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# Around One Michael's Theorem on Selections

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In this paper, using the language and techniques of set-valued mappings, we give a new characteristic of paracompactness (Theorems 2, 3 and 3) and fully paracompactness (Theorems 4 and 4). This improves, in particular, some results of E.Michael.

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## 1 Introduction

In [9] Michael has proved the following:

**Theorem 1.1** *If $\theta : X \to Y$ is a lower semi-continuous closed-valued mapping of a paracompact space $X$ into a complete metric space $Y$, then there exist a lower semi-continuous compact-valued mapping $\phi : X \to Y$ and an upper semi-continuous compact-valued mapping $\psi : X \to Y$ such that $\phi(x) \subseteq \psi(x) \subseteq \theta(x)$ for every $x \in X$.*

Let $X$ and $Y$ be non-empty topological spaces. A *set-valued mapping* $\theta : X \to Y$ assigns to every $x \in X$ a non-empty subset $\theta(x)$ of $Y$. If $\phi, \psi : X \to Y$ are set-valued mappings and $\phi(x) \subseteq \psi(x)$ for every $x \in X$, then $\phi$ is called *a selection* of $\psi$.

Let $\theta : X \to Y$ be a set-valued mapping and let $A \subseteq X$ and $B \subseteq Y$. The set $\theta^{-1}(B) = \{x \in X : \theta(x) \bigcap B \neq \varnothing\}$ is *the inverse image* of $B$, $\theta(A) = \theta^1(A) = \bigcup\{\theta(x) : x \in A\}$ is *the image* of the set $A$ and $\theta^{n+1}(A) = \theta(\theta^{-1}(\theta^n(A)))$ is *the $n+1$-image* of the set $A$. The set $\theta^\infty(A) = \bigcup\{\theta^n(A) : n \in \mathbb{N}\}$ is *the largest image* of the set $A$ (see [5]). A set-valued mapping $\theta : X \to Y$ is called *lower (upper) semi-continuous* if for every open (closed) subset $H$ of $Y$ the set $\theta^{-1}(H)$ is open (closed) in $X$.

In [2, 3, 4] was proved the following conversion of Theorem 1:

**Theorem 1.2** *For a $T_1$-space $X$ the following are equivalent:*

*1. $X$ is paracompact.*

*2. For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space $Y$ there exists an upper semi-continuous compact-valued mapping $\psi : X \to Y$ such that $\psi(x) \subseteq \theta(x)$ for every $x \in X$.*

*3. For every lower semi-continuous mapping closed-valued mapping $\theta : X \to Y$ into a discrete space $Y$ there exists an upper semi-continuous compact-valued mapping $\psi : X \to Y$ such that $\psi(x) \subseteq \theta(x)$ for every $x \in X$.*

The aim of the present article is to extend the assertions of Theorem 1 and to propose a characteristic of fully paracompactness (see Theorem 4). The article continues the investigations from earlier articles [5, 6] of the authors. Any paracompact space is considered to be Hausdorff. We use the terminology from [7, 14]. By $|A|$ we denote the cardinality of a set $A$ and $\mathbb{N} = \{1, 2, ...\}$. We consider only the covering dimension of spaces.

## 2 On paracompact spaces

A set-valued mapping $\theta$ is called *simple* if $\theta^\infty(x) = \theta(x)$ for every $x \in X$.

**Remark 2.1** For a set-valued mapping $\theta : X \to Y$ the following 1,2 and 3 are equivalent:

1. $\theta^\infty(x) = \theta(x)$ for every $x \in X$;
2. $\theta^2(x) = \theta(x)$ for every $x \in X$;
3. For every $x, y \in X$ either $\theta(x) \cap \theta(y) = \varnothing$ or $\theta(x) = \theta(y)$.

**Remark 2.2** Let$\theta : X \to Y$ be a set-valued mapping. Then $\theta^\infty(x) = \theta(x)$ is a simple set-valued mapping.

**Proposition 2.3** *Let $\theta : X \to Y$ be a simple closed-valued mapping of a space $X$ into a space $Y$. Then:*

*1. If $\theta$ is lower semi-continuous and $Y$ is first-countable, then $\theta^{-1}(\theta(x))$ is a $G_\delta$-set of $X$ for every $x \in X$. 2. If $\theta$ is upper semi-continuous and $Y$ is a metric space, then $\theta^{-1}(\theta(x))$ is a $G_\delta$-set of $X$ for every $x \in X$.*

**Theorem 2.4** *Let $\theta : X \to Y$ be a lower semi-continuous closed-valued mapping of a paracompact space $X$ into a complete metric space $Y$. Then there exist a lower semi-continuous compact-valued mapping $\phi :\ X \to Y$, an upper semi-continuous compact-valued mapping $\psi :\ X \to Y$, a metric space $Z$ and a continuous single-valued mapping $g : X \to Z$ such that:*

*1. $\phi(x) \subseteq \psi(x) \subseteq \theta(x)$ for every $x \in X$;*

*2. $\nu : X \to Y \times Z$, where $\nu(x) = \phi(x) \times \{g(x)\}$ for every $x \in X$, is a lower semi-continuous compact-valued mapping;*

*3.* $\lambda : X \to Y \times Z$*, where* $\lambda(x) = \psi(x) \times \{g(x)\}$ *for every* $x \in X$*, is an upper semi-continuous compact-valued mapping;*

*4.* $\lambda^{\infty}(x)$ *is a separable metric subspace of* $Y \times Z$ *for every* $x \in X$*.*

Proof. Let $d$ be a complete metric on $Y$. Put $V_n(y) = \{z \in Y : d(y,z) < 2^{-n}\}$ for every $y \in Y$ and every $n \in \mathbb{N}$. If $y \in Y$ and $F$ is an non-empty subset of $Y$, then $diam(F) = sup\{d(y,z) : y, z \in F\}$ and $d(y,F) = inf\{d(y,z) : z \in F\}$. Assume that $diam(\varnothing) = 0$ and $d(y,\varnothing) = \infty$. Fix a sequence $\{\xi_n = \{V_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open locally finite covers of $Y$ such that $diam(V_\alpha) < 2^{-n}$ for every $\alpha \in A_n$ and every $n \in \mathbb{N}$.

By virtue of Theorem 1 there exist two lower semi-continuous compact-valued mappings $\phi_1, \phi : X \to Y$ and two upper semi-continuous compact-valued mappings $\psi_1, \psi : X \to Y$ such that $\phi(x) \subseteq \psi(x) \subseteq \phi_1(x) \subseteq \psi_1(x) \subseteq \theta(x)$ for every $x \in X$. Put $W_\alpha = \phi_1^{-1}(V_\alpha)$ for every $\alpha \in A_n$ and every $n \in \mathbb{N}$.

Since $\psi_1$ is an upper semi-continuous compact-valued mapping, then $\xi'_n = \{\psi_1^{-1}(V_\alpha) : \alpha \in A_n\}$ is a locally finite open cover of $X$. By construction, $W_\alpha \subseteq \psi_1^{-1}(V_\alpha)$. Thus $\gamma_n = \{W_\alpha : \alpha \in A_n\}$ is an open locally finite cover of $X$ for every $n \in \mathbb{N}$. Since $\psi^{-1}(V_\alpha)$ is an $F_\sigma$-set and $\psi^{-1}(V_\alpha) \subseteq W_\alpha$, there exists an open $F_\sigma$-set $U_\alpha$ of $X$ such that $\psi^{-1}(V_\alpha) \subseteq U_\alpha \subseteq W_\alpha$. Therefore $\eta_n = \{U_\alpha : \alpha \in A_n\}$ is an open locally finite cover of $X$ for every $n \in \mathbb{N}$.

There exists a family $\{f_\alpha : X \to [0,\ 1] : \alpha \in A_n,\ n \in \mathbb{N}\}$ of continuous functions on $X$ such that $\sum\{f_\alpha(x) : \alpha \in A_n\} = 2^{-n}$ and $X \setminus U_\alpha = f_\alpha^{-1}(0)$ for every $\alpha \in A_n$ and $n \in \mathbb{N}$. Put $\overline{\rho}(x,z) = \sum\{\mid f_\alpha(x) - f_\alpha(z) \mid : \alpha \in A_n,\ n \in \mathbb{N}\}$ for $x, z \in X$. By construction, $\overline{\rho}$ is a continuous pseudometric on $X$. There exist a metric space $(Z, \rho)$ and a continuous mapping $g : X \to Z$ such that $\rho(g(x), g(z)) = \overline{\rho}(x,z)$ for every $x, z \in X$.

Fix $x \in X$. Put $A_n(x) = \{\alpha \in A_n : x \in U_\alpha\}$. Thus $\psi(x) \subseteq \cup\{U_\alpha : \alpha \in A_n(x)\}$ for every $n \in \mathbb{N}$. Let $\lambda(x) = \psi(x) \times \{g(x)\}$ and $\nu(x) = \phi(x) \times \{g(x)\}$. The mapping $\lambda : X \to Y \times Z$ is upper semi-continuous compact-valued and the mapping $\nu : X \to Y \times Z$ is lower semi-continuous compact-valued. If $x \in X$ and $\Psi(x) = \psi(g^{-1}(g(x)))$, then $\mathcal{B}(x) = \{U_\alpha : \alpha \in A_n, n \in \mathbb{N}\}$ is a countable base of the subspace $\Psi(x)$ in $Y$. Let $p : Y \times Z \to Y$ and $q : Y \times Z \to Z$ be the natural projections, $z \in Z$ and $H \subseteq q^{-1}(z)$. Fix $x \in \lambda^{-1}(H)$. Then $\lambda(x) = \psi(x) \times \{g(x)\}$ and $\lambda(x) \cap H \neq \varnothing$. Thus there exists $(y,z) \in \lambda(x) \cap H$. Hence $g(x) = z$. Therefore $\lambda(\lambda^{-1}(H)) \subseteq q^{-1}(z)$. Hence, by construction, $\lambda^{\infty}(x)$ is a separable metric subspace of $Y \times Z$ for every $x \in X$. ■

Let $\{\varphi, \psi, \phi, \theta\}$ be set-valued mappings of a space $X$ into a space $Y$. We say that:

– $\{\varphi, \psi\}$ is a Michael pair of mappings of $X$ into $Y$ if $\varphi : X \leftrightarrow Y$ is a lower semi-continuous compact-valued mapping, $\psi : X \leftrightarrow Y$ is a upper semi-continuous compact-valued mapping and $\varphi(x) \subseteq \psi(x)$ for every $x \in X$;

– $\{\psi, \phi, \theta\}$ is a Michael triple of mappings $X$ into $Y$ if $\phi : X \leftrightarrow Y$ is a lower semi-continuous compact-valued mapping, $\psi, \theta : X \leftrightarrow Y$ are upper semi-continuous compact-valued mappings and $\psi(x) \subseteq \phi(x) \subseteq \theta(x)$ for every $x \in X$;

– $\{\varphi, \psi, \phi, \theta\}$ is a Michael quadruple of mappings of $X$ into $Y$ if $\varphi, \phi : X \leftrightarrow Y$ are lower semi-continuous compact-valued mappings, $\psi, \theta : X \leftrightarrow Y$ are upper semi-continuous compact-valued mappings and $\varphi(x) \subseteq \psi(x) \subseteq \phi(x) \subseteq \theta(x)$ for every $x \in X$.

In the proof of Theorem 2 we establish the validity of the following assertion.

**Proposition 2.5** *Let $\psi, \phi, \theta$ be a Michael triple of mappings of a normal space $X$ into a metric space $Y$ . Then there exist a metric space $Z$ and a continuous single-valued mapping $g : X \to Z$ such that:*

*1. $\nu : X \to Y \times Z$, where $\nu(x) = \phi(x) \times \{g(x)\}$ for every $x \in X$, is a lower semi-continuous compact-valued mapping;*

*2. $\lambda : X \to Y \times Z$, where $\lambda(x) = \psi(x) \times \{g(x)\}$ for every $x \in X$, is an upper semi-continuous compact-valued mapping;*

*3. $\lambda^\infty(x)$ is separable for every $x \in X$.*

## 3 On finite-dimensional paracompact spaces

**Theorem 3.1** *Let $\theta : X \to Y$ be a lower semi-continuous closed-valued mapping of a paracompact space $X$ into a complete metric space $Y$ and $dimX \leq n$. Then there exist two metric spaces $Z$ and $S$, a continuous single-valued mapping $g : Z \to Y$, a continuous single-valued mapping $h : S \to Z$ and an upper semi-continuous mapping $\psi : X \to Z$, such that:*

*1. $dimZ \leq n$ and $dimS = 0$.;*

*2. $h : S \to Z$ is a closed mapping and with the fibers $h^{-1}(z)$ of cardinality at most $n + 1$;*

*3. $|\psi(x)| \leq n + 1$ and $g(\psi(x)) \subseteq \theta(x)$ for any $x \in X$;*

*4. $\lambda : X \to S$, where $\lambda(x) = h^{-1}(\psi(x))$ for every $x \in X$, is an upper semi-continuous finite-valued mapping;*

*5. $\lambda^\infty(x)$ is a separable subspace for every $x \in X$.*

P r o o f. There exists a complete metric space $(B, d)$ and a continuous open mapping $f : B \to Y$ such that $dimB = 0$ (see[1]). The mapping $\varphi : X \to B$, where $\varphi(x) = f^{-1}(\theta(x))$ for each $x \in X$, is lower semi-continuous and closed-valued. Since $X$ is a paracompact space and $dimX \leq n$,there exist (see [2, 3, 13]) a sequence $\{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open locally finite covers of $X$, a sequence $\{\eta_n = \{F_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of closed locally finite covers of $X$, a sequence $\{\xi_n = \{V_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of families of open subsets of $B$ and a sequence $\{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\}$ of single-valued mappings such that:

1. $F_\alpha \subseteq U_\alpha \subseteq cl_X U_\alpha \subseteq \varphi^{-1}(V_\alpha)$ and $diam V_\alpha < 2^{-n}$ for all $n \in \mathbb{N}$ and $\alpha \in A_n$;

2. $U_\alpha = \cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$, $\cup\{cl_B V_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq V_\alpha$ and $F_\alpha = \cup\{F_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for all $n \in \mathbb{N}$ and $\alpha \in A_n$;

3. If $x \in X$, $n \in \mathbb{N}$ and $A_n(x) = \{\alpha \in A_n : x \in F_\alpha\}$, then $|A_n(x)| \leq n+1$;

4. There exists a family of continuous non-negative functions $\{u_\alpha : X \to [0,1] : n \in \mathbb{N}, \alpha \in A_n\})$ such that $U_\alpha = u_\alpha^{-1}(0)$ and $\Sigma\{u_\beta(x) : \beta \in A_n\} = 2^{-1}$ for all $x \in X$, $n \in \mathbb{N}$ and $\alpha \in A_n$.

Put $A(x) = \{\alpha = (\alpha_n :\in \mathbb{N}) : \alpha_n \in A_n(x), n \in \mathbb{N}\}$. If $\alpha = (\alpha_n :\in \mathbb{N}) \in A(x)$, then $V(\alpha) = \cap\{V_{\alpha_n} :\in \mathbb{N}\}$. For any $x \in X$ we put $\psi_1(x) = \cup\{V(\alpha) : \alpha A(x)\}$. Then $\psi_1 : X \to B$ is an upper semi-continuous mapping, $|\psi_1(x)| \leq n+1$ and $\psi_1(x)) \subseteq \varphi(x)$ for any $x \in X$.

Put $\rho_1(x,z) = \sum\{|u_\alpha(x) - u_\alpha(z)| : \alpha \in A_n, n \in \mathbb{N}\}$ for all $x, z \in X$. By construction $\rho_1$ is a continuous pseudometric on $X$. There exist a metric space $(Z_1, \rho)$ and a continuous mapping $g_1 : X \to Z_1$ such that $\rho(g_1(x), g_1(z)) = d_1(x,z)$ for every $x, z \in X$. By virtue of Mardešić's factorization theorem, there exist a metric space $Z_2$ and two continuous single-valued mappings $g_2 : X \to Z_2$ and $g_3 : Z_2 \to Z_1$ such that $dim Z_2 \leq n$ and $g_1 = g_3 \cdot g_2$ (see [8, 10]). One can assume that the spaces $Z_1$, $Z_2$ and $Z_3$ are complete metrizable.

Put $Z = B \times Z_2$ and $\psi(x) = \psi_1(x) \times \{g_2(x)\}$ for any $x \in X$. Let $g(s, z\times) = f(s)$ for any point $(s,z) \in Z$. By construction, $\psi : X \to Z$ is a upper semi-continuous mapping, $|\psi(x)| \leq n+1$, $\psi(x) \subseteq \varphi(x)$ and $g(\psi(x)) \subseteq \theta(x)$ for any $x \in X$. Let $x \in X$, $n \in \mathbb{N}$ and $L_n(x) = \{\alpha \in A_n : x \in F_\alpha\}$. If $x, z \in X$ and $g_1(x) = g_1(z)$ (i.e. $\rho_1(x,z) = 0$), then $L_n(x) = L_n(z)$ for any $n \in \mathbb{N}$. Let $q : B \times Z_2 \to Z_2$ be the natural projections, $z \in Z_2$ and $H \subseteq q^{-1}(z) = B \times \{z\}$. Then $\psi(\psi^{-1}(H)) \subseteq q^{-1}(z)$. Thus $\psi^\infty(x)$ is a separable subspace of $Z$ and $\psi^\infty(x) \subseteq B \times \{g(x)\}$ for each $x \in X$. By virtue of Morita's theorem, there exist a metric space $S$ and a closed continuous single-valued mapping $h : S \to Z$ such that $dim S = 0$ and the fibers $h^{-1}(z)$ are of cardinality at most $n+1$(see [8, 11]). ■

If the space $Y$ is discrete, then $B = Y$, $\psi^\infty(x)$ is a discrete subspace of $Z$ and $\psi^\infty(x) \subseteq Y \times \{g(x)\}$ for each $x \in X$. In fact we have

**Theorem 3.2** *Let $\theta : X \to Y$ be a lower semi-continuous closed-valued mapping of a paracompact space $X$ into a discrete space $Y$ and $dim X \leq n$. Then there exist two metric spaces $Z$ and $S$, a continuous single-valued mapping*

*$g : Z \to Y$, a continuous single-valued mapping $h : S \to Z$ and a upper semi-continuous mapping $\psi : X \to Z$, such that:*

*1. $dimZ \leq n$ and $dimS = 0$;*

*2. $h : S \to Z$ is a closed mapping and with the fibres $h^{-1}(z)$ of cardinality at most $n+1$;*

*3. $|\psi(x)| \leq n+1$ and $g(\psi(x)) \subseteq \theta(x)$ for any $x \in X$;*

*4. $\lambda : X \to S$, where $\lambda(x) = h^{-1}(\psi(x))$ for every $x \in X$, is an upper semi-continuous finite-valued mapping;*

*5. $\lambda^{\infty}(x)$ is a countable discrete subspace for every $x \in X$.*

## 4 Fully paracompact spaces

A space $X$ is called *fully paracompact* if $X$ is a regular space and for every open cover $\omega$ of $X$ there exist an open refinement $\xi$ of $X$ and a sequence $\{\xi_n : n \in \mathbb{N}\}$ of open star-finite covers of $X$ such that $\xi \subseteq \bigcup\{\xi_n : n \in \mathbb{N}\}$. Recall that a family $\zeta$ of subsets of $X$ is *star-finite* if the set $\{H \in \zeta : H \cap L \neq \varnothing\}$ is finite for every $L \in \zeta$. Every fully paracompact space is paracompact.

A space $X$ is called *strongly paracompact* if $X$ is a Hausdorff space and every open cover $\omega$ of $X$ has an open star-finite refinement. Every strongly paracompact space is fully paracompact.

Let $\tau$ be a cardinal number. Denote by $B(\tau)$ the topological product of a countably family of discrete spaces of cardinality $\tau$. The space $B(\tau)$ is called the Baire space of weight $\tau$ ([7], Example 4.2.12). If $\mathbb{R}$ is the space of reals and the cardinal number $\tau$ is uncountable, then the space $B(\tau) \times \mathbb{R}$ is fully paracompact and not strongly paracompact [12].

Recall that a family $\{U_\alpha : \alpha \in A\}$ is called *$\sigma$-discrete* (*$\sigma$-locally finite*) if $A = \bigcup\{A_n : n \in \mathbb{N}\}$ and the family $\{U_\alpha : \alpha \in A_n\}$ is discrete (locally finite) for every $n \in \mathbb{N}$.

A family $\{H_\alpha : \alpha \in A\}$ is said *to have a $\sigma$-discrete decomposition* if there exist a sequence $\{\{P_\beta : \beta \in B_n\} : n \in \mathbb{N}\}$ of discrete families of sets and a sequence $\{p_n : B_n \to A : n \in \mathbb{N}\}$ of single-valued mappings such that $A = \bigcup\{p_n(B_n) : n \in \mathbb{N}\}$ and $\bigcup\{P_\beta : \beta \in p_n^{-1}(\alpha), n \in \mathbb{N}\} = H_\alpha$ for every $\alpha \in A$.

**Theorem 4.1** *Let $\theta : X \to Y$ be a lower semi-continuous closed-valued mapping of a fully paracompact space $X$ into a complete metric space $Y$. Then there exist a lower semi-continuous compact-valued mapping $\phi : X \to Y$, an upper semi-continuous compact-valued mapping $\psi : X \to Y$, a metric space $Z$ and a continuous single-valued mapping $g : X \to Z$ such that:*

*1. $\phi(x) \subseteq \psi(x) \subseteq \theta(x)$ for every $x \in X$;*

*2. $\nu : X \to Y \times Z$, where $\nu(x) = \phi(x) \times \{g(x)\}$ for every $x \in X$, is a lower semi-continuous compact-valued mapping;*

*3.* $\lambda : X \to Y \times Z$*, where* $\lambda(x) = \psi(x) \times \{g(x)\}$ *for every* $x \in X$*, is an upper semi-continuous compact-valued mapping;*

*4.* $\lambda^\infty(x)$ *is a separable metric subspace of* $Y \times Z$ *for every* $x \in X$*;*

*5.* $dimZ = 0$.

P r o o f. Let $d$ be a complete metric on $Y$. Fix a sequence $\{\xi_n = \{V_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ of open locally finite covers of $Y$ such that $diam(V_\alpha) < 2^{-n}$ for every $\alpha \in A_n$ and every $n \in \mathbb{N}$.

By virtue of Theorem 1 there exists an upper semi-continuous compact-valued mapping $\psi_1 : X \to Y$ such that $\psi_1(x) \subseteq \theta(x)$ for every $x \in X$. Fix a mapping $\psi : X \to Y$ such that $\psi(x) \subseteq \psi(x)$ for every $x \in X$. Put $W_\alpha = \psi^{-1}(V_\alpha)$ for every $\alpha \in A_n$ and every $n \in \mathbb{N}$. Since $\psi_1$ is an upper semi-continuous compact-valued mapping, then $\xi'_n = \{\psi_1^{-1}(V_\alpha) : \alpha \in A_n\}$ is a locally finite open cover of $X$. By construction, $W_\alpha \subseteq \psi_1^{-1}(V_\alpha)$ and $\gamma_n = \{W_\alpha : \alpha \in A_n\}$ is a locally finite cover of $X$ for every $n \in \mathbb{N}$.

There exist a sequence $\{\xi_{nm} = \{U_\beta : \beta \in B_{nm}\} : n, m \in \mathbb{N}\}$ of open star-finite covers of $X$ and a sequence $\{\xi_n = \{U_\beta : \beta \in B_n\} : n \in \mathbb{N}\}$ of open locally finite covers of $X$ such that:

1. $B_n \subseteq \cup\{B_n m : m \in \mathbb{N}\}$ for any $n \in \mathbb{N}$;
2. For each $n \in \mathbb{N}$ and $\beta \in B_n$ the set $A(n, \beta) = \{\alpha \in A_n : U_\beta \cap W_\alpha \neq \varnothing\}$ is finite.

For each $n \in \mathbb{N}$ there exists a decomposition $\{B_\mu : \mu \in Z_{nm}\}$ of the set $B_{nm}$ such that:

– each set $B_\mu$ is countable and the set $H_\mu = \cup\{U_\beta : \beta \in B_\mu\}$is open-and-closed in $X$;

– $H_\mu \cap H_\nu = \varnothing$ for all distinct elements $\mu, \nu \in Z_n m$.

By construction, $H_\mu = \cup\{cl_X U_\beta : \beta \in B_\mu\}$. For each $x \in X$ and $n, m \in \mathbb{N}$ there exists a unique element $\mu(x, n, m) \in Z_{nm}$ such that $x \in H_{\mu(x,n,m)}$. On each $Z_{nm}$ we consider the discrete topology and put $Z = \Pi\{M_{nm} : n, m \in \mathbb{N}\}$ and $g(x) = (\mu(x, n, m) : n, m \in \mathbb{N})$ for each $x \in X$. Since the covers $\{H_\mu : \mu \in M_n\}$ are discrete, $g : X \to Z$ is a single-valued continuous mapping.

Fix $x \in X$. Let $B_n(x) = \cup\{B_n \cap B_{\mu(x,n,m)} : m \in \mathbb{N}\}$ and $A_n(x) = \cup\{A(n, \beta) : \beta \in B_n(x)\}$ for every $n \in \mathbb{N}$, $A(x) = \cup\{A_n(x) : n \in \mathbb{N}\}$ and $\mathcal{B}(x) = \{V_\alpha : \alpha \in A(x)\}$. Then $M(x) = M(z)$ provided $g(x) = g(z)$. The family $\mathcal{B}(x)$ is a countable base of the set $\psi(g^{-1}(g(x)))$. Hence, by construction, $\lambda^\infty(x)$ is a separable metric subspace of $Y \times Z$ for every $x \in X$. Suppose that$\{\phi, \psi, \psi_1\}$ is a Michael triple of mappings. The proof is complete. ■

Now we continue the arguments from the proof of Theorem 4. Fix $n, m \in \mathbb{N}$. We put $B(n, m, \mu) = B_n \cap B_\mu$ and $A(n, m, \mu) = \{\alpha \in A_n : W_\alpha \cap U_\beta \neq \varnothing$ *for some* $\beta \in B(n, m, \mu)\}$ for any $\mu \in Z_{nm}$. The set $A(n, m, \mu)$ is countable or finite. Thus $A(n,m,\mu) = \{\alpha_k(\mu) : k \in N(\mu) \subseteq \mathbb{N}\}$. Put $C(nmk) = \{(\mu, \alpha_k(\mu)) : \mu \in Z_{nm}, \alpha_k(\mu) \in A(n,m,\mu)\}$ and $V_\beta = W_{\alpha_k(\mu)} \cap H_\mu$ for any $\beta = (\mu, \alpha_k(\mu)) \in C(nmk)$. Consider the single-valued mapping $p_{nmk} : C(nmk) \to A_n$, where $p_{nmk}(\mu, \alpha) = \alpha$ for any $(\mu, \alpha) \in C(nmk)$. By construction, we have:

– the family $\gamma_{nmk} = \{V_\beta : \beta \in C(nmk)\}$ is open and discrete in $X$;

– the family $\xi_{nmk} = \{V'_\beta = g(V_\beta) : \beta \in C(nmk)\}$ is open and discrete in $Z$;

– $W_\alpha = \cup\{V_\beta : \beta \in C(nmk), m, k \in \mathbb{N}\}$.

Therefore we prove the following assertion.

**Proposition 4.2** *Let $\varphi, \psi : X \to Y$ be two set-valued mappings of a fully paracompact space $X$ into a metric space $Y$, the mapping $\psi$ is upper semicontinuous and $\varphi(x) \subseteq \psi(x)$ for each $x \in X$. Then for any $\sigma$-locally finite family $\{V_\alpha : \alpha \in A\}$ of the space $Y$ there exist a metric space $Z$ and a continuous single-valued mapping $g : X \to Z$ such that:*

*1. $dimZ = 0$;*

*2. $\lambda : X \to Y \times Z$, where $\lambda(x) = \psi(x) \times \{g(x)\}$ for every $x \in X$, is an upper semi-continuous compact-valued mapping;*

*3. $\lambda^\infty(x)$ is a separable metric subspace of $Y \times Z$ for every $x \in X$;*

*4. The family $\{g(\varphi^{-1}(V_\alpha)) : \alpha \in A\}$ has a $\sigma$-discrete decomposition in $Z$.*

**Corollary 4.3** *Let $\theta : X \to Y$ be a lower semi-continuous closed-valued mapping of a fully paracompact space $X$ into a complete metric space $Y$. Then there exist a metric space $Z$, a lower semi-continuous compact-valued mapping $\phi : X \to Z$, an upper semi-continuous compact-valued mapping $\psi : X \to Z$, and a continuous single-valued mapping $g : Z \to Y$ such that:*

*1. $g(\phi(x)) \subseteq g(\psi(x)) \subseteq \theta(x)$ for every $x \in X$;*

*2. $\psi^\infty(x)$ is a separable metric subspace of $Z$ for every $x \in X$;*

*3. $dimZ = 0$.*

**Theorem 4.4** *If $X$ is a fully paracompact space, then for every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space $Y$ there exist a discrete space $D$, a single-valued mapping $f : D \to Y$, an upper semi-continuous finite-valued mapping $\psi : X \to D$, a lower semi-continuous mapping $\varphi : X \to D$, a metric space $Z$ and a single-valued continuous mapping $g : X \to Z$ such that:*

*1. $\varphi(x) \subseteq \psi(x)$ and $f(\psi(x)) \subseteq \theta(x)$ for every $x \in X$;*

*2. For every $x \in X$ the set $\psi(g^{-1}(x))$ is countable;*

*3. If $\lambda(x) = \psi(x) \times \{g(x)\}$ and $\phi(x) = \varphi(x) \times \{g(x)\}$, then $\lambda : X \to D \times Z$ is upper semi-continuous and $\phi : X \to D \times Z$ is lower semi-continuous, moreover $\lambda^\infty(x)$ is a countable discrete subspace of $D \times Z$ for every $x \in X$;*

*4. $dimZ = 0$.*

P r o o f. Assume that $X$ is is fully paracompact and $\theta : X \to Y$ is a lower semi-continuous mapping into a discrete space $Y$. Then $\gamma = \{Uy = \theta^{-1}(y) : y \in Y\}$ is an open cover of the space $X$ and there exist an open refinement $\xi$ of $X$ and a sequence $\{\xi_n = \{V_\alpha : \alpha A_n : n \in \mathbb{N}\}$ of open star-finite covers of $X$ such that $\xi \subseteq \bigcup\{\xi_n : \ n \in \mathbb{N}\}$. We can consider that $\xi = \{V_\alpha : \alpha D\}$, $D \subseteq \cup\{A_n : n \in \mathbb{N}\}$ and the cover $\xi$ is locally finite. For any $\alpha \cup \{A_n : n \in \mathbb{N}\}$ there exists an open subset $H_\alpha$ of $X$ such that:

– $clXH_\alpha \subseteq V_\alpha$;

– $\eta = \{H_\alpha : \alpha D\}$ is a cover of $X$;

– $\eta_n = \{H_\alpha : \alpha A_n\}$ is a cover of $X$ for any $n \in \mathbb{N}$.

On $D$ we consider the discrete topology. For each $\alpha \in D$ we fix a point $y = f(\alpha) \in Y$ such that $V_\alpha \subseteq Uy$.

For each $x \in X$ we put $\psi(x) = \{\alpha \in D : x \in cl_X H_\alpha\}$ and $\varphi(x) = \{\alpha \in D : x \in H_\alpha\}$. By construction, $\psi : X \to D$ is upper semi-continuous, $\varphi : X \to A$ is lower semi-continuous, $\varphi(x) \subseteq \psi(x)$ and $f(\psi(x)) \subseteq \theta(x)$ for every $x \in X$. For each $n \in \mathbb{N}$ there exists a decomposition $\{A_\mu : \mu \in M_n\}$ of the set $A_n$ such that: - each set $A_\mu$ is countable and the set $W_\mu = \cup\{H_\alpha : \alpha \in A_\mu\}$ is open-and-closed in $X$; - $W_\mu \cap W_\nu = \varnothing$ for all distinct elements $\mu, \nu \in M_n$.

By construction, $W_\mu = \cup\{cl_X H_\alpha : \alpha \in A_\mu\}$ and for each point $x \in X$ there exists a minimal number $l(x) \in \mathbb{N}$ such that $\psi(x) \subseteq \cup\{A_n : n \leq l(x)\}$. On each $M_n$ we consider the discrete topology and put $Z = \Pi\{M_n : n \in \mathbb{N}\}$ and $g(x) = \{\mu(x, n) : n \in \mathbb{N}\}$ for each $x \in X$. Since the covers $\{W_\mu : \mu \in M_n\}$ are discrete, $g : X \to Z$ is a single-valued continuous mapping. Fix $x \in X$. Put $Z(x) = \{(\mu_n : n \in \mathbb{N}) \in Z : \mu_i = \mu(x, i) \textit{ for any } i \leq l(x)\}$. The set $Z(x)$ is closed-and-open and $g^{-1}(W(x) = \cap\{W_{\mu(x,n)} : n \leq l(x)\}$. Let $D(x) = D \cap (\cup\{A_{\mu(x,n)} : n \in \mathbb{N}\})$. The set $D(x)$ is countable and $\psi(x) \subseteq D(x)$. Fix $s \in X$. Then $g(s) = g(x)$ if and only if $D(s) = D(x)$. Thus $\psi(g^{-1}(x)) \subseteq D(x)$. This complete the proof. ■

Assume that for any sequence $\{U_n : n \in \mathbb{N}\}$ of open-and-closed subsets of $X$ the set $\cap\{U_n : n \in \mathbb{N}\}$ is open. In this case the mapping $g : X \to Z$ is continuous if on $Z$ we consider the discrete topology. Let $h(\alpha, z) = f(\alpha)$. Then $h : D \times Z \to Y$ is a single-valued mapping, the mapping $\lambda : X \to D \times Z$ is upper semi-continuous and finite-valued, $h(\lambda(x) \subseteq \theta(x)$ and $\lambda^\infty(x)$ is a countable subspace of $D \times Z$ for every $x \in X$. Therefore from ([5], Theorem 2) it follows

**Corollary 4.5** *Let $X$ be a fully paracompat space and for any sequence $\{U_n : n \in \mathbb{N}\}$ of open-and-closed subsets of $X$ the set $\cap\{U_n : n \in \mathbb{N}\}$ is open. Then the space $X$ is strongly paracompact.*

**Theorem 4.5** *For a $T_1$-space $X$ the following are equivalent:*
*1. $X$ is fully paracompact.*

*2. For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space $(Y, d)$ and an open locally finite cover $\{H_\alpha : \alpha \in A\}$ of of the space $Y$ there exist a lower semi-continuous compact-valued mapping $\phi : X \to Y$, an upper semi-continuous compact-valued mapping $\psi : X \to Y$, a metric space $Z$ and a single-valued continuous mapping $g : X \to Z$ such that:*

*2.1. $\phi(x) \subseteq \psi(x) \subseteq \theta(x)$ for every $x \in X$;*

*2.2. $\phi(X) \subseteq \bigcup\{H_\alpha : \alpha \in A\}$;*

*2.3. The family $\{g(\psi^{-1}(H_\alpha)) : \alpha \in A\}$ has a $\sigma$-discrete decomposition in $Z$;*

*2.4. $dimZ = 0$;*

*2.5. For each point $x \in X$ the subspace $\psi(g^{-1}(g(x)))$ is separable.*

*3. For every lower semi-continuous closed-valued mapping $\theta : X \to Y$ into a complete metric space $(Y, d)$ there exist a lower semi-continuous compact-valued mapping $\phi : X \to Y$, an upper semi-continuous compact-valued mapping $\psi : X \to Y$, a metric space $Z$, a single-valued continuous mapping $g : X \to Z$ and an open base $\{H_\alpha : \alpha \in A\}$ of $Y$ such that:*

*3.1. $\phi(x) \subseteq \psi(x) \subseteq \theta(x)$ for every $x \in X$;*

*3.2. $\psi(g^{-1}(g(x)))$ is a separable metric space for every $x \in X$;*

*3.3. The family $\{g(\psi^{-1}(H_\alpha)) : \alpha \in A\}$ has a $\sigma$-discrete decomposition in $Z$;*

*3.4. $dimZ = 0$.*

*4. A space $X$ is regular and for every lower semi-continuous mapping $\theta : X \to Y$ into a complete metric space $Y$ and an open locally finite cover $\{H_\alpha : \alpha \in A\}$ of of the space $Y$ there exist a lower semi-continuous mapping $\psi : X \to Y$, a metric space $Z$ and a simple-valued continuous mapping $g : X \to Z$ such that:*

*4.1. $\psi(x) \subseteq \theta(x)$ for every $x \in X$;*

*4.2. $\psi(g^{-1}(g(x)))$ is a separable metric space for every $x \in X$;*

*4.3. The family $\{g(\psi^{-1}(H_\alpha)) : \alpha \in A\}$ has a $\sigma$-discrete decomposition in $Z$;*

*4.4. $dimZ = 0$.*

*5. For every lower semi-continuous mapping $\theta : X \to Y$ into a complete metric space $Y$ and an open locally finite cover $\{H_\alpha : \alpha \in A\}$ of of the space $Y$ there exist an upper semi-continuous mapping $\psi : X \to Y$, a metric space $Z$ and a simple-valued continuous mapping $g : X \to Z$ such that:*

*5.1. $\psi(x) \subseteq \theta(x)$ for every $x \in X$;*

*5.2. $\psi(g^{-1}(g(x)))$ is a separable metric space for every $x \in X$;*

*5.3. The family $\{g(\psi^{-1}(H_\alpha)) : \alpha \in A\}$ has a $\sigma$-discrete decomposition in $Z$;*

*5.4.* $dim Z = 0$.

*6. A space* $X$ *is regular and for every lower semi-continuous mapping* $\theta : X \to Y$ *into a discrete space* $Y$ *there exist a set-valued mapping* $\psi : X \to Y$*, a metric space* $Z$ *and a simple-valued continuous mapping* $g : X \to Z$ *such that:*

*6.1.* $\psi(x) \subseteq \theta(x)$ *for every* $x \in X$*;*

*6.2.* $\psi(g^{-1}(g(x)))$ *is a separable metric space for every* $x \in X$*;*

*6.3. The family* $\{g(\psi^{-1}(y)) : y \in Y\}$ *has a* $\sigma$*-discrete decomposition in* $Z$*;*

*6.4.* $dim Z = 0$.

P r o o f. Implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ follows from Proposition 4. Implications $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 6$ and $3 \Rightarrow 5 \Rightarrow 6$ are obvious.

($6 \Rightarrow 1$): Let $\omega = \{U_\alpha : \alpha \in Y\}$ be an open cover of $X$. Introduce on $Y$ the discrete topology with the metric $d$, where $d(y, z) = 1$ for every $y, z \in Y$, $y \neq z$. The mapping $\theta : X \to Y$, where $\theta(x) = \{y \in Y : x \in U_y\}$ is lower semi-continuous. Let $\varphi : X \to Y$ be a set-valued mapping, $Z$ be a zero-dimensional space, $g : X \to Z$ be a single-valued mapping and there exist a sequence $\{\xi_n = \{P_\beta : \beta \in B_n\} : n \in \mathbb{N}\}$ of discrete families of subsets of $Z$ and a sequence $\{p_n : B_n \to Y : n \in \mathbb{N}\}$ of single-valued mappings such that $Y = \bigcup\{p_n(B_n) : n \in \mathbb{N}\}$ and $\bigcup\{P_\beta : \beta \in p_n^{-1}(y), n \in \mathbb{N}\} = g(\psi^{-1}(y))$ for every $y \in Y$, a $\sigma$-discrete decomposition of $\{g(\psi^{-1}(y)) : y \in Y\}$ in $X$.

Fix $n \in \mathbb{N}$. There exists a discrete cover $\{H_\beta : \beta \in B_n\}$ of $Z$ such that $P_\beta \subseteq H_\beta$ for every $\beta \in B_n$. Fix an open subset $V_\beta$ of $X$ such that $g^{-1}(P_\beta) \cap \theta^{-1}(p_n(\beta)) \subseteq V_\beta \subseteq U_{p_n(\beta)} \cap g^{-1}(H_\beta)$ for every $\beta \in B_n$. Then $\gamma_n = \{g^{-1}(H_\beta) : \beta \in B_n\} \cup \{V_\beta : \beta \in B_n\}$ is a star-finite open cover of $X$. By construction $\gamma = \{V_\beta : \beta \in \bigcup\{B_n : n \in \mathbb{N}\}\}$ is a refinement of $\omega$. Thus $X$ is fully paracompact. ∎

**Question 4.5.** Suppose that for a $T_1$-space $X$ and for every lower semi-continuous mapping $\theta : X \to Y$ into a discrete space $Y$ there exist an upper semi-continuous compact-valued mapping $\psi : X \to Y$, a metric space $Z$ and a single-valued continuous mapping $g : X \to Z$ such that $\psi(x) \subseteq \theta(x)$ for every $x \in X$, $dim Z = 0$ and the set $\psi(g^{-1}(z))$ is countable for every $z \in Z$. Is it true that $X$ is fully paracompact?

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