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Cesaro Summability in Some Orthogonal Systems

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The jump of a function f belonging to the Wiener class $\mathcal{V}_p$, $p > 1$, can be determined through (C,α), $\alpha > 1 - \frac{1}{p}$, summability of the sequence of terms of its differentiated Fourier-Jacobi series. Consesequently, the corresponding (C,α) summability result holds for the Waterman classes $\{n^\beta\}BV$ and the Chanturiya classes $V[n^\beta]$ if $\alpha > \beta$, $0 < \beta < 1$.
A maximal converse ineqality enables one to characterize the Hardy space H^1 on a bounded Vilenkin group by means of a $(C,1)$ related operator $\sigma^\dagger f = \sup_n |\sigma_{M_n} f|$.

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1 Introduction

The first of the three parts of [14] concerns with Cesàro summability of Fourier series with respect to a system of generalized Jacobi polynomials of a function of generalized bounded variation and accordingly gives new ways for determination of jump discontinuities. The problem of determination of jump discontinuities in piecewise smooth functions from their spectral data is relevant in signal processing.

While the Jacobi polynomials are related to irreducible unitary representations of the group $SU(2)$, the Vilenkin system appears as the set of characters of a compact, totally disconnected Abelian group satisfying the second axiom of countability. The second part of [14] concerns with certain aspects of harmonic analysis on Vilenkin groups that are present in the general setting, i.e. without the boundedness assumption related to the sequence of subgroups that determines the topology of a group under consideration.

Hardy spaces have an atomic structure and can be determined by means of different norms.The norm obtained from the maximal function and maximal operators provides the easiest and more natural way to test whether a function belongs to a Hardy space. New characterizations of the Hardy spaces got by investigations of different equivalent norms of the Cesaro means with respect to the Vilenkin system, in the bounded setting, form the content of the last part of [14].

2 Determination of Jumps by Fourier-Jacobi Coefficients

We say that a function w is a generalized Jacobi weight and write $w \in GJ$, if

$$w(t) = h(t)(1-t)^{\alpha}(1+t)^{\beta}|t-x_1|^{\delta_1}...|t-x_M|^{\delta_M},$$

$h \in C[-1,1]$, $h(t) > 0\ (|t| \leq 1), -1 < x_1 < ... < x_M < 1,\ \ \alpha, \beta, \delta_1, ..., \delta_M > -1.$

By $\sigma(w) = (P_n(w;x))_{n=0}^{\infty}$ we denote the system of algebraic polynomials $P_n(w;x) = \gamma(w)x^n +$ lower degree terms with positive leading coefficients $\gamma_n(w)$, which are orthonormal on $[-1,1]$ with respect to the weight $w \in GJ$ i.e. $\int_{-1}^{1} P_n(w;t)P_m(w;t)w(t)dt = \delta_{nm}$.

Such polynomials are called the generalized Jacobi polynomials [11].

If $fw \in L[-1,1]$, $w \in GJ$, then the n-th partial sum of the Fourier series of f with respect to the system $\sigma(w)$ is given by

$$S_n(w;f;x) = \sum_{k=0}^{n-1} a_k(w;f)P_k(w;x) = \int_{-1}^{1} f(t)K_n(w;x;t)w(t)dt,$$

where $a_k(w;f) = \displaystyle\int_{-1}^{1} f(t)P_k(w;t)w(t)dt$ is the k-th Fourier coefficient of the function f and $K_n(w;x;t) = \sum_{k=0}^{n-1} P_k(w;x)P_k(w;t)$ is the Dirichlet kernel of the system $\sigma(w)$.

In the next theorem we prove the corresponding (C,α) summability result for the Wiener classes $\mathcal{V}_p$ (see [1] for notation) in the case of Fourier-Jacobi series.

Theorem 2.1 (M.Avdispahić-S.Pirić) *Let f be a function of bounded p-variation, i.e. $f \in \mathcal{V}_p$, $p > 1$, such that $fw \in L[-1,1], w \in GJ$.*

Then the sequence $(a_n(w;f)P_n'(w;x))$ is (C,α), $\alpha > 1 - \dfrac{1}{p}$ summable to $\dfrac{(1-x^2)^{-\frac{1}{2}}}{\pi}(f(x+0) - f(x-0))$ for every $x \in (-1,1)$, $x \neq x_1, \ldots, x_M$, where $a_n(w;f)P_n'(w;x)$ is the n-th term of the differentiated Fourier-Jacobi series of f.

In the proof we use [2, Theorem 3] and [8, Theorem 7]. By a theorem of Avdispahic [1], there exist the following inclusion relations between the classes $\mathcal{V}_p$, ΛBV and $V[\nu]$ of generalized bounded variation in the sense of Wiener, Waterman and Chanturiya

$$\{n^\alpha\}BV \subset \mathcal{V}_{\frac{1}{1-\alpha}} \subset V[n^\alpha] \subset \{n^\beta\}BV,$$

for $0 < \alpha < \beta < 1$. Therefore, we have

Corollary 2.2 *[13] If f belongs to $\{n^\beta\}BV$ or $V[n^\beta]$, $0 < \beta < 1$, then the claim of Theorem 1 is valid for (C, α), $\alpha > \beta$.*

Similar identities hold if we consider the integrated rather than the differentiated Fourier series.

By $R_n(x, f)$ we denote the n-th order tail of the Fourier series of a function f.

For any function f, integrable on $[-\pi, \pi]$, $f^{(-r)}$, $r \in \mathbb{N}_0 = \mathbb{N} \cup 0$, is defined as follows $f^{(-r-1)} = \int f^{(-r)}$, where $f^{(0)} = f$.

First, we obtain new identity which determines the jumps of a periodic function of $\mathcal{V}_p, 1 \leqslant p < 2$, class with a finite number of discontinuities, by means of the tail of its integrated Fourier-Jacobi series and then we establish (C, α), $\alpha > 1 - \dfrac{1}{p}$, summability of the sequence $(n^2 a_n(w; f) \int P_n(w; x)dx)$, where $a_n(w; f) \int P_n(w; x)dx$ is the n-th term of the integrated Fourier-Jacobi series of f.

Theorem 2.3 *[14] Suppose a function $f \in \mathcal{V}_p$, $1 \leqslant p < 2$, has a finite number of discontinuities and $fw \in L[-1, 1], w \in GJ$. Then the identity*

$$\lim_{n\to\infty} nR_n^{(-1)}(w; f; x) = -\frac{1}{\pi}(1 - x^2)^{\frac{1}{2}}(f(x+) - f(x-))$$

is valid for each fixed $x \in [-\pi, \pi]$, where $R_n^{(-1)}(w; f; x)$ is the n-th order tail of the integrated Fourier-Jacobi series of the function f.

In the proof, [7, Theorem 4] is used.

Theorem 2.4 *[14] Let f be a function of bounded p-variation, i.e. $f \in \mathcal{V}_p$, $1 \leqslant p < 2$, which has a finite number of discontinuities such that $fw \in L[-1, 1], w \in GJ$. Then the sequence $\left\{n^2 a_n(w; f) \int P_n(w; x)dx\right\}$ is (C, α),*

$\alpha > 1 - \frac{1}{p}$ *summable to* $\frac{(1-x^2)^{\frac{1}{2}}}{\pi}(f(x+0) - f(x-0))$ *for every* $x \in [-\pi, \pi]$, *where* $\left\{a_n(w; f) \int P_n(w; x)dx\right\}$ *is the n-th term of the integrated Fourier-Jacobi series of* f.

3 Vilenkin Systems

Let $(m_0, m_1, \ldots, m_n, \ldots)$ be a sequence of integers each of them not less than 2 and let $\mathbb{Z}_{m_n}$ denote the discrete cyclic group of order m_n, with addition mod m_n. Since the group is discrete, then every subset is open. Let $G := \prod_{n=0}^{\infty} \mathbb{Z}_{m_n}$. Then each element from G can be represented as a sequence $(x_n)_n$, where $x_n \in \{0, 1, \ldots, m_n - 1\}$. Addition in G is obtained coordinatewise. The measure is the normalized Haar measure. Consequently, G is a compact Abelian group. If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G a bounded Vilenkin group. The topology on G is generated by the subgroups
$I_n := \{x = (x_i)_i \in G, x_i = 0 \text{ for } i < n\}$ and their translations
$I_n(y) := \{x = (x_i)_i \in G, x_i = y_i \text{ for } i < n\}$.
Define the sequence $(M_n)_n$ as follows: $M_0 = 1$ and $M_{n+1} = m_n M_n$. Then each natural number n can be uniquely represented in the following form

$$n = \sum_{i=0}^{\infty} n_i M_i, \; n_i \in \{0, 1, \ldots, m_i - 1\}, \tag{3.1}$$

where only a finite number of $n_i s$ differ from zero. G has a countable collection of characters, i.e., continuous complex valued functions χ, that satisfy the following condition

$$\chi(x+y) = \chi(x)\chi(y), \; (\forall x, y \in G).$$

That collection is denoted by Γ. The characters form an Abelian group with respect to the pointwise product of functions. It is known [6] that $(\Gamma, \cdot)$ is a discrete, countable and Abelian group with torsion.
A common way to define the Vilenkin system is from the so-called basis of Rademacher functions $(r_n)_n$. The generalized Rademacher functions are defined as

$$r_n(x) = e^{\frac{2\pi i x_n}{m_n}}, n \in \mathbb{N} \cup \{0\}, x \in G.$$

The n-th Vilenkin function is

$$\psi_n(x) = \prod_{i=0}^{\infty} r_i^{n_i}(x), n \in \mathbb{N} \cup \{0\}, x \in G.$$

The system $\psi = (\psi_n, n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G, and all the characters of G are of this form.

We recall Lemma 7 [16] obtained for the group $\prod_{k=0}^{\infty} \mathbb{Z}_{n_k}$. For each $n \in \mathbb{N}$ let $\tilde{n}$ denote the positive integer for which $\chi_n(x)\chi_{\tilde{n}}(x) = 1$, for every $x \in G$.

Lemma 3.1 *[16] If $n \in [M_j, M_{j+1})$, that is $n = \sum_{i=0}^{j} n_i M_i$, $1 \le n_j < m_{j+1}$; $0 \le n_i < m_{i+1}$, $i = 0, 1, \ldots, j-1$, then $\tilde{n} \in [M_j, M_{j+1})$ and $\tilde{n} = (m_{j+1} - n_j)M_j + \sum_{i=0, n_i \neq 0}^{j-1} (m_{i+1} - n_i)M_i = M_{j+1} + \sum_{i=0, n_i \neq 0}^{j-1} M_{i+1} - n$.*

The following result gives a general formula which expresses indices of inverse characters $\tilde{n}$ in the Vilenkin system, in the general setting. The character $\chi_{m_N}^{p_{N+1}}$ is of the form $\chi_{m_N}^{p_{N+1}} = \prod_{j=0}^{N-1} \chi_{m_j}^{\alpha_j^N}$, where the nonnegative integers $\alpha_j^N \le p_{j+1}, j \le N-1$, are uniquely identified.

Theorem 3.2 *[9] Let G be any Vilenkin group. If $n = \sum_{i=0}^{N} n_i M_i$, then $\tilde{n} = \sum_{l=0}^{N} c_l M_l$, where c_l satisfy $\sum_{i=l}^{N} b_l^i n_i + \sum_{i=l+1}^{N} F_i \alpha_l^i = F_l m_{l+1} + c_l$, for some explicit positive integers $(F_i)_i$, and b_l^i, $i \le l$ satisfy the equations*

$$b_i^i = m_{i+1} - 1,$$

$$b_l^i + \sum_{t=l+1}^{i} R_{t+1} \alpha_l^t = R_{l+1} m_{l+1},$$

for $l < i$, where the positive integers R_j, $j \ge 0$, are recursively uniquely determined as $0 \le b_l^i, \alpha_l^t \le m_{l+1} - 1$, for every $0 \le l \le i, t$.

Lemma 7 in [16] and Theorem 1 in [12] are direct consequences of Theorem 3.2. This can be seen in the following corollaries, applicable in two different speciffic situations.

Corollary 3.3 *[9] Let G be a Vilenkin group. Using the above notation, let $(\chi_n)_n$ be a Vilekin system such that $\alpha_j^N = 0$, for every $N \ge 1$, $j = 0, \ldots, N-1$. Then, $c_l = m_{l+1} - n_l$ if $n_l \neq 0$ and $c_l = 0$ if $n_l = 0$.*

Corollary 3.4 *[9] Let G be a Vilenkin group. Using the above notation, let $(\chi_n)_n$ be a Vilenkin system such that $\alpha_j^N = 0$, for every $N \ge 1$, $j = 0, \ldots, N-1$. Then, $c_N = m_{N+1} - 1$ and $c_l = m_{l+1} - n_l - 1$, for every $l \le N-1$.*

4 Characterizations of Hardy spaces by Cesaro means

For bounded Vilenkin groups, the space H^1 can be characterized in several ways. Many equivalent norms are defined in [15], [17] and [3].

The Fourier coefficients, the partial sums of the Fourier series and the Fejér means with respect to the Vilenkin system are respectively defined as follows $\hat{f}(n) = \int f(x)\bar{\psi}_n(x)dx$, $S_n f = \sum_{k=0}^{n-1} \hat{f}(k)\psi_k$ and $\sigma_n f = \frac{1}{n}\sum_{k=1}^{n} S_k f$, for every $f \in L^1(G)$.

We introduce the maximal function and the maximal operators

$$f^*(x) = \sup_n |I_n|^{-1} \left| \int\limits_{I_n(x)} f(t)dt \right|, \qquad \sigma^* f(x) = \sup_n |\sigma_n f(x)| \qquad \text{and}$$

$$\sigma^\dagger f(x) = \sup_n |\sigma_{M_n} f(x)| .$$

The Hardy space $H^p, p > 0$ consists of integrable functions f for which $f^* \in L^p$. The norm used in this paper is

$$\|f\|_{H^p} = \|f^*\|_p.$$

The boundedness of σ^* from H^1 to L^1 for bounded groups was established in [4]. Moreover, it was proved in [15] that the boundedness of the group is necessary and sufficient for the boundedness of the maximal operator σ^*. According to [10], the L^1 norms of the operators σ^* and $\sigma^\dagger$ also define norms for the space H^1 in the bounded case.

First, we prove that the mean value of f on the coset $I_{N-1}(x)$ is dominated by either $\sigma_{M_{N-1}}$ or σ_{M_N} on some translated element.

Lemma 4.1 *[10] Let $x \in G$, $N \in \mathbb{N}$. Then,*

$$|S_{M_{N-1}} f(x)| \leq 4 \max(|\sigma_{M_{N-1}} f(x + je_{N-1})|, |\sigma_{M_N} f(x + je_{N-1})|),$$

for at least some $j \in \{0, 1, \ldots, m_{N-1} - 1\}$.

A maximal converse inequality which characterizes the space H^1 on bounded groups by means of the operator $\sigma^\dagger f = \sup_n |\sigma_{M_n} f|$, is given by the following theorem.

Theorem 4.2 *[10] Let G be a bounded Vilenkin group, and $f \in L^p$, for some $p > 0$, with $\sigma^* f \in L^p$. Then, $f \in H^p$. Moreover, there exists a positive constant $C > 0$ depending only on the sequence $(m_n)_n$ such that*

$$\|f^*\|_p^p \leq C \|\sigma^\dagger f\|_p^p.$$

[5, Theorem 1] is used in the proof. An immediate consequence is the next result.

Corollary 4.3 *[10] If G is a bounded Vilenkin group, then the following norms are equivalent in H^1*

$$\|f\|_{H^1} \sim \|\sigma^\dagger f\|_1 \sim \|\sigma^* f\|_1.$$

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