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One Problem for the Generalized Complex Differential Vekua Equation

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In the paper are examined the conditions that are necessary for the generalized complex differential Vekua equation to be finite integrable.

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1. Introduction

In his most popular monograph [1] I.Vekua studied to the details the elliptic system of partial equations given below

$$\begin{aligned} u'_x - v'_y &= a(x,y)\,u + b(x,y)\,v + f(x,y) \\ u'_y + v'_x &= c(x,y)\,u + d(x,y)\,v + g(x,y) \end{aligned} \tag{1.1}$$

where $a(x,y)\,,\ b(x,y)\,, c(x,y)\,,\ d(x,y)\,, f(x,y)$ and $g(x,y)$ are given continuous functions in some simple, single connected area Ω. This system has a big theoretic and practical meaning, as well as the many applications in various areas of mechanics. If the second equation in (1.1) multiplies with i and adds to the first we get the so cold Vekua complex differential equation

$$U'_{\bar{z}} = MU + N\overline{U} + L \tag{1.2}$$

where

$$\begin{aligned} &M(z,\bar{z}) = (a + d + ic - ib)/4, \\ &N(z,\bar{z}) = (a - d + ic + ib)/4, \\ &L(z,\bar{z}) = (f + ig)/2, \\ &U(z,\bar{z}) = u + iv. \end{aligned}$$

With the substitution $U = wU_0$ where U_0 is one regular particular solution of the equation $U'_{\bar{z}} = MU$ which can be easily determined (see [2]) the equation (1.2), transfers in the canonical shape

$$w'_{\bar{z}} = A\bar{w} + B \tag{1.3}$$

where $A = \left(N\overline{U_0}\right)/U_0\ ,\ B = L/U_0$.

In many cases it is possible with different procedures to determine one particular solution w_0 of the equation (1.3). With the substitution $w = w_0 + V$ where V is the new unknown function, the equation (1.3) transforms itself in a homogeneous equation

$$V'_{\bar{z}} = A\bar{V}. \tag{1.4}$$

Vekua [1] has shown that the general solution of the differential equation (1.3) has the next shape

$$\begin{aligned} &w\left(z,\bar{z}\right) = \phi\left(z\right) + \iint\limits_T \Gamma_1\left(z,t\right)\phi\left(t\right)dT + \iint\limits_T \Gamma_2\left(z,t\right)\overline{\phi\left(t\right)}dT - \\ &-\tfrac{1}{\pi}\iint\limits_T \Omega_1\left(z,t\right)\mathrm{B}\left(t\right)dT - \tfrac{1}{\pi}\iint\limits_T \Omega_2\left(z,t\right)\overline{\mathrm{B}\left(t\right)}dT \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} &\Gamma_1\left(z,t\right) = \sum_{j=1}^{\infty} K_{2j}\left(z,t\right), \\ &\Gamma_2\left(z,t\right) = \sum_{j=1}^{\infty} K_{2j+1}\left(z,t\right), \\ &\mathrm{K}_1\left(z,t\right) = -\tfrac{A(t)}{\pi(t-z)}, \\ &\mathrm{K}_n\left(z,t\right) = \iint\limits_T K_1(z,\sigma)\overline{K_{n-1}\left(\sigma,t\right)}dT_\sigma, \\ &\Omega_1\left(z,t\right) = \tfrac{1}{(t-z)} + \iint\limits_T \tfrac{\Gamma_1(z,\sigma)}{t-\sigma}dT_\sigma, \\ &\Omega_2\left(z,t\right) = \iint\limits_T \tfrac{\Gamma_2(z,\sigma)}{\bar{t}-\bar{\sigma}}dT_\sigma. \end{aligned}$$

The solution (1.5) practically can not be used because besides infinite series and recurrent relations it also contains double singular integrals from Cauchy type, which are very difficult to be solved. There for many mathematicians consider qualitative and quantitative approach to this kind of problems. From the other side, they deal with various solving treatments.

There are various generalizations of the equation (1.2). For example D.Dimitrovski, M.Rajović and D.Stoiljković (see [3]) examined the differential equation

$$aV'_{\bar{z}} + bV'_z + c\overline{V}'_{\bar{z}} + d\overline{V}'_z = AV + B\overline{V} + F. \tag{1.6}$$

In many problems in physics, mechanics and technic the real and the imaginary part of the solution has particular physical meaning. If the solution of the Vekua complex differential equation (basic or generalized) can be presented in a "nice", finite, closed and explicit shape

$$w = w\left(z, \bar{z}, Q\left(z\right), \overline{Q\left(z\right)}\right) \tag{1.7}$$

where $Q(z)$ is an arbitrary analytic function, than could be possible to separate its real and imaginary part.

One important, but degenerative case when it is possible is the differential equation

$$U'_{\bar{z}} = MU + L,\ \left(N = 0\right). \tag{1.8}$$

This equation has solved S.Fempl (see [2]). He showed that its general solution has the following form

$$U\left(z, \bar{z}\right) = e^{J(M)}\left[Q\left(z\right) + J\left(Le^{-J(M)}\right)\right] \tag{1.9}$$

where J is the one integral operator which is inverse to the operator $\frac{\partial}{\partial \bar{z}}$. Unfortunately, in the equation (1.8) is missing the member with $\overline{U}$, which is characteristic for the Vekua equation.

In this paper we introduce at first so cold quasi-Vekua differential equation as a first class of the finite integrable generalized Vekua equations. Than we determine the particular solutions with the Mitrinović method and is got one sufficient condition for their existence. In the fifth part the class of the generalized finite integrable Vekua equations is expanded. In the sixth part one special symmetric generalized Vekua equation is introduced which is different from the others, and at the end it is given one application in the theory of the p-analytic functions.

2. About the quasi-Vekua differential equation

We will observe the general Vekua differential equation in the shape:

$$w'_{\bar{z}} + A\bar{w}'_{\bar{z}} = Bw + C\bar{w} + D \tag{2.1}$$

where the coefficients $A = A\left(z, \bar{z}\right)$, $B = B\left(z, \bar{z}\right)$, $C = C\left(z, \bar{z}\right)$ and $D = D\left(z, \bar{z}\right)$ are continuous functions in some area Ω. In the special case when $A\left(z, \bar{z}\right) = 0$ the equation (2.1) is reduced to simple Vekua equation (1.2). Based to the common principle that the general solution of one complex differential equation from first order contains one arbitrary analytic function, we introduce the following definitions:

Definition 2.1: The function $w = w(z, \bar{z}, Q(z))$ represents $\mathfrak{F}$-general solution of the general Vekua equation (2.1) in some area Ω, if it is continuous in that area and differentiable and if identically satisfies this equation. In the function w besides an arbitrary analytic function, $\overline{Q(z)}$, $Q'(z)$ can take this role also, as well.

Definition 2.2: The general Vekua differential equation (2.1) is $\mathfrak{F}$-finite integrable, if it contains an $\mathfrak{F}$-general solution in a finite and closed shape $w = w(z, \bar{z}, Q(z))$.

Lets introduce in the equation (2.1) the next substitution

$$w + A\bar{w} = f \tag{2.2}$$

where $f = f(z, \bar{z})$ is a new unknown function. Making a conjugation of the equation (2.2) we get

$$\bar{w} + \bar{A}w = \bar{f}. \tag{2.3}$$

The solution of the system (2.2) – (2.3) has the next shape

$$w = \frac{A\bar{f} - f}{A\bar{A} - 1}. \tag{2.4}$$

If we substitute the values (2.4) in the equation (2.1), after some short calculations it transfers in the general Vekua equation.

$$f'_{\bar{z}} = \frac{(C + A'_{\bar{z}})\,\bar{A} - B}{A\bar{A} - 1} f + \frac{BA - (C + A'_{\bar{z}})}{A\bar{A} - 1} \bar{f} + D. \tag{2.5}$$

Based on the observations from the previous part, the differential equation (2.5) is finite integrable if we annul the coefficient in front of $\bar{f}$. From there we can formulate the next

Theorem 2.1: *The general Vekua differential equation (2.1) is $\mathfrak{F}$-finite integrable in the case when it is fulfilled the condition*

$$BA - C - A'_{\bar{z}} = 0. \tag{2.6}$$

Note 2.1: If the condition (2.6) is fulfilled, then the equation (2.1) transfers in

$$w'_{\bar{z}} + A\bar{w}'_{\bar{z}} = Bw + \left(BA - A'_{\bar{z}}\right)\bar{w} + D. \tag{2.7}$$

The general solution of the equation (2.5) has the following shape

$$f = e^{J(B)}\left[Q(z) + J\left(De^{-J(B)}\right)\right]. \tag{2.8}$$

With a substitution of these values in (2.4) we get the $\mathfrak{F}$-general solution of the equation (2.7).

Note 2.2: The complex differential equation (2.7) is not the real generalization of the general Vekua equation (1.2) because for $A = 0$ the coefficient in front of $\bar{w}$ is annulled. There for the equation (2.7) we call quasi-Vekua differentiable equation. It represents one important $\mathfrak{F}$-finite integrable case of generalized Vekua equation (2.1).

3. Method of Mitrinović for determining of the particular solution of the generalized Vekua differential equation

In his work [4] D. Mitrinović studied the following problem: Let be given the equation

$$F(u, v, x, y) = 0 \tag{3.1}$$

where F is a real function from real variables u, v, x and y. We should determine harmonic conjugated functions $u = u(x, y)$ and $v = v(x, y)$ which satisfy the condition (3.1).

If the equation (3.1) is derivated by x and y and here we use the Cauchy-Riemann conditions, after a short calculation, we get equations as follows

$$v'_x = f_1(v, x, y), \quad v'_y = f_2(v, x, y). \tag{3.2}$$

Makind a derivative of the first relation in (3.2) by y, and the other by x and if we equal the other partial derivatives, we get the following equality

$$f'_{1_y} + f_2 \cdot f'_{1_\nu} = f'_{1_x} + f_1 \cdot f'_{2_\nu}. \tag{3.3}$$

Mitrinović mentioned a few different cases:

I) The equality (3.3) is identically satisfied. The equations (3.2) than have infinitely many solutions expressed by the function which contains one arbitrary real constant. The system (3.2) is fully integrable.
II) The equality (3.3) is not identically satisfied and for v we can take the function that is defined with this equality. Checking we determine that the relations (3.2) has one common solution.
III) The equality (3.3) is not identically satisfied and checking we determine that the relations (3.2) doesn't have common solution. In that case we say that the equation (3.1) doesn't define any analytic function.

This method can be used for determination of the particular solution of the basic or generalized Vekua differential equation. Here often is used general linear relation

$$\bar{w} = Mw + N \tag{3.4}$$

where the coefficients $M = M(z, \bar{z})$ and $N = N(z, \bar{z})$ are chosen in correlation with the coefficients of the Vekua equation.

Example: We want to determine one real particular solution of the generalized Vekua differential equation

$$w'_{\bar{z}} + 3\bar{w}'_{\bar{z}} = 5w - \bar{w}. \tag{3.5}$$

For the real solution we have

$$w = \bar{w} \quad (M = 1,\ N = 0) \tag{3.6}$$

and the equation (3.5) transfers into

$$w'_{\bar{z}} - w = 0. \tag{3.7}$$

Its general solution has the next shape

$$\begin{aligned} &w(z, \bar{z}) = Q(z)\, e^{\bar{z}} = (Q_1 + iQ_2)\, e^{x-iy} = \\ &= (Q_1 + iQ_2)\, e^{x} (\cos y - i \sin y) = \\ &= e^{x} (Q_1 \cos y + Q_2 \sin y) + ie^{x} (Q_2 \cos y - Q_1 \sin y)\,. \end{aligned}$$

Since this solution has to be real, we have that its imaginary part is 0 i.e.

$$Q_2 \cos y - Q_1 \sin y = 0. \tag{3.8}$$

The relation (3.8) represents the problem of Mitrinović for determination of the analytic function $Q(z) = Q_1 + iQ_2$. Using the method of Mitrinović we get $Q(z) = c \cdot e^{z}$ where c is an arbitrary real constant and one class of the solution of the generalized Vekua equation (3.5) is in shape

$$w = c \cdot e^{z+\bar{z}}.$$

4. One sufficient condition for existence of the partial solution of the generalized Vekua differential equation

In the case of the Mitrinović method we don‘t know up front weather we will succeed to determine one particular solution of the generalized Vekua differential equation. In this part we will prove one sufficient condition, when this equation has particular solution in certain shape and determine that solution.

Lets observe the generalized homogenous Vekua differential equation

$$w'_{\bar{z}} + A\bar{w}'_{\bar{z}} = Bw + C\bar{w} \tag{4.1}$$

whose coefficients $A = A(z,\bar{z})$, $B = B(z,\bar{z})$ and $C = C(z,\bar{z})$ are continuous and differentiable functions by $\bar{z}$ in some area Ω. With the substitution $w + A\bar{w} = f$ (similar as before) this equation transfers in basic Vekua equation

$$f'_{\bar{z}} = Pf + R\bar{f}. \tag{4.2}$$

where

$$P = \frac{(C + A'_{\bar{z}})\,\bar{A} - B}{A \cdot \bar{A} - 1}, \quad R = \frac{BA - (C + A'_{\bar{z}})}{A\bar{A} - 1}$$

With the new substitution $f = U \cdot e^{J(P)}$ where $U = U(z,\bar{z})$ is a new unknown function, the equation (4.2) transfers in the canonical shape

$$U'_{\bar{z}} = \mathcal{R}\bar{U} \tag{4.3}$$

where

$$\mathcal{R} = Re^{\overline{J(A)} - J(A)}.$$

Let us choose now in the relation (3.4) to be $M = \frac{1}{R \cdot \bar{z}}$, $N = \frac{1}{R \cdot \bar{z}}$. Then it transfers in

$$\bar{U} = \frac{1}{\mathcal{R} \cdot \bar{z}}\, U + \frac{1}{\mathcal{R} \cdot \bar{z}} \tag{4.4}$$

and the equation (4.3) in the equation

$$U'_{\bar{z}} = \frac{1}{\bar{z}}\, U + \frac{1}{\bar{z}}. \tag{4.5}$$

The general solution of the equation (4.5) is

$$U(z,\bar{z}) = \bar{z}Q(z) - 1. \tag{4.6}$$

If the value (4.6) further more substitute in the relation (4.4) after a little work we get

$$\mathcal{R}(z,\bar{z}) = \frac{Q(z)}{z\overline{Q(z)} - 1}. \tag{4.7}$$

and on this observations we can formulate the next theorem.

Theorem 4.1: *If the coefficient $\mathcal{R}$ of the Vekua equation (4.3) has the shape (4.7) where $Q(z)$ is an arbitrary analytic function, than there exists a particular solution of the equation (4.3) in the shape (4.6) as well as the particular solution of the corresponding Vekua homogeneous differential equation (4.1) in the shape (see (2.4)).*

$$w = \frac{A\bar{f} - f}{A\bar{A} - 1}, \quad f = U \cdot e^{J(P)} \tag{4.8}$$

Example: We want to determine one particular solution of the generalized Vekua differential equation

$$w'_{\bar{z}} + 2\bar{w}'_{\bar{z}} = \left(z + \frac{2e^z}{ze^{\bar{z}} - 1}\right) w + \left(2z + \frac{e^z}{ze^{\bar{z}} - 1}\right) \bar{w} \tag{4.9}$$

Solution: With the substitution

$$w + 2\bar{w} = f, \quad w = \frac{2\bar{f} - f}{3}, \quad \bar{w} = \frac{2f - \bar{f}}{3}$$

the equation (4.9) transfers in a basic Vekua equation

$$f'_{\bar{z}} = zf + \frac{e^z}{ze^{\bar{z}} - 1}\bar{f}. \tag{4.10}$$

With the new substitution $f = e^{z\bar{z}} \cdot V$ the equation (4.10) transfers in canonical form

$$V'_{\bar{z}} = \frac{e^z}{ze^{\bar{z}} - 1}\bar{V}. \tag{4.11}$$

Here we see that the condition of the theorem (4.1) is fulfilled and that $Q(z) = e^z$. Because of this the particular solution of the equation (4.11) based on (4.6) has the shape

$$V(z, \bar{z}) = \bar{z}Q(z) - 1 = \bar{z}e^z - 1. \tag{4.12}$$

Then

$$f(z, z) = e^{z\bar{z}}V = e^{z\bar{z}}(\bar{z} \cdot e^z - 1)$$

and

$$w(z, \bar{z}) = \frac{2\bar{f} - f}{3} = \frac{e^{z\bar{z}}(2ze^{\bar{z}} - \bar{z}e^z - 1)}{3} \tag{4.13}$$

what is the solution that we was looking for.

5. One new class of generalized finite integrable Vekua differentiable equations

We saw that the quasi Vekua differentiable equation has to satisfy the condition (2.6). Now we will construct one wider class of $\mathfrak{F}$-finite integrable generalized Vekua differential equations, and here the condition (2.6) doesn‘t have to be considered.

In the previous part we have shown that every generalized homogeneous Vekua differential equation with two continuous substitutions can be reduced to canonical form

$$U'_{\bar{z}} = \mathcal{R}\overline{U} \tag{5.1}$$

where $\mathcal{R} = \mathcal{R}(z, \bar{z})$ is continuous and differentiable function in some area Ω. But in his work [5] M. Čanak proved the next theorem

Theorem 5.1: *Let $\varphi(z)$ is given analytic function. Then* $\mathfrak{F}$-general solution of the Vekua differentiable equation

$$U'_{\bar{z}} = -\frac{\varphi'(z)}{\varphi + \bar{\varphi}}\overline{U} \tag{5.2}$$

has a shape

$$U(z, \bar{z}) = Q'(z) - \left(Q + \bar{Q}\right)\frac{\varphi'(z)}{\varphi + \bar{\varphi}} \tag{5.3}$$

where $Q(z)$ is an arbitrary analytic function and $\varphi(z), Q(z) \in \mathcal{A}'$.

Note 5.1: Let $\mathcal{A}$ is the set of all analytic functions in Ω and $\mathcal{A}'$ is the set of those analytic functions whose Taylor development contains only real coefficients and whitch satisfy the $\overline{Q(z)} = Q(\bar{z})$ (conjugation "peace by peace"). So we have

$$\overline{Q(z)} = \overline{\sum_{k=0}^{\infty} c_k z^k} = \sum_{k=0}^{\infty} c_k \bar{z}^k = Q(\bar{z}), \text{ as well as}$$

$$\overline{Q'(z)} = \overline{\sum_{k=1}^{\infty} k c_k z^{k-1}} = \sum_{k=1}^{\infty} k c_k \bar{z}^{k-1} = Q'(\bar{z}) = \bar{Q}'_{\bar{z}}.$$

Based on the theorem 5.1 we have that the corresponding generalized Vekua equation for the equation (5.2), for example

$$w'_{\bar{z}} + A\bar{w}'_{\bar{z}} = -A\frac{\varphi'(z)}{\varphi + \bar{\varphi}}w - \left(A'_{\bar{z}} + \frac{\varphi'(z)}{\varphi + \bar{\varphi}}\right)\bar{w} \tag{5.4}$$

will be finite integrable and will have $\mathfrak{F}$-general solution in the shape

$$w = \frac{A\overline{U} - U}{A\bar{A} - 1}$$

(see (2.4)).

6. One symmetric generalized Vekua differentiable equation

Solving the generalized Vekua differentiable equations can be made by performing or reducing to the basic Vekua equation, as in the previous part or directly. Here we will see one of those methods.

We will observe the Vekua differentiable equation

$$w'_{\bar{z}} + A\bar{w}'_{\bar{z}} = Bw + C\bar{w} \tag{6.1}$$

whose coefficients are continuous and differentiable functions by $\bar{z}$ in some area Ω. Let‘s look for its general solution in symmetric form

$$w = u\left(Q'(z) + \overline{Q'(z)}\right) + \nu\left(Q(z) + \overline{Q(z)}\right) \tag{6.2}$$

where $u = u(z, \bar{z})$ and $v = v(z, \bar{z})$ are unknown, complex functions, and $Q(z)$ is an arbitrary analytic function from the class $\mathcal{A}'$. So we have

$$\begin{aligned} \bar{w} &= \bar{u}\left(\overline{Q'(z)} + Q'(z)\right) + \bar{v}\left(\overline{Q(z)} + Q(z)\right) \\ w'_{\bar{z}} &= u'_{\bar{z}}\left(Q' + \bar{Q}'\right) + u\bar{Q}'' + v'_{\bar{z}}\left(Q + \bar{Q}\right) + v\bar{Q}' \\ \bar{w}'_{\bar{z}} &= \bar{u}'_{\bar{z}}\left(Q' + \bar{Q}'\right) + \bar{u}\bar{Q}'' + \bar{v}'_{\bar{z}}\left(Q + \bar{Q}\right) + \bar{v}\bar{Q}'. \end{aligned}$$

Substituing of these values in the equation (6.1) and equalling the corresponding coefficients we get the next system of equations:

$$\begin{aligned} &u + A\bar{u} = 0 \\ &v + A\bar{v} = 0 \\ &u'_{\bar{z}} + A\bar{u}'_{\bar{z}} = Bu + C\bar{u} \\ &v'_{\bar{z}} + A\bar{v}'_{\bar{z}} = Bv + C\bar{v}. \end{aligned} \tag{6.3}$$

We will observe only the special case when: $A(z, \bar{z}) = -1$, $B(z, \bar{z}) = -C(z, \bar{z})$. Then we get the next solution of this system:

$u = u(x, y)$ $v = v(x, y)-$ arbitrary, real, differentiable functions. Then $\mathcal{F}$ - general solution of the generalized Vekua equation

$$w'_{\bar{z}} - \bar{w}'_{\bar{z}} = B(w - \bar{w}) \tag{6.4}$$

has the following shape

$$\begin{aligned} &w(z, \bar{z}) = u(x, u)\left(S(z) + \overline{S(z)}\right) \\ &\left(Q'(z) + Q(z) = S(z) \in \mathcal{A}'\right) \end{aligned} \tag{6.5}$$

Here we will not go in observation of the general shape of the system of the equations (6.3). But in our special case the generalized Vekua equation is symmetric according to w and $\bar{w}$, and its general solution is also symmetric according to the arbitrary analytic function $S(z)$ and its conjugated function $\overline{S(z)}$. The treatment (6.2) is characteristic for the generalized Vekua equation and it can not be used at the basic Vekua equation (5.1).

7. One application in the theory of the p-analytic functions

In his monograph [6] Položij introduced the next definition of the **p**-analytic functions: The function $f(z,\bar{z}) = u + iv$ is called **p**-analytic function with characteristic $p = p(x,y)$ in the area Ω, if it is defined in that area and its real and imaginary part have continuous partial derivatives from first order by x and y and are satisfying the next system of partial differential equations

$$u'_x = \frac{1}{p}v'_y, \quad u'_y = -\frac{1}{p}v'_x. \tag{7.1}$$

Položij showed that it exists deep structure and qualitative connection between the solutions of the (7.1) and the analytic functions. Also he researched some applications of p-analytic functions in the theory of filtration, theory of torsion of the rotating bodies, the theory of elastic shells etc.

Here we meet the next problem. We have to express one p-analytic function in finite and explicit form through an arbitrary analytic function and the characteristic. It is often possible using common and generalized $\mathfrak{F}$-finite integrable Vekua differential equations, what shows us the next example.

Example: The system (7.1) can be written in the next shape

$$u'_x - v'_y = \frac{1-p}{p}v'_y, \quad u'_y + v'_x = -\frac{1-p}{p}v'_x. \tag{7.2}$$

If the second equation of (7.2) multiplies with $\boldsymbol{i}$ and adds to the first we get the next complex differentiable equation

$$w'_{\bar{z}} = -i\frac{1-p}{p}v'_{\bar{z}}, \quad (w = u + iv)$$

or

$$w'_{\bar{z}} + \frac{p-1}{p+1}\bar{w}'_{\bar{z}} = 0. \tag{7.3}$$

The equation (7.3) represents one generalized Vekua differentiable equation. In the case when $p = p(x,y) = const.$ it is fulfilled the condition (2.6) and the equation (7.3) transfers in a quasi - Vekua equation. Its general solution has the following shape

$$w(z,\bar{z}) = \frac{(p+1)^2}{4p}Q(z) - \frac{p^2-1}{4p}\overline{Q(z)}. \tag{7.4}$$

With this formula the p-analytic function with constant characteristic is expressed through an arbitrary analytic function $Q(z)$ and the characteristic p.

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