation of a given function $F \in \mathcal{S}^\natural$.

The problem concerning the set of basic invariants has the following solution.

Theorem 3.2. *[5] The H-invariants $H_F(n)$, $n \geq 0$, the conductor q_F and the root number ω_F^* form a global set of basic invariants.*

We need to notice that if we drop the condition that the basic invariants are numerical invariants, then finite global sets of basic invariants are easily detected. In [9] it is pointed out that there exist global sets of basic invariants formed by a single numerical invariant. However these invariants are quite artificial, while the invariants from Theorem 3.2 are definitely more interesting.

Another interesting problem closely related to the problem of finding the set of basic invariants is to construct a form of the functional equation where all data are invariants, called an invariant form. This problem is discussed in [6].

From Theorem 3.2 it is obvious that H-invariants are particulary interesting for investigation. In [6], Kaczorowski and Perelli obtained an interpretation of H-invariants and the conductor as coefficients in a certain asymptotic expansion of the gamma factor of the functional equation. Precisely they derived the following expression

$$\begin{aligned}\log\gamma(s) &= \frac{1}{2}H_F(0)\,s\,\log s + \frac{1}{2}\left(\log q_F - H_F(0)\log 2\pi\, e\right)s \\ &\quad + \frac{1}{2}H_F(1)\log s + c(\gamma) \\ &\quad + \frac{1}{2}\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n(n+1)}H_F(n+1)s^{-n} + \mathcal{O}\left(|s|^{-N-1}\right),\end{aligned}$$

for all $N \in \mathbb{N}$, $|\arg(s)| < \pi$, where $c(\gamma) = \sum_{j=1}^{r}(\mu_j - \frac{1}{2})\log\lambda_j + \frac{r}{2}\log(2\pi)$ and $\gamma(s)$ is γ factor of the functional equation for $F \in \mathcal{S}^\natural$.

They also raised a question of interpretation of $H_F(n)$, $n \geq 2$ in terms of F alone, without explicit reference to the functional equation. That question is posed as a problem in [9, Problem 4.1.]. The solution to this problem is obtained in [8] using "superzeta" functions constructed over trivial and non-trivial zeros of $F \in \mathcal{S}^\natural$. They are defined as follows

$$\mathcal{Z}_F(s,z) = \sum_{\rho}(z-\rho)^{-s} \qquad (\mathrm{Re}s > 1),$$

where the sum is taken over all nontrivial zeros ρ (counted with multiplicities) of the function F , $z \in X = \{z \in \mathbb{C} : (z-\rho) \notin \mathbb{R}^-(\forall \rho)\}$ and

$$\mathbf{Z}_F(s,z) = \sum_{\eta_k}(z-\eta_k)^{-s} - m_F z^{-s} \qquad (\mathrm{Re}s > 1),$$

where the sum is taken over zeros $\eta_k = \eta_{n,j} = -\frac{n+\mu_j}{\lambda_j}$, $n = 0,1,2,\ldots, j = 1,\ldots,r$ of G, counted with their multiplicities and $z \in X_1 = \{z \in \mathbb{C} : (z-\eta_k) \notin \mathbb{R}^-(\forall k)\}$

$\mathcal{Z}_F(s,z)$ is called the zeta function constructed over non trivial zeros of F, while $\mathbf{Z}_F(s,z)$ is the zeta function constructed over trivial zeros since it can be shown that $\mathbf{Z}_F(s,z)$ is equal to the sum $\sum_{\kappa}(z-\kappa)^{-s}$ over all trivial zeros κ of F (including the zero $\kappa = 0$, if present).

The following theorem gives us representation of the H-invariants using zeta function constructed over trivial zeros.

Theorem 3.3. *[8] Let $F \in \mathcal{S}^\natural$ and $z \in X_2 = \{z \in X_1 : \mathrm{Re}(\lambda_j z + \mu_j) > 0, \ \ j = 1,\ldots,r\}$. Then*

$$H_F(n) = -2n\mathbf{Z}_F(1-n,0),$$

for $n \in N$, $n \geq 2$,

$$H_F(1) = -2(\mathbf{Z}_F(0,0) + m_F)$$

and

$$H_F(0) = 2\operatorname*{Res}_{s=1}\mathbf{Z}_F(s,z)\,.$$

Let us mention here that this theorem provides only a partial solution to [9, Problem 4.1.], since $\mathbf{Z}_F(s,z)$ depends directly on the factor of the functional equation. The complete solution is contained in the following theorem.

Theorem 3.4. *[8] Let $F \in \mathcal{S}$. Then*

(a) For $n \in \mathbb{N}$ and $z \in X \cap X_2 \setminus (-\infty, 1]$ one has

$$\mathcal{Z}_F(-n,z) = \frac{H_F(n+1)}{2(n+1)} + \frac{1}{2(n+1)}\sum_{k=0}^{n}\binom{n+1}{k}H_F(k)z^{n+1-k}$$
$$+m_F(z-1)^n + m_F z^n.$$

(b) *For a fixed integer $n \geq 0$, the function $\mathcal{Z}_F(-n,z)$ has an analytic continuation to the whole z-plane and*

$$H_F(n) = 2n(\mathcal{Z}_F(1-n,0) + (-1)^n m_F),$$

for $n \geq 2$, while

$$H_F(1) = 2(\mathcal{Z}_F(0,0) - 2m_F).$$

The proof of this theorem is derived using the theory of zeta functions and zeta regularized products developed by Voros [13]. Details can be found in [8].

Comparing the polar structure of the functions $\mathbf{Z}_F$ and $\mathcal{Z}_F$ it is possible to derive representation of $H_F(0)$ and conductor q_F using the zeta function constructed over nontrivial zeros of F.

Corollary 3.5. *[8] Let $F \in \mathcal{S}$. For $z \in X \cap X_2 \setminus (-\infty, 1]$ one has*

(a) $H_F(0) = -2 \operatorname*{Res}_{s=1}(\mathcal{Z}_F(s,z))$

(b) $\log q_F = 2\left[\mathcal{Z}_F^*(1,z) - \log 2\pi \operatorname*{Res}_{s=1} \mathcal{Z}_F(s,z) - \operatorname*{FP}_{s=1} \mathcal{Z}_F(s,z)\right]$.

Here, $\mathcal{Z}_F^*(1,z) := \frac{(\Phi_F^c)'}{\Phi_F^c}(z)$, $\Phi_F^c(z) = (z-1)^{m_F} z^{m_F} \Phi_F(z)$ and $\operatorname*{FP}_{s=1} \mathcal{Z}_F(s,z)$ denotes the constant term in the Laurent series expansion of $\mathcal{Z}_F(s,z)$ at the pole $s=1$.

Let us emphasize here that the definition and the properties of invariants are stated in the framework of the extended Selberg class. Obviously, they can be stated in the framework of the Selberg class as well.

However the results concerning the representation of H-invariants using "superzeta" functions are written in the framework of the Selberg class. It can be shown that these results remain valid for all $F \in \mathcal{S}^\natural$ having an Euler product, convergent in some half plane $\operatorname{Re} z > \sigma \geq 1$, without additional bounds on the coefficients $b_F(n)$. The result may not be extended to the class $\mathcal{S}^\natural$, as pointed out in [8].

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