Mathematica
Balkanica

New Series Vol. 25, 2011, Fasc. 5

Numerical Integration Using Cardinal B-Splines

Zlatko Udovičić

In this paper, we construct a new formula for approximate computation of the integral

$$\int_a^b f(x)dx, \tag{1}$$

where $f(\cdot)$ is a given function.

MSC2010: 65D32, 65D07

Key Words: cardinal B-spline, quadrature, nodes with multiplicity two

1 Introduction

One of the classical problems in mathematical analysis is, of course, calculation of the integral (1). Since this integral cannot, in general, be calculated exactly, there is a large number of formulas for its approximate computation. The most used idea in construction of those formulas is to approximate function $f(\cdot)$ by some function $\widehat{f}(\cdot)$ such that the integral $\int_a^b \widehat{f}(x)dx$ is easy to calculate. This paper is inspired by the recently published paper [4]. Namely, in [4], Daubechies refinable function of arbitrary order was used as a basic function for approximation of the function $f(\cdot)$. Since cardinal B-spline of arbitrary order is a typical example of the refinable function, we specifically used cardinal B-spline of arbitrary order as a basic function for approximation of the function $f(\cdot)$.

The paper is organized as follows; in the preparatory section, we give the definition of the cardinal B-spline, a list of its basic properties and we also describe the calculation of the so-called shortened spline moments. In the second section, we give the main result of this paper, i.e. we describe construction of the quadrature formula. Some comments, as well as a couple of numerical examples, are given in the last section.

Definition 1 *Cardinal B-spline of the first order, denoted by $\varphi_1(\cdot)$, is a characteristic function of the interval $[0,1)$, i.e.*

$$\varphi_1(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{otherwise} \end{cases}.$$

Cardinal B-spline of order $m, m \in \mathbb{N}$, denoted by $\varphi_m(\cdot)$, is

$$\begin{aligned} \varphi_m(x) &= (\varphi_{m-1} * \varphi_1)(x) = \int_{\mathbb{R}} \varphi_{m-1}(x-t)\varphi_1(t)dt \\ &= \int_0^1 \varphi_{m-1}(x-t)dt. \end{aligned}$$

Theorem 1 *Cardinal B-spline of order $m, m \in \mathbb{N}$, has the following properties:*

1. $(\forall m \in \mathbb{R})\, \varphi_m(\cdot) \in C^{m-2}[0,m]$;
2. *At each interval $[k, k+1], 0 \leq k \leq m-1$, cardinal B-spline $\varphi_m(\cdot)$ is a polynomial of degree not greater than $m-1$;*
3. $supp\varphi_m(\cdot) = [0,m]$;
4. $\int_{\mathbb{R}} \varphi_m(x)dx = 1$;
5. *Cardinal B-spline is symmetric with respect to the midpoint of the interval $[0,m]$, i.e.*
$$(\forall x \in [0,m])\, \varphi_m(x) = \varphi_m(m-x);$$
6. $(\forall t \in [0,m])\, \varphi_m(t) = \dfrac{t}{m-1}\varphi_{m-1}(t) + \dfrac{m-t}{m-1}\varphi_{m-1}(t-1), m \geq 2$;
7.
$$(\forall x \in [0,m])\, \varphi_m'(x) = \varphi_{m-1}(x) - \varphi_{m-1}(x-1), m \geq 2; \tag{2}$$

Definition 2 *For each $m \in \mathbb{Z}$, the space of the cardinal B-splines of order m, with knot sequence $\mathbb{Z}$, denoted by S_m, is a collection of all functions $f(\cdot) \in C^{m-2}$ such that restriction of the function $f(\cdot)$ at each interval $[k, k+1), k \in \mathbb{Z}$, is a polynomial of degree not greater than $m-1$.*

In accordance with the previous, it is obvious that the collection

$$\{\varphi_m(\cdot - k) \big| k \in \mathbb{Z}\}$$

generates the basis of the space S_m. This, of course, means that every function $f(\cdot) \in S_m$ can be represented as a linear combination

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \varphi_m(x-k).$$

Therein, we do not have to worry about the convergence of the series on the right hand side of the previous equality, since for every $x \in \mathbb{R}$ all members of this series, except finitely many of them, are equal to zero.

To obtain better approximation, we introduce spaces $S_m^{(j)}, j \in \mathbb{N}$, (so-called spaces with finer resolution) with bases

$$\{\varphi_m^{(j,k)}(\cdot) \big| k \in \mathbb{Z}\},$$

where

$$\varphi_m^{(j,k)}(x) = 2^{\frac{j}{2}} \varphi_m(2^j x - k).$$

Proofs of the given assertions, and much more details about cardinal B-splines can be found in [1] or [2].

Furthermore, by using the definition of the cardinal B-spline of the first order and equality (2), after integration by part, we obtain recurrent relations for calculation of the shortened spline moments.

$$\begin{aligned}
\mu_{n,1}^x &= \int_0^x \varphi_1(t)t^n dt = \begin{cases} 0, & x<0 \\ \frac{x^{n+1}}{n+1}, & x\in[0,1] \\ \frac{1}{n+1} & x>1 \end{cases}, \\
\mu_{n,m}^x &= \int_0^x \varphi_m(t)t^n dt \\
&= \frac{x^{n+1}\varphi_m(x)}{n+1} - \frac{1}{n+1}\int_0^x \varphi_{m-1}(t)t^{n+1}dt \\
&\quad + \frac{1}{n+1}\int_0^{x-1} \varphi_{m-1}(t)(t+1)^{n+1}dt \\
&= \frac{1}{n+1}\left(x^{n+1}\varphi_m(x) - \mu_{n+1,m-1}^x + \sum_{k=0}^{n+1}\binom{n+1}{k}\mu_{k,m-1}^{x-1}\right).
\end{aligned}$$

2 Construction of the formula

Without loss of generality, we will construct the formula for approximate computation of the integral

$$\int_0^m f(x)dx, \tag{3}$$

where $m \in \mathbb{N}$ and $f(\cdot)$ is a given function.

Let $j \in \mathbb{N}$ be fixed and let $\widehat{f}(\cdot)$ be a projection of the function $f(\cdot)$ on the space $S_m^{(j)}$. Hence,

$$\widehat{f}(x) = \sum_{k \in \mathbb{Z}} c_k \varphi_m^{(j,k)}(x).$$

Having in mind that $supp \varphi_m^{(j,k)}(\cdot) = [2^{-j}k, 2^{-j}(k+m)], k \in \mathbb{Z}$, the last equality, for $x \in [0, m]$, reduces to

$$\widehat{f}(x) = \sum_{k=-m+1}^{2^j m-1} c_k \varphi_m^{(j,k)}(x) = 2^{\frac{j}{2}} \sum_{k=-m+1}^{2^j m-1} c_k \varphi_m(2^j x - k). \tag{4}$$

Now, in equality (4), we have to determine $(2^j + 1)m - 1$ unknown coefficients $c_k, -m+1 \le k \le 2^j m - 1$. To determine those coefficients, we will use the technique similar to the technique used in [4].

The first $2^j m + 1$ linear equations will be obtained from the interpolatory conditions at the diadic points (exactly those conditions were used in [4]), i.e. from the condition that the equalities

$$f\left(\frac{l}{2^j}\right) = \widehat{f}\left(\frac{l}{2^j}\right), 0 \le l \le 2^j m.$$

hold. In accordance with the definition of functions $\varphi_m^{(j,k)}(\cdot)$ and equality (4) we have

$$\begin{aligned}
\widehat{f}\left(\frac{l}{2^j}\right) &= 2^{\frac{j}{2}} \sum_{k=-m+1}^{2^j m-1} c_k \varphi_m(l-k) \\
&= 2^{\frac{j}{2}} \sum_{k=l-m+1}^{l-1} c_k \varphi_m(l-k) \\
&= 2^{\frac{j}{2}} \sum_{k=1}^{m-1} c_{k+l-m} \varphi_m(m-k) \\
&= 2^{\frac{j}{2}} \sum_{k=1}^{m-1} c_{k+l-m} \varphi_m(k), 0 \le l \le 2^j m.
\end{aligned}$$

It remains to choose additional $m-2$ conditions for determining the unknown coefficients. Those conditions, of course, should be chosen depending on a concrete situation, i.e. depending on the function $f(\cdot)$. In numerical examples, which are given in the last section of this paper, we impose the additional conditions for integer nodes to have multiplicity two. Namely, we claim that

$$f'(l) = \widehat{f}'(l), 0 \le l \le \left[\frac{m}{2}\right] - 2 \wedge \left[\frac{m}{2}\right] + 2 \le l \le m,$$

where $[a]$ denotes the largest integer, which is not greater than $a \in \mathbb{R}$.

Now, in accordance with the equalities (4) and (2), and with the definition of the functions $\varphi_m^{(j,k)}(\cdot)$ we have

$$\begin{aligned}
\widehat{f}'(l) &= 2^{\frac{3j}{2}} \sum_{k=-m+1}^{2^j m-1} c_k \left(\varphi_{m-1}(2^j l - k) - \varphi_{m-1}(2^j l - k - 1)\right) \\
&= 2^{\frac{3j}{2}} \sum_{k=2^j l-m+1}^{2^j l-1} c_k \left(\varphi_{m-1}(2^j l - k) - \varphi_{m-1}(2^j l - k - 1)\right) \\
&= 2^{\frac{3j}{2}} \sum_{k=0}^{m-2} c_{k+2^j l-m+1} \left(\varphi_{m-1}(m - 1 - k)\right. \\
&\quad \left. -\varphi_{m-1}(m - 1 - k - 1)\right) \\
&= 2^{\frac{3j}{2}} \sum_{k=0}^{m-2} c_{k+2^j l-m+1} \left(\varphi_{m-1}(k) - \varphi_{m-1}(k+1)\right), \\
&\quad 0 \le l \le \left[\frac{m}{2}\right] - 2 \wedge \left[\frac{m}{2}\right] + 2 \le l \le m.
\end{aligned}$$

So, the coefficients c_k, $-m+1 \le k \le 2^j m-1$, are a solution of the system of linear equations

$$\Phi c = f,$$

where

$$c = \begin{pmatrix} c_{-m+1} & c_{-m+2} & \cdots & c_{2^j m-2} & c_{2^j m-1} \end{pmatrix}^T,$$

then

$$f = \begin{pmatrix} f_1 & f_2 \end{pmatrix}^T,$$

with

$$f_1 = \left(f(0) \quad f\left(\tfrac{1}{2^j}\right) \quad \cdots \quad f\left(\tfrac{2^j m}{2^j}\right) \right)^T$$

and

$$f_2 = \left(f'(0) \quad \ldots \quad f'\left(\left[\tfrac{m}{2}\right] - 2\right) \quad f'\left(\left[\tfrac{m}{2}\right] + 2\right) \quad \ldots \quad f'(m) \right)^T.$$

The matrix Φ has the following form

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}.$$

Furthermore, the matrix Φ_1 has $2^j m + 1$ rows and $(2^j + 1)m - 1$ columns. The first row of this matrix is

$$2^{\frac{j}{2}} \begin{pmatrix} \varphi_m(1) & \varphi_m(2) & \dots & \varphi_m(m-2) & \varphi_m(m-1) & 0 & \dots & 0 \end{pmatrix},$$

while the other rows are obtained by rotation of this row to the right for $l, 1 \le l \le 2^j m$, positions. Matrix Φ_2 has $m - 2$ rows and $(2^j + 1)m - 1$ columns. The first row of this matrix is

$$2^{\frac{3j}{2}} \begin{pmatrix} -\varphi_{m-1}(1) & -\Delta\varphi_{m-1}(1) & \dots & -\Delta\varphi_{m-1}(m-2) & & & \\ & & & \varphi_{m-1}(m-2) & 0 & \dots & 0 \end{pmatrix}$$

($\Delta\varphi_{m-1}(k) = \varphi_{m-1}(k+1) - \varphi_{m-1}(k)$), while the other rows are obtained by rotation of this row to the right for $2^j l, 1 \le l \le \left[\frac{m}{2}\right] - 2 \wedge \left[\frac{m}{2}\right] + 2 \le l \le m$, positions.

After determining the coefficients c_k, we have to calculate

$$\int_0^m \widehat{f}(x)dx.$$

For the calculation of this integral, we will divide the sum (4) in three parts. Namely, let

$$\widehat{f}(x) = \sum_{k=-m+1}^{-1} c_k \varphi_m^{(j,k)}(x) + \sum_{k=0}^{(2^j-1)m} c_k \varphi_m^{(j,k)}(x) + \sum_{k=(2^j-1)m+1}^{2^j m - 1} c_k \varphi_m^{(j,k)}(x),$$

i.e. let

$$\begin{aligned} \widehat{f}(x) &= 2^{\frac{j}{2}} \Bigg(\sum_{k=-m+1}^{-1} c_k \varphi_m(2^j x - k) + \sum_{k=0}^{(2^j-1)m} c_k \varphi_m(2^j x - k) \\ &\quad + \sum_{k=(2^j-1)m+1}^{2^j m - 1} c_k \varphi_m(2^j x - k) \Bigg). \end{aligned} \tag{5}$$

If $-m + 1 \le k \le -1$ (the first sum on the right hand side of the equality (5)), then

$$\int_0^m \varphi_m(2^j x - k)dx = 2^{-j} \int_{-k}^m \varphi_m(x)dx = 2^{-j}\left(1 - \mu_{0,m}^{-k}\right),$$

and then

$$\int_0^m \sum_{k=-m+1}^{-1} c_k\varphi_m(2^jx-k)dx = \sum_{k=-m+1}^{-1} c_k \int_{-k}^m \varphi_m(x)dx$$

$$(6) \qquad = 2^{-j} \sum_{k=-m+1}^{-1} c_k\left(1-\mu_{0,m}^{-k}\right).$$

Furthermore, if $0 \le k \le (2^j-1)m$ (the second sum on the right hand side of the equality (5)), then

$$\int_0^m \varphi_m(2^jx-k)dx = 2^{-j}\int_0^m \varphi_m(x)dx = 2^{-j},$$

and then

$$(7) \qquad \int_0^m \sum_{k=0}^{(2^j-1)m} c_k\varphi_m(2^jx-k)dx = 2^{-j}\sum_{k=0}^{(2^j-1)m} c_k.$$

Finally, if $(2^j-1)m+1 \le k \le 2^jm-1$ (the last sum on the right hand side of the equality (5)), then

$$\int_0^m \varphi_m(2^jx-k)dx = 2^{-j}\int_0^{2^jm-k} \varphi_m(x)dx = 2^{-j}\mu_{0,m}^{2^jm-k},$$

and then

$$\int_0^m \sum_{k=(2^j-1)m+1}^{2^jm-1} c_k\varphi_m(2^jx-k)dx = \sum_{k=(2^j-1)m+1}^{2^jm-1} c_k\int_0^{2^jm-k} \varphi_m(x)dx$$

$$(8) \qquad = 2^{-j}\sum_{k=(2^j-1)m+1}^{2^jm-1} c_k\mu_{0,m}^{2^jm-k}.$$

The formula for the approximate computation of the integral (3) will be obtained from the equalities (5), (6), (7) and (8). Hence,

$$\int_0^m f(x)dx \approx \int_0^m \widehat{f}(x)dx = 2^{-\frac{j}{2}}\left(\sum_{k=-m+1}^{-1} c_k\left(1-\mu_{0,m}^{-k}\right)\right.$$

$$(9) \qquad \left. + \sum_{k=0}^{(2^j-1)m} c_k + \sum_{k=(2^j-1)m+1}^{2^jm-1} c_k\mu_{0,m}^{2^jm-k}\right).$$

At the end of this section let us say that we omitthe error estimation, since such estimation depends on the choice of the mentioned additional $m-2$ conditions for determining the coefficients c_k.

3 Comments and numerical examples

Basic disadvantage of the formula (9) is, of course, the fact that the coefficients c_k depend on the function under the integral sign, and in order to determine them, we have to solve the system of linear equations. However, this disadvantage has the formula constructed in [4], too. Also, when using the formula (9), we have to calculate shortened spline moments $\mu_{0,m}^k, 1 \leq k \leq m-1$. More precisely, because of the symmetry ($\mu_{0,m}^k = 1 - \mu_{0,m}^{m-k}$), we only have to calculate the shortened moments $\mu_{0,m}^k, 1 \leq k \leq \left[\frac{m}{2}\right]$. Since those moments do not depend on the function under the integral sign, this can not be considered as disadvantage of the proposed formula.

On the other hand, formula (9) has two important properties with respect to the formula proposed in [4]. First, since it is possible to recover polynomials of degree no greater than $m-1$ from the interpolating conditions, formula (9) integrates exactly polynomials of degree no greater than od $m-1$. Besides, as one can see from the following numerical examples, the formula (9) has significantly better accuracy than the formula proposed in [4].

Let us also say that the choice of the numerical examples was influenced by the numerical examples given in [4] and in [3]. In every example, we approximately computed an integral

$$\int_0^1 f(x)dx. \tag{10}$$

As a basic function for approximation of the function $f(\cdot)$, we used cardinal B-splines of order $3, 5$ i 7. Of course, by the appropriate transformation, we mapped the interval $[0,1]$ on the corresponding interval of integration. For each example, in the corresponding table, we give a relative error. The rows of every table correspond to the level of resolution (denoted by j), while the columns correspond to the order of the cardinal B-spline (denoted by m). As usual, the numbers in the square parentheses indicate decimal exponents.

Example 1 *We begin with the partial sums of the exponential series, i.e. we take in (10)*

$$f(x) = \sum_{i=0}^{s} \frac{x^i}{i!},$$

for $s \in \{3, 9, 15\}$. *Of course, for* $s = 3$ *and* $m = 5$ *or* $m = 7$, *exact values are already obtained for* $j = 0$.

	$s=3$	$s=9$		
j	$m=3$	$m=3$	$m=5$	$m=7$
0	0.100[−3]	0.201[−3]	0.162[−7]	0.473[−11]
1	0.000	0.427[−5]	0.105[−9]	0.911[−14]
2	0.000	0.268[−6]	0.164[−11]	0.967[−15]

	$s=15$		
j	$m=3$	$m=5$	$m=7$
0	0.201[−3]	0.162[−7]	0.504[−11]
1	0.427[−5]	0.106[−9]	0.101[−13]
2	0.268[−6]	0.165[−11]	0.108[−14]

Example 2 *Now we take in (10)* $f(x) = \exp(x)$.

j	$m=3$	$m=5$	$m=7$
0	0.201[−3]	0.162[−7]	0.504[−11]
1	0.427[−5]	0.106[−9]	0.101[−13]
2	0.268[−6]	0.165[−11]	0.108[−14]

The last two examples are the same as the first two examples in [4].

Example 3 *Hence, we take in (10)* $f(x) = \sqrt{x^2 - 4x + 13}$.

j	$m=3$	$m=5$	$m=7$
0	0.446[−5]	0.342[−9]	0.238[−12]
1	0.383[−8]	0.219[−11]	0.906[−15]
2	0.253[−9]	0.341[−13]	0.938[−16]

Example 4 *In the last example we take in (10)* $f(x) = \cos(x^2)$.

j	$m=3$	$m=5$	$m=7$
0	0.173[−3]	0.230[−5]	0.148[−8]
1	0.145[−5]	0.191[−7]	0.928[−11]
2	0.337[−7]	0.295[−9]	0.168[−11]

In the end, let us say that we can conclude from the given examples that the exactness of the proposed formula increases if the level of resolution and/or degree of cardinal B-spline increase (as we expect). Furthermore, we can imply that it is better to increase order of the cardinal B-spline, since dimension of the corresponding system of linear equations increase slower. For example, in the case $m = 5$ and $j = 2$ we have the system with 24 unknowns, but in the case

$m = 7$ and $j = 1$ we get better accuracy, even if we have to solve the system with 20 unknowns. Moreover, in the case $m = 7$ and $j = 0$, we have to solve the system with 13 unknowns while the accuracy is negligibly lower than in the case $m = 5$ and $j = 2$.

Finally, let us mention that in all examples we used cardinal B-splines as basis functions. These have the same size of support as Daubechies refinable functions which were used, as basis functions, in examples in [4]. Obtained results suggest that formula (9) is significantly better than the formula proposed in the mentioned paper in a sense that much better accuracy was obtained at much lower level of resolution.

References

[1] Chui, C., *An Introduction to Wavelets*, Academic Press, inc., Boston San Diego New York London Sydney Tokyo Toronto, 1992.

[2] Chui, C., *Wavelets: A mathematical tool for signal analysis*, Society for Industrial and Applied Mathematics, Philadelphia, 1997.

[3] Gautschi, W., Gori, L., Pitolli, F., *Gauss quadrature for refinable weight functions*, Appl. Comput. Harmon. Anal. 8 (2000), no. 3, pp. 249–257.

[4] Hashish, H., Behiry, S.H., El-Shamy N.A. *Numerical Integration Using Wavelets*, Appl. Math. Comput. (2009)

University of Sarajevo,
Department of Mathematics
Zmaja od Bosne 35
71000 Sarajevo
Bosnia and Herzegovina
zzlatko@pmf.unsa.ba