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Two Theorems for the Vekua Equation

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In this paper are proved two theorems for the Vekua equation. In the first theorem the canonical Vekua equation $\frac{\hat{d}W}{d\bar{z}} = B(z)\overline{W} + F(z)$, with antyanalytic coefficients in the area $D \subseteq \mathbb{C}$ with appropriate substitution is reduced to basic Vekua equation $\frac{\hat{d}\omega}{d\bar{z}} = \overline{B}(z)\overline{\omega}$ with analytic coefficient. This result is used for finding the general solution of one special, but general enough, class of the canonical Vekua equation $\frac{\hat{d}W}{d\bar{z}} = a\bar{z}^m\overline{W} + F(z)$. In the second theorem it is proved that if $W_j = W_j(z), j = 1, 2$ are two solutions of the Vekua equation $\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, than $W = \frac{\lambda_1 W_1 + \lambda_2 W_2}{\lambda_1 + \lambda_2}$ is also a solution of the mentioned equation. Here, some corollaries of this theorem are formulated.

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1. Introduction

G. V. Kolosov [1] in 1909 introduced the expressions

$$\frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{dz} \tag{1}$$

and

$$\frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{d\bar{z}} \tag{2}$$

known as operatory derivatives of the function $W = W(z) = u(x, y) + iv(x, y)$ by the variable $z = x + iy$ and $\bar{z} = x - iy$ correspondingly. The operatory rules for this derivetives are formulated and proved in the monograph of G. N. Položii

[2] page 18–31. In the mentioned monograph are also defined so cold operatory integrals

$$\overset{\wedge}{\int} f(z)dz = F(z) + C \quad (\frac{\hat{d}F}{dz} = f(z) \text{ in } D \wedge \frac{\hat{d}C}{dz} = 0) \tag{3}$$

and

$$\overset{\wedge}{\int} f(z)d\bar{z} = F(z) + \widetilde{C} \quad (\frac{\hat{d}F}{d\bar{z}} = f(z) \text{ in } D \wedge \frac{\hat{d}\widetilde{C}}{d\bar{z}} = 0) \tag{4}$$

by z and $\bar{z}$ correspondingly. Here their operatory rules are proven page 32-41. We should notice that the Cauchy-Riemann conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Leftrightarrow \frac{\hat{d}W}{\partial \bar{z}} = 0$ in the class of the functions $W = W(z) = u(x,y) + iv(x,y)$, whose real and imaginary parts have continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in the area $D \subseteq \mathbb{C}$, defines the analyticity of the function $W = W(z)$ in D.

In the second half of the XIX century E.Beltrami (1868) and E.Picard (1891) came to the idea to replace the Cauchy-Riemann conditions with more general systems of partial differential equations according to the unknown functions $u = u(x,y)$ and $v = v(x,y)$ and they cold the functions $W = W(z) = u(x,y) + iv(x,y)$, whose real and imaginary part is a solution of the mentioned systems, generalized analytic functions [3].

I. N. Vekua [4] – [5], introduced the operatory differential equation

$$\frac{\hat{d}W}{d\bar{z}} = A(z)W + B(z)\overline{W} + F(z) \tag{5}$$

which is complex writing to the system of partial differential equations

$$\begin{cases} u'_x - v'_y = a(x,y)u + b(x,y)v + f(x,y) \\ u'_y + v'_x = c(x,y)u + d(x,y)v + g(x,y) \end{cases} \tag{6}$$

The equation (5), known as Vekua equation, depending on the coefficients $A = A(z)$, $B = B(z)$ and $F = F(z)$ defines various classes of generalized analytic functions [2]: In the case when $A = A(z) \equiv 0$, $B = B(z) \equiv 0$ and $F = F(z) \equiv 0$ in D the equation (5), as we already mentioned, defines the class of the analytic functions; In the case when $A = A(z) \equiv 0$ and $F = F(z) \equiv 0$ the equation (5) defines the so cold class of generalized analytic functions of third class or $(r + is)$-analytic functions; In the case when $F = F(z) \equiv 0$ (5) defines the class of the generalized analytic functions of IV class; In the case when $B = B(z) \equiv 0$ in D (5) reduces to the so cold areolar linear differential equation which is solvable with quadratures $W = \exp\int A(z)d\bar{z}\left[\Phi(z) + \int F(z)\exp(-\int A(z)d\bar{z})d\bar{z}\right]$ i.e. (5) is a type of a Vekua equation which is finite integrable [5]. This kind of type of Vekua equation, i.e. finite integrable is the equation (5) in the case when $B = B(z) \equiv 0$ and $F = F(z) \equiv 0$ in D. Finaly, lets mention that in the general case the Vekua equation (5) is not quadrature solvable i.e. it is not finite integrable (case when $B = B(z) \not\equiv 0$ in D).

In this work we are considering the no homogeneous equation

$$\frac{\hat{d}W}{d\bar{z}} = B(z)\overline{W} + F(z), \quad z \in D \tag{7}$$

Theorem 1. *If $B = B(z)$ and $F = F(z)$ are antyanalytic functions of z, then with the substitute*

$$\omega = \bar{B}(z)W + \bar{F}(z) \tag{8}$$

the no homogeneous equation (7) is transformed in the basic Vekua equation with analytic coeffitients i.e. in the equation

$$\frac{\hat{d}\omega}{d\bar{z}} = \bar{B}(z)\bar{\omega}. \tag{9}$$

Proof. $B = B(z)$ and $F = F(z)$ are antyanalytic functions of $z \in D \Rightarrow$ that $\bar{B} = \bar{B}(z)$ and $\bar{F} = \bar{F}(z)$ are analytic functions in D i.e.

$$\frac{\hat{d}\bar{B}(z)}{d\bar{z}} = 0 \text{ and } \frac{\hat{d}\bar{F}(z)}{d\bar{z}} = 0, \quad z \in D. \tag{10}$$

From the substitute (8), according to the operatory rules for the operatory derivative from $\bar{z}$ we have

$$\frac{\hat{d}\omega}{d\bar{z}} = \frac{\hat{d}\bar{B}(z)}{d\bar{z}}W + \bar{B}(z)\frac{\hat{d}W}{d\bar{z}} + \frac{\hat{d}\bar{F}(z)}{d\bar{z}},$$

and according to (10)

$$\frac{\hat{d}\omega}{d\bar{z}} = \bar{B}(z)\frac{\hat{d}W}{d\bar{z}},$$

where from

$$\frac{\hat{d}W}{d\bar{z}} = \frac{1}{\bar{B}(z)}\frac{\hat{d}\omega}{d\bar{z}}. \tag{11}$$

According to (8) and (11) the equation (7), previously written in the shape

$$\frac{\hat{d}W}{d\bar{z}} = \overline{\bar{B}\left(z\right)W + \bar{F}\left(z\right)},$$

is transformed in

$$\frac{1}{\bar{B}\left(z\right)}\frac{\hat{d}\omega}{d\bar{z}} = \bar{\omega},$$

where from we get the equation (9). ∎

Corollary. *The no homogeneous Vekua equation*

$$\frac{\hat{d}W}{d\bar{z}} = a\bar{z}^m\overline{W} + F\left(z\right), \tag{7'}$$

where $a \in \mathbb{C}$, $m \in \mathbb{N}$ and $F = F(z)$ is antianalytic function from $z \in D \subseteq \mathbb{C}$, with the substitution:

$$\omega = \bar{a}z^m W + \bar{F}(z) \tag{8'}$$

is transformed to the basic Vekua equation

$$\frac{\hat{d}\omega}{d\bar{z}} = \bar{a}z^m\bar{\omega} \tag{9'}$$

by the unknown function $\omega = \omega\left(z\right)$. In the paper [7], with so cold method of the areolar series is found the general solution (9') in the shape

$$\begin{aligned}\omega\left(z\right) = \Phi\left(z\right) + \bar{a}z^m \overset{\wedge}{\int}\overline{\Phi}\left(z\right)d\bar{z} + \\ + z^m\sum_{n=1}^{\infty}\tfrac{|a|^{2n}}{(m+1)^n n!}\Big[\bar{z}^{n(m+1)}\underbrace{\overset{\wedge}{\int} z^m dz \overset{\wedge}{\int} z^m dz \ldots \overset{\wedge}{\int} z^m dz \overset{\wedge}{\int}}_{n-\text{integrals}}\Phi\left(z\right)dz + \\ + \bar{a}z^{n(m+1)}\underbrace{\overset{\wedge}{\int}\bar{z}^m d\bar{z}\overset{\wedge}{\int}\bar{z}^m d\bar{z}\ldots\overset{\wedge}{\int}\bar{z}^m d\bar{z}\overset{\wedge}{\int}}_{n+1-\text{integrals}}\overline{\Phi}\left(z\right)d\bar{z}\Big]\end{aligned} \tag{12}$$

Here $\Phi = \Phi\left(z\right)$ is an arbitrary analytic function from $z = x+iy$ in the considered area $D \subseteq \mathbb{C}$ in the role of an integral constant. According to the substitution

(8‘), the general solution of the no homogeneous equation (7’) is

$$W=\frac{1}{\bar{a}z^m}\left\{\Phi(z)+\bar{a}z^m\hat{\int}\overline{\Phi}(z)\,d\bar{z}+\right.$$
$$+z^m\sum_{n=1}^{\infty}\frac{|a|^{2n}}{(m+1)^n n!}\Big[\bar{z}^{n(m+1)}\underbrace{\hat{\int}z^m dz\hat{\int}z^m dz\ldots\hat{\int}z^m dz\hat{\int}\Phi(z)\,dz}_{n-\text{integrals}}+ \qquad (13)$$
$$\left.+\bar{a}z^{n(m+1)}\underbrace{\hat{\int}\bar{z}^m d\bar{z}\hat{\int}\bar{z}^m d\bar{z}\ldots\hat{\int}\bar{z}^m d\bar{z}\hat{\int}\overline{\Phi}(z)\,d\bar{z}}_{n+1-\text{integrals}}\Big]-\overline{F}(z)\right\}.$$

In the paper [8] it is shown that the solution (12) of the basic Vekua equation (9’) can be written through single parameter integrals

$$\omega=\Phi(z)+\bar{a}z^m\hat{\int}\overline{\Phi}(z)\,d\bar{z}+$$
$$+z^m\left\{\hat{\int}\left[\sum_{n=1}^{\infty}\frac{|a|^{2n}(z-\zeta)^{(m+1)(n-1)}\bar{z}^{n(m+1)}}{(m+1)^n n!(m+1)^{n-1}(n-1)!}\right]\Phi(\zeta)\,d\zeta+\right.$$
$$\left.+\bar{a}\hat{\int}\left[\sum_{n=1}^{\infty}\frac{|a|^{2n}\left(\bar{z}-\bar{\zeta}\right)^{(m+1)n}z^{n(m+1)}}{(m+1)^n n!(m+1)^n n!}\right]\overline{\Phi}(\zeta)\,d\bar{\zeta}\right\}.$$

This means that the solution (13) of the no homogeneous Vekua equation (7’) with coefficient exponential function $a\bar{z}^m$, can be written in the next shape

$$W=\frac{1}{\bar{a}z^m}\left\{\Phi(z)+\bar{a}z^m\hat{\int}\overline{\Phi}(z)\,d\bar{z}+\right.$$
$$+z^m\hat{\int}\left[\sum_{n=1}^{\infty}\frac{|a|^{2n}(z-\zeta)^{(m+1)(n-1)}\bar{z}^{n(m+1)}}{(m+1)^n n!(m+1)^{n-1}(n-1)!}\right]\Phi(\zeta)\,d\zeta+ \qquad (14)$$
$$\left.+\bar{a}z^m\hat{\int}\left[\sum_{n=1}^{\infty}\frac{|a|^{2n}\left(\bar{z}-\bar{\zeta}\right)^{(m+1)n}z^{n(m+1)}}{((m+1)^n n!)^2}\right]\overline{\Phi}(\zeta)\,d\bar{\zeta}\;-\overline{F}(z)\right\}.$$

Let us notice that the sum

$$S=\sum_{n=1}^{\infty}\frac{|a|^{2n}\left(\left(\bar{z}-\bar{\zeta}\right)z\right)^{(m+1)n}}{((m+1)^n n!)^2}=$$
$$=\sum_{n=1}^{\infty}\frac{\left[|a|^{\frac{2}{(m+1)}}\left(\bar{z}-\bar{\zeta}\right)z\right]^{(m+1)n}}{((m+1)^n n!)^2}=\sum_{n=1}^{\infty}\frac{u^{(m+1)n}}{((m+1)^n n!)^2}=S(u),$$
$$u=|a|^{\frac{2}{(m+1)}}\left(\bar{z}-\bar{\zeta}\right)z,$$

in the second row in (14), is a solution of the common differential equation of second order $u\frac{d^2S}{du^2}+\frac{dS}{du}=u^m\left(S\left(u\right)+1\right)$.

Note 1: The equation (7') is general enough class of a Vekua equation and for her we can find a general solution, because it contains arbitrary parameters $a\in\mathbb{C}$, $m\in\mathbb{N}$ and an arbitrary antianalytic function $F=F\left(z\right)$.

Note 2: The results of this paper are generalization of the results in the paper [9]. Here, if in the equation (7) of this paper, we put $B\left(z\right)=a\in\mathbb{C}$ we get a no homogeneous Vekua equation which is studied in the paper [9].

Theorem 2: *If* $W_j=W_j\left(z\right),\quad j=1,2$ *are solutions of the complex Vekua differential equation (5) and if* $\lambda_1,\lambda_2\in\mathbb{R}$ *are an arbitrary real constants, than the function* $W_0=\frac{\lambda_1W_1+\lambda_2W_2}{\lambda_1+\lambda_2}$ *is a solution of the same equation as well.*

Proof. From the conditions of the theorem, i.e. from the fact that $W_1=W_1\left(z\right)$ and $W_2=W_2\left(z\right)$ are solutions of the equation (5) we have

$$\frac{\hat{d}W_1}{d\bar{z}}=AW_1+B\bar{W}_1+F$$

and (15)

$$\frac{\hat{d}W_2}{d\bar{z}}=AW_2+B\bar{W}_2+F$$

If the first equation in (15) multiplies with $\lambda_1\in\mathbb{R}$, and the second one with $\lambda_2\in\mathbb{R}$, we get:

$$\begin{aligned}\lambda_1\tfrac{\hat{d}W_1}{d\bar{z}}&=A\left(\lambda_1W_1\right)+B\left(\lambda_1\overline{W_1}\right)+F\lambda_1\\ \lambda_2\tfrac{\hat{d}W_2}{d\bar{z}}&=A\left(\lambda_2W_2\right)+B\left(\lambda_2\overline{W_2}\right)+F\lambda_2\end{aligned}\tag{16}$$

i.e.

$$\begin{aligned}\tfrac{\hat{d}(\lambda_1W_1)}{d\bar{z}}&=A\left(\lambda_1W_1\right)+B\overline{\left(\lambda_1W_1\right)}+F\lambda_1\\ \tfrac{\hat{d}(\lambda_2W_2)}{d\bar{z}}&=A\left(\lambda_2W_2\right)+B\overline{\left(\lambda_2W_2\right)}+F\lambda_2\end{aligned}\tag{17}$$

Adding the equations (17) and according to the operatory rules for the operatory derivative from $\bar{z}$ we get:

$$\frac{\hat{d}}{d\bar{z}}\left(\lambda_1W_1+\lambda_2W_2\right)=A\left(\lambda_1W_1+\lambda_2W_2\right)+B\overline{\left(\lambda_1W_1+\lambda_2W_2\right)}+F\left(\lambda_1+\lambda_2\right)\tag{18}$$

i.e.

$$\frac{\hat{d}}{d\bar{z}}\left(\frac{\lambda_1W_1+\lambda_2W_2}{\lambda_1+\lambda_2}\right)=A\left(\frac{\lambda_1W_1+\lambda_2W_2}{\lambda_1+\lambda_2}\right)+B\left(\overline{\frac{\lambda_1W_1+\lambda_2W_2}{\lambda_1+\lambda_2}}\right)+F,$$

where we divided (18) with the expression $\lambda_1 + \lambda_2 \neq 0$. This means that the function $W_0 = \frac{\lambda_1 W_1 + \lambda_2 W_2}{\lambda_1 + \lambda_2}$ is a solution of the Vekua equation (5) which we wanted to prove. ■

Corollary 1: *If $W_1 = W_1(z)$ and $W_2 = W_2(z)$ are solutions of the Vekua equation (5), than their "arithmetic middle value" is a solution of the same equation, too.*

Corollary 2: *If $W_1 = W_1(z)$ and $W_2 = W_2(z)$ are solutions of the homogeneous Vekua equation (5) where $F = F(z) \equiv 0$ in the considered area $D \subseteq \mathbb{C}$ and if $\lambda_1, \lambda_2 \in \mathbb{R}$ are an arbitrary real constants, than $W_0 = \lambda_1 W_1 + \lambda_2 W_2$ is a solution of the mentioned equation, too.*

We get this immediately from the equation (18).

Corollary 3: *If $W_1 = W_1(z)$ and $W_2 = W_2(z)$ are solutions of the Vekua equation (5) and $\lambda \in \mathbb{R}$ is an arbitrary real constant, than the function $W_0 = = \lambda W_1 + (1 - \lambda) W_2$ is a solution of the mentioned equation, too. The proof follows immediately from the theorem 2, if we put $\lambda_1 = \lambda, \quad \lambda_2 = 1 - \lambda$.*

Note 1: According to the theorem 2, if we know two particular solutions of the equation (5), than we can construct a family of solutions of the mentioned equation. Every affine combination of these solutions is a solution of the no homogeneous Vekua equation, and in the case of a homogeneous Vekua equation, i.e. if $F \equiv 0$, than this family is a linear combination of the given two solutions.

Note 2: At the beginning of this paper it is stressed that if $F \equiv 0$, the Vekua equation defines the class of the generalized analytic functions IV class. This means that the Corollary 2 of the theorem 2 concerns this class of functions.

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