Mathematica
Balkanica

New Series Vol. 26, 2012, Fasc. 3–4

Iterative Function Systems With Affine Invariance Property

Elena Babače[1], *Sonja Gegovska-Zajkova*[2], *Ljubiša Kocić*[3]

Presented at MASSEE International Conference on Mathematics MICOM-2009

One of the most powerful methods for studying fractal sets consists in using the constructive approach through the Iterative Function Systems (IFS). On the other hand, the IFS has a substantial shortcoming: it does not allow prediction of the fractal attractor's shape. The improved concept, the AIFS (Affine invariant IFS) removes this weakness partially. Here, we contribute to the theory of AIFS by considering some elementary affine transformations, seen from the point of view of AIFS geometry. Several examples support the theory.

Key Words: fractal sets, iterative function systems, affine invariant IFS.

1. Introduction

Definition 1.1. The convex hull of m+1 affinly independent points is called m-dimensional simplex (m-simplex). Standard m-simplex is the convex hull of the points $\mathbf{e}_i \in \mathbf{R}^{m+1}$, $i = 1, 2, \ldots, m+1$, where $\mathbf{e}_i = (\delta_{ij})_{i,j=1}^{m+1}$, δ_{ij} is Kronecker's delta.

Definition 1.2. Let $\hat{\mathbf{P}}_m$ be an arbitrary m-simplex. The iterated function system (IFS) $\Omega(\hat{\mathbf{P}}_m) = \{\hat{\mathbf{P}}_m; \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m\}$, where $\mathbf{S}_i$ are row-stochastic matrices that define contractive linear mappings $\mathcal{L}_i : \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$, $i = 1, 2, \ldots, m$, such that $\mathcal{L}_i(\mathbf{x}) = \mathbf{S}_i^T \mathbf{x}$, is called affine invariant iterated function system (AIFS).

The Hutchinson operator for $\Omega(\hat{\mathbf{P}}_m)$ is defined as follows

$$W_{\Omega(\hat{\mathbf{P}}_m)}(\hat{\mathbf{P}}_m) = \bigcup_{i=1}^{n} \mathcal{L}_i(\hat{\mathbf{P}}_m).$$

The AIFS got its name because it possesses the *affine invariance property*. This means that if $\mathcal{A}$ is some affine mapping, then $\mathcal{A}(\mathrm{att}(\Omega(\hat{\mathbf{P}}_m))) = \mathrm{att}(\Omega(\mathcal{A}(\hat{\mathbf{P}}_m)))$, where the abbreviation "att" stands for the particular attractor. The key issue that endows the AIFSs with the affine invariance property is the change of the coordinate system; instead of Descartes system, we work in barycentric system w.r.t the affine base determined by the vertices of the simplex $\hat{\mathbf{P}}_m$. The affine invariance property is one of the most important properties in CAGD since it allows prediction of the attractor's shape. This subject is worked out in [5–7] as well as in [4], [1].

The usual notation for the standard m-simplex, $\mathrm{conv}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1}\} \subset \mathbf{R}^{m+1}$ is $\hat{\mathbf{T}}_m$. The AIFS $\Omega(\hat{\mathbf{T}}_m) = \{\hat{\mathbf{T}}_m; \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n\}$ is called *standard AIFS*. The affine space induced by the standard m-simplex is denoted by V^m, $\mathrm{V}^m = \mathrm{aff}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1}\} \subset \mathbf{R}^{m+1}$. V^m can be treated as a vector space (unless it is otherwise specified) with origin $\mathbf{e}_{m+1}$ provided that it is parallel with the vector space $\mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{m+1}\} \subset \mathbf{R}^{m+1}$, $\mathbf{u}_i = \mathbf{e}_i - \mathbf{e}_{i+1}, i = 1, 2, \ldots, m$.

The set of linearly independent vectors that spans the vector space V^m is neither orthogonal nor normalized, since $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \left\{ \begin{array}{ll} 1, & i \neq j \\ 2, & i = j \end{array} \right.$. By means of the Gram-Schmidt procedure the set $\mathcal{U}_m$ is transformed into an orthonormal basis $\mathcal{V}_m = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \ldots\,, \mathbf{v}_m\}$ of V^m. This new orthonormal basis is represented by the $m \times (m+1)$ matrix

$$\mathbf{V}_m = \begin{bmatrix} v_{11} & v_{12} & & v_{1,m+1} \\ v_{21} & v_{22} & & v_{2,m+1} \\ & & \ddots & \\ v_{m,1} & v_{m,2} & & v_{m,m+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \\ \\ \mathbf{v}_m^{\mathrm{T}} \end{bmatrix}. \tag{1.1}$$

The relation between the areal coordinates of a point $\mathbf{r} \in \mathrm{V}^m$ w.r.t. the affine basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1}\}$, $\mathbf{r} = [\rho_1\, \rho_2\, \ldots\, \rho_{m+1}]^{\mathrm{T}}$, $(\rho_{m+1} = \sum_{i=1}^{m} \rho_i)$ and the coordinates in the orthonormal basis $\mathcal{V}_m$, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ of the same point is given by

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \\ \rho_m \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & & v_{m,1} \\ v_{12} & v_{22} & & v_{m,2} \\ & & \ddots & \\ v_{1,m} & v_{2,m} & & v_{m,m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_m \end{bmatrix}, \quad \text{i.e.} \quad \bar{\mathbf{r}} = \mathbf{Q}_m \mathbf{x}, \tag{1.2}$$

where $\mathbf{Q}_m = (\overline{\mathbf{V}}_m)^{\mathrm{T}}$, $\overline{\mathbf{V}}_m$ is the truncated matrix $\mathbf{V}_m$, i.e. the matrix $\mathbf{V}_m$ given by (1.1), with the last column dropped and $\bar{\mathbf{r}}$ is a truncated vector $\mathbf{r}$. The inverse transformation of (1.2) exists and $\mathbf{x} = \mathbf{Q}_m^{-1}\bar{\mathbf{r}}$ ([8]). In [8], the items of the matrices $\mathbf{Q}_m = [q_{ij}]_{i,j=1}^m$ and $\mathbf{Q}_m^{-1} = [q'_{ij}]_{i,j=1}^m$ are explicitly calculated,

$$q_{ij} = \begin{cases} \dfrac{-1}{\sqrt{j(j+1)}}, & i < j \\ \sqrt{\dfrac{i+1}{i}}, & i = j \\ 0, & i > j \end{cases}, \qquad q'_{ij} = \begin{cases} \dfrac{1}{\sqrt{i(i+1)}}, & i < j \le m \\ \sqrt{\dfrac{i+1}{i}}, & i = j \le m \\ 0, & m \ge i > j \end{cases}.$$

2. Relation between classical IFS and standard AIFS, both defined on V^m

Wishing to join the deeply developed theory of the classical IFS described in [2] and practically useful standard AIFS, we tried to find a relation between them.

Let $\mathbf{x} \in \mathrm{V}^m$ be transformed by an affine mapping $w : \mathbf{x} \mapsto \mathbf{Ax} + \mathbf{b}$ defined on V^m ($\mathbf{A}$ is an $m \times m$ matrix and $\mathbf{b}$ is a translation vector), so that

$$\mathbf{x}' = \mathbf{Ax} + \mathbf{b}. \tag{2.1}$$

By (1.2), $\bar{\mathbf{r}}' = \mathbf{Q}_m\mathbf{x}'$, which by (2.1) gives $\bar{\mathbf{r}}' = \mathbf{Q}_m(\mathbf{Ax} + \mathbf{b})$. Since $\mathbf{x} = \mathbf{Q}_m^{-1}\bar{\mathbf{r}}$, the penultimate formula is changed into

$$\bar{\mathbf{r}}' = \mathbf{Q}_m(\mathbf{A}\mathbf{Q}_m^{-1}\bar{\mathbf{r}} + \mathbf{b}) = \mathbf{Q}_m\mathbf{A}\mathbf{Q}_m^{-1}\bar{\mathbf{r}} + \mathbf{Q}_m\mathbf{b}. \tag{2.2}$$

On the other hand, it is easy to see that, if $\mathbf{r} = [\rho_1\,\rho_2\,\ldots\,\rho_{m+1}]^T$ is the "complete" areal coordinate vector, then $\bar{\mathbf{r}}$ and $\mathbf{r}$ are related to each other as follows:

$$\bar{\mathbf{r}} = \mathbf{K}_m \cdot \mathbf{r}, \tag{2.3}$$

$$\mathbf{r} = \mathbf{J}_m \cdot \bar{\mathbf{r}} + \mathbf{e}_{m+1}, \tag{2.4}$$

where

$$\mathbf{K}_m = [\mathbf{I}_m \,|\, \mathbf{0}] \quad \text{and} \quad \mathbf{J}_m = \left[\begin{array}{c} \mathbf{I}_m \\ \hline -\mathbf{1} \end{array}\right], \tag{2.5}$$

are block-matrices obtained by the identity matrix by adding a zero-column or $(-\mathbf{1})$-row. Also $\mathbf{e}_{m+1}$ is the ultimate unit vector in the $\mathbf{R}^{m+1}$ Descartes system. So, combining (2.2) with (2.3) and (2.4) gives

$$\mathbf{r}' = \tilde{\mathbf{A}}\mathbf{r} + \tilde{\mathbf{b}}, \tag{2.6}$$

where

$$\tilde{\mathbf{A}} = \mathbf{J}_m\mathbf{Q}_m\mathbf{A}\mathbf{Q}_m^{-1}\mathbf{K}_m$$

is $(m+1)\times(m+1)$ matrix, and

$$\tilde{\mathbf{b}} = \mathbf{J}_m\mathbf{Q}_m\mathbf{b} + \mathbf{e}_{m+1}$$

is $(m+1)$ - dimensional vector. Now the following theorem is valid.

Theorem 2.1. *The row stochastic matrix $\mathbf{S}$, defining the linear transformation $\mathbf{r} \mapsto \mathbf{S}^T\mathbf{r}$, $\mathbf{r} \in \mathrm{V}^m$, that is equivalent to a given affine mapping $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{b}$, $\mathbf{x} \in \mathrm{V}^m$, is given by*

$$\mathbf{S}^T = \tilde{\mathbf{A}} + [\tilde{\mathbf{b}}\,\tilde{\mathbf{b}}\,\ldots\,\tilde{\mathbf{b}}], \tag{2.9}$$

where $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{b}}$ are given by (2.7) *and* (2.8).

P r o o f. Setting $\mathbf{r} = \mathbf{e}_i$, $i = 1, 2, \ldots, m+1$ in (2.6), the standard simplex vertices $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1}\}$ are transformed into $\{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_{m+1}\}$,

$$\boldsymbol{\rho}_i = \tilde{\mathbf{A}}\mathbf{e}_i + \tilde{\mathbf{b}}, \quad i = 1, 2, \ldots, m+1,$$

or, in matrix form

$$[\boldsymbol{\rho}_1\,\boldsymbol{\rho}_2\,\ldots\,\boldsymbol{\rho}_{m+1}] = \tilde{\mathbf{A}}[\mathbf{e}_1\,\mathbf{e}_2\,\ldots\,\mathbf{e}_{m+1}] + [\tilde{\mathbf{b}}\,\tilde{\mathbf{b}}\,\ldots\,\tilde{\mathbf{b}}].$$

Since $[\boldsymbol{\rho}_1\,\boldsymbol{\rho}_2\,\ldots\,\boldsymbol{\rho}_{m+1}] = \mathbf{S}^T$ and $[\mathbf{e}_1\,\mathbf{e}_2\,\ldots\,\mathbf{e}_{m+1}] = \mathbf{I}_{m+1}$, (2.9) follows. ∎

Theorem 2.2. *Given the row stochastic matrix $\mathbf{S}$, defining the linear mapping $\mathbf{r} \mapsto \mathbf{S}^T\mathbf{r}$, $\mathbf{r} \in \mathrm{V}^m$. The corresponding linear transformation $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{b}$, $\mathbf{x} \in \mathrm{V}^m$, is then given by*

$$\mathbf{A} = \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{J}_m\mathbf{Q}_m, \quad \mathbf{b} = \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{e}_{m+1}. \tag{2.10}$$

P r o o f. Let $\mathbf{r}' = \mathbf{S}^T\mathbf{r}$. Combining it with (2.3) gives $\bar{\mathbf{r}'} = \mathbf{K}_m\mathbf{S}^T\mathbf{r}$. On the other hand, $\mathbf{x}' = \mathbf{Q}_m^{-1}\bar{\mathbf{r}'}$, so that $\mathbf{x}' = \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{r}$. By inserting (2.4) one gets

$$\mathbf{x}' = \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{J}_m\bar{\mathbf{r}} + \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{e}_{m+1}$$

and by (1.2) $\mathbf{x}' = (\mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{J}_m\mathbf{Q}_m)\mathbf{x} + \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{e}_{m+1}$, which gives (2.10). ■

3. Examples The following examples illustrate the conclusions given in the theorems.

Example 3.1. If $\mathbf{S}$ is the matrix

$$\mathbf{S} = \begin{bmatrix} 1 + \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{6}} & \sqrt{\frac{2}{3}}b & -\frac{a}{\sqrt{2}} - \frac{b}{\sqrt{6}} \\ \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{6}} & 1 + \sqrt{\frac{2}{3}}b & -\frac{a}{\sqrt{2}} - \frac{b}{\sqrt{6}} \\ \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{6}} & \sqrt{\frac{2}{3}}b & 1 - \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{6}} \end{bmatrix}$$

Then, by the Theorem 1.2., the pair $(\mathbf{A}, \mathbf{b})$ can be determined:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} a \\ b \end{bmatrix},$$

which means that the transformation defined in V^m, by the linear mapping associated to $\mathbf{S}$, is a translation. On the Fig. 1, the standard simplex (whose vertices have $\mathcal{V}_m$-coordinates: $(\sqrt{2}, 0), (\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}), (0,0))$ is translated by the vector $\mathbf{b}$. The same image of the standard simplex is gained if it is mapped with the linear transformation associated to $\mathbf{S}$, but in this case, we work with the areal coordinates w.r.t. the vertices of the standard simplex.

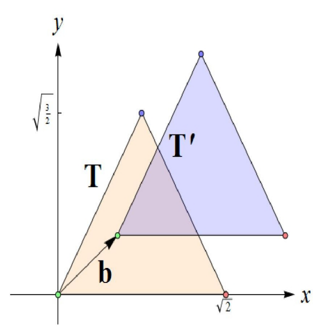

Figure 1: $\mathbf{T}'$ is obtained from $\mathbf{T}$ either by the mapping $\mathbf{r} \mapsto \mathbf{S}^T\mathbf{r}$, or by the mapping $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{b}$, defined in example 1.

Example 3.2. If the pair $(\mathbf{A}, \mathbf{b})$ determines symmetry w.r.t. arbitrary line $p : y = kx + r$,

$$\mathbf{A} = \frac{1}{1+k^2}\begin{bmatrix} 1-k^2 & 2k \\ 2k & k^2-1 \end{bmatrix}, \quad \mathbf{b} = \frac{2k}{1+k^2}\begin{bmatrix} -k \\ 1 \end{bmatrix}, \qquad ([3])$$

then, according to the Theorem 2.2., the matrix $\mathbf{S}$ obtains the form

$$\mathbf{S}=\begin{bmatrix} \frac{3-2\sqrt{3}k-3k^2-\sqrt{6}r-3\sqrt{2}kr}{3(1+k^2)} & \frac{4\sqrt{3}k+2\sqrt{6}r}{3(1+k^2)} & -\frac{(\sqrt{3}-3k)(2k+\sqrt{2}r)}{3(1+k^2)} \\ \frac{3+2\sqrt{3}k-3k^2-\sqrt{6}r-3\sqrt{2}kr}{3(1+k^2)} & \frac{-3+2\sqrt{3}k+3k^2+2\sqrt{6}r}{3(1+k^2)} & \frac{3-4\sqrt{3}k+3k^2-\sqrt{6}r+3\sqrt{2}kr}{3(1+k^2)} \\ -\frac{\sqrt{2}(\sqrt{3}+3k)r)}{3(1+k^2)} & \frac{2\sqrt{\frac{2}{3}}r}{1+k^2} & \frac{3+3k^2-\sqrt{6}r+3\sqrt{2}kr}{3(1+k^2)} \end{bmatrix}$$

Again, we check the influence of these mappings on the standard simplex. The effect is shown on the Fig.2, it is the same for both mappings. First time we work with $\mathcal{V}_m$-coordinates and the second, with areal coordinates w.r.t. the vertices of the standard simplex.

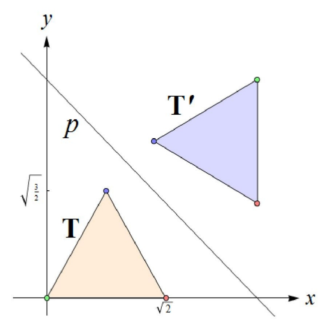

Figure 2: $\mathbf{T}'$ and $\mathbf{T}$ are symmetrical w.r.t the line p

4. Conclusion The Theorems 2.1 and 2.2 give a relation between the IFS defined on V^m and standard AIFS. Two examples of typical transformations, translation and symmetry w.r.t. arbitrary line, are given to depict the theorems. Working with the AIFS gives opportunity for predictable modeling of attractor, which is especially important when the attractor has fractal structure.

References

[1] E. Babače. *Contributions to iterated function systems with affine invariance property.* // Doctoral dissertation, Skopje, 2009.
[2] M. F. Barnsley. *Fractals Everywhere*, Academic Press, 1993.
[3] Lj. M. Kocić. *Geometric modeling.* (In Serbian), (in print).
[4] Lj. M. Kocić, E. Babače, S. Gegovska-Zajkova. Introducing of affine invariance to IFS. // *International conference FAA 2009*, (in print).
[5] Lj. M. Kocić, A. C. Simoncelli. Shape predictable IFS representations. // M. M. Novak (ed.), *Emergent Nature,* World Scientific, 2002, 435–436.

[6] Lj. M. Kocić, A. C. Simoncelli. Cantor Dust by AIFS. // FILOMAT (Niš), **15**, 265–276.

[7] Lj. M. Kocić, A. C. Simoncelli. Stochastic approach to affine invariant IFS. // M. Hruskova, P. Lachout, J. A. Visek (eds) *Prague Stochastics'98*, Proc. 6-th Prague Symp., Aug. 23–28, Vol II. Charles Univ. and Academy of Sciences of Czech Republic, Union of Czech Mathematicians and Physicists, Prague, 1998.

[8] Lj. M. Kocić, S. Gegovska-Zajkova, E. Babače. Orthogonal decompostiton of fractal sets. // *International Conference Approximation and Computation, dedicated to 60-th anniversary of Prof. Gradimir Milovanović, August 25–29, 2008*, Nish, Serbia (to appear).

[1] *Ss Cyril and Methodius University*
Faculty of Electrical Engineering and Information Tehcnologies
P.O. Box 574
Skopje 1000, MACEDONIA
E-MAIL: elena.babace@feit.ukim.edu.mk

Received 04.02.2010

[2] *University of Niš*
Faculty of Electronic Engineering
POBox 73
Niš 18 000, SERBIA
E-MAIL: ljubisa.kocic@elfak.ni.ac.yu

[3] *Ss Cyril and Methodius University*
Faculty of Electrical Engineering and Information Tehcnologies
P.O. Box 574
Skopje 1000, MACEDONIA
E-MAIL: szajkova@feit.ukim.edu.mk