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On the Solvability of Discrete Nonlinear Hammerstein Systems in $l_{p,\sigma}$ Spaces

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In this article we observe a discrete nonlinear Hammerstein system of equation

$$x = KFx + g, (x, g \in l_{p,\sigma})$$

in weighted Banach spaces and establish some results about its unique solvability.

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We study a solvability of a discrete nonlinear Hammerstein system of equations (see [7]):

$$x(t) = \sum_{i=1}^{\infty} k(s,t) f(s, x(s)) + g(t), (t \in \mathbf{N}) \tag{1}$$

with non-symmetric kernel $k(s,t)$ $(s, t \in \mathbf{N})$ in weighted Banach spaces of sequences - $l_{p,\sigma}$. By $l_{p,\sigma}$ $(1 \leq p < \infty)$ we denote spaces of functions $x : \mathbf{N} \to \mathbf{R}$ (real functions of natural argument) for which norm

$$\|x\|_{p,\sigma} = \left(\sum_{s \in \mathbf{N}} |x(s)|^p \sigma(s) \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty) \tag{2}$$

makes sense and it is finite. Here σ is a weight function ($\sigma(s)$ is a sequence of positive numbers).In the case $p = \infty$ we can introduce the space $l_{\infty,\sigma}$ with norm

$$\|x\|_{\infty,\sigma} = sup_{s \in \mathbf{N}} |x(s)| \sigma(s)$$

but then $\sigma \in l_\infty$ and in fact $l_{\infty,\sigma} = l_\infty$. The scalar product and norm in $l_{2,\sigma}$ are given by

$$\langle x, y\rangle = \sum_{s=1}^{\infty} x(s)y(s)\sigma(s), \quad \|x\|_{2,\sigma} = \left(\sum_{s\in\mathbf{N}} |x(s)|^2\right)^{\frac{1}{2}} = \langle x, x\rangle^{\frac{1}{2}}.$$

We shall suppose that weight function is given and satisfies the condition $\sigma(s) \geq 1$ $(\forall n \in \mathbf{N})$ which is both necessary and sufficient condition that $p < q$ implies $l_{p,\sigma} \subset l_{q,\sigma}$ ([5]). It is important to know this relation because we are searching the existence of solution of system (1) in the "smallest" space and show its uniqueness in the "largest" space.
In system (1) $k : \mathbf{N} \times \mathbf{N} \to \mathbf{R}$ defines a linear bounded operator

$$Kx(t) = \sum_{s=1}^{\infty} k(s,t)x(s) \tag{3}$$

and $f : \mathbf{N} \times \mathbf{R} \to \mathbf{R}$ is a real function which generates a nonlinear operator superpositon F :

$$Fx(s) = f(s, x(s)), \quad x \in l_{p,\sigma}. \tag{4}$$

The problem of solvability of the system (1) is equivalent to the problem of solvability of the operator equation

$$x = KFx + g, \quad (x, g \in l_{p,\sigma}). \tag{5}$$

The function g is a real function of a natural argument; the function x is an unknown real sequence. Operator KF is, of course, nonlinear and it is called discrete Hammerstein operator. Let the linear operator K defined by non-symmetric matrix-kernel $k : \mathbf{N} \times \mathbf{N} \to \mathbf{R}$ acts from $l_{p',\tau}$ into $l_{p,\sigma}$ space where $1 \leq p \leq \infty$, $\sigma(s) \geq 1$, $\tau(s) \geq 1$ and $p' = p(p-1)^{-1}, \infty' = 1$. Operator K is a bounded operator in $l_{p',\tau}$ space and at the same time a compact operator in those spaces if holds ([8])

$$\sum_{t=1}^{\infty}\sum_{s=1}^{\infty} |k(s,t)|^{\max\{2,p\}}\sigma(t) < \infty. \tag{6}$$

Suppose that operator (3) acts not only in $l_{2,\tau}$, but also from $l_{p',\tau}$ into $l_{p,\sigma}$ where $2 \leq p \leq \infty$. Let

$$A = \frac{1}{2}(K + K^*). \tag{7}$$

denotes the self-adjoint part of K, where K^* is the adjoint operator defined by $K^*x(t) = \sum_{s=1}^{\infty} k(t,s)x(s)$. Observe also operator $B = \dfrac{1}{2}(K - K^*)$; see [2]. Both

operators A and B act from the space $l_{p',\tau}$ into the space $l_{p,\sigma}$. Let $A = UL$ be a polar decomposition of the operator A into a superposition of an unitary operator U, acting in $l_{2,\sigma}$ and the positive defined operator L, $C = L^{\frac{1}{2}}$. Assume that A is a positive defined operator (A is positive defined operator in $l_{2,\tau}$ iff $\langle Ax, x\rangle \geq 0$, $\forall x \in l_{2,\tau}$).
As we have $p' \leq 2 \leq p$ and $\tau(s) \geq 1$ ($\forall s \in \mathbf{N}$), it implies $l_{p',\tau} \subseteq l_{2,\tau} \subseteq l_{p,\tau}$. Let $\overline{\lim_{n\to\infty}} \dfrac{\sigma(n)}{\tau(n)} \in (0,\infty)$ then $l_{p,\tau} \subseteq l_{p,\sigma}$ (see [1], [5]) and $l_{p',\tau} \subseteq l_{2,\tau} \subseteq l_{p,\tau} \subseteq l_{p,\sigma}$ ([9]).
Now the operator A can be represented in the form $A = CC^*$ where $C = A^{\frac{1}{2}}$ is the square root of A acting from $l_{2,\tau}$ into $l_{p,\tau}$ and from $l_{2,\tau}$ into $l_{p,\sigma}$as well. The adjoint operator C^* acts from $l_{p',\tau}$ into $l_{2,\tau}$.

Definition 1 We call the operator K to be **P** - positive if satisfies the angle-bounded inequality

$$|\langle Kx, y\rangle - \langle x, Ky\rangle| \leq \beta\sqrt{\langle Kx, x\rangle}\sqrt{\langle Ky, y\rangle} \qquad (x, y \in l_{2,\tau}) \tag{8}$$

where $\beta \in \mathbf{R}^+$ and operator A (7) is a positive defined operator.([2])
Observe now operators $M = C^{-1}K(C^*)^{-1}$ and $N = K(C^*)^{-1}$ and note that, under our assumptions, both M and N act in the space $l_{2,\tau}$.

Lema 1 *If the operator K is **P**-positive then the operators M and N are bounded.*

P r o o f. The operator M is bounded in $l_{2,\tau}$ if and only if $C^{-1}B(C^*)^{-1}$ is bounded in $l_{2,\tau}$. The same statement is valid for the operators $N = B(C^*)^{-1}$. For any $h \in l_{2,\tau}$, we have

$$\langle Mh, h\rangle + \langle h, Mh\rangle - 2\langle h, h\rangle =$$

$$\left\langle K(C^*)^{-1}h, (C^*)^{-1}h\right\rangle + \left\langle K^*(C^*)^{-1}h, (C^*)^{-1}h\right\rangle - 2\left\langle C^*(C^*)^{-1}h, C^*(C^*)^{-1}h\right\rangle =$$

$$\left\langle K(C^*)^{-1}h, (C^*)^{-1}h\right\rangle + \left\langle K^*(C^*)^{-1}h, (C^*)^{-1}h\right\rangle - 2\left\langle A(C^*)^{-1}h, (C^*)^{-1}h\right\rangle = 0$$

so one can conclude that holds

$$\langle Mh, h\rangle = \|h\|^2, \quad (h \in l_{2,\tau}). \tag{9}$$

This relation, in particular, means that both operators M and M^* have a trivial null-space. On the other side, for arbitrary $h_1, h_2 \in l_{2,\tau}$, is

$$\left|\left\langle C^{-1}B(C^*)^{-1}h_1, h_2\right\rangle\right| = \left|B(C^*)^{-1}h_1, (C^*)^{-1}h_2\right| = |\langle B\phi, \theta\rangle| =$$

$$\left|\frac{1}{2}\langle (K-K^*)\phi,\theta\rangle\right| = \left|\frac{1}{2}\langle K\phi,\theta\rangle - \frac{1}{2}\langle K^*\phi,\theta\rangle\right| = \left|\frac{1}{2}\langle K\phi,\theta\rangle - \frac{1}{2}\langle \phi,K\theta\rangle\right| \leq$$
$$\frac{1}{2}\beta\sqrt{\langle K\phi,\phi\rangle}\sqrt{\langle K\theta,\theta\rangle} =$$
$$\frac{1}{2}\beta\sqrt{\langle K(C^*)^{-1}h_1,(C^*)^{-1}h_1\rangle}\sqrt{\langle K(C^*)^{-1}h_2,(C^*)^{-1}h_2\rangle} =$$
$$\frac{1}{2}\beta\sqrt{\langle C^{-1}K(C^*)^{-1}h_1,h_1\rangle}\sqrt{\langle C^{-1}K(C^*)^{-1}h_2,h_2\rangle} =$$
$$\frac{1}{2}\beta\sqrt{\langle Mh_1,h_1\rangle}\sqrt{\langle Mh_2,h_2\rangle} = \frac{1}{2}\beta\|h_1\|\|h_2\|$$

therefore the operator $M = C^{-1}K(C^*)^{-1}$ is bounded. For the operator $B(C^*)^{-1}$ we have

$$\left|\left\langle B(C^*)^{-1}h,g\right\rangle\right| = \left|\frac{1}{2}\left\langle (K-K^*)(C^*)^{-1}h,g\right\rangle\right| =$$

$$\left|\frac{1}{2}\left\langle K(C^*)^{-1}h,g\right\rangle - \frac{1}{2}\left\langle K^*(C^*)^{-1}h,g\right\rangle\right| = \left|\frac{1}{2}\left\langle K(C^*)^{-1}h,g\right\rangle - \frac{1}{2}\left\langle (C^*)^{-1}h,Kg\right\rangle\right| \leq$$
$$\frac{1}{2}\beta\sqrt{\langle K(C^*)^{-1}h,(C^*)^{-1}h\rangle}\sqrt{\langle Kg,g\rangle} = \frac{1}{2}\beta\sqrt{\langle C^{-1}K(C^*)^{-1}h,h\rangle}\sqrt{\langle Kg,g\rangle} =$$
$$= \frac{1}{2}\beta\sqrt{\langle Mh,h\rangle}\sqrt{\langle Kg,g\rangle} \leq \frac{1}{2}\widetilde{\beta}\|h\|\|g\|$$

so $B(C^*)^{-1}$ is bounded operator as well as the operator N. ■

The operator N has also a trivial null-space, since $N = CM$ and the operator C has a trivial null space as well. Now we can keep the same notation M and N for the closure (continuous extension) in $l_{2,\tau}$ of the operators $C^{-1}K(C^*)^{-1}$ and $K(C^*)^{-1}$, respectively. This closure, in our position, coincides with $l_{2,\tau}$ ([8],[9]). Operator K has two essential decompositions

$$K = CMC^*, \qquad K = NC^*. \tag{10}$$

On the other hand, operators M, N and K are related by

$$N = CM, \qquad N^* = M^*C^*. \tag{11}$$

Denote by μ_K the smallest positive number of all μ for which is

$$\|Kh\|^2 \leq \mu\langle Kh,h\rangle \qquad (h \in l_{2,\tau}). \tag{12}$$

It holds $\mu_K = \|N\|^2$. Indeed,

$$\langle Kh,Kh\rangle = \langle NC^*h,NC^*h\rangle = \|NC^*h\|^2 \leq \|N\|^2\langle C^*h,C^*h\rangle =$$

$$\|N\|^2\langle CC^*h,h\rangle = \|N\|^2\langle Ah,h\rangle = \frac{1}{2}\|N\|^2\langle (K+K^*)h,h\rangle = \|N\|^2\langle Kh,h\rangle.$$

The nonlinear part of the equation (5) is the operator superposition F which acts from $l_{p,\sigma}$ into $l_{p',\tau}$. Suppose that $f(s,0)=0$, though this condition could be easily omitted if we replace the operator F by the operator $\widetilde{F}x = F(x+\widetilde{x}) - F\widetilde{x}$. Due to ([3], [1]), operator F, in the case $1 \le p < \infty$, acts from $l_{p,\sigma}$ into $l_{q,\tau}$ if and only if there exist a function a $a \in l_{q,\tau}$, and constants $b \ge 0, \delta > 0$ such that estimate

$$|f(s,u)| \le a(s) + b\sigma^{\frac{1}{q}}(s)\tau^{-\frac{1}{q}}(s)|u|^{\frac{p}{q}} \tag{13}$$

holds for all pairs $(s,u) \in \mathbf{N}\times\mathbf{R}$, for which is $\sigma(s)|u|^p \le \delta^p$. When $\frac{1}{p}+\frac{1}{q}=1$ we have

$$|f(s,u)| \le a(s) + b\sigma^{\frac{p-1}{p}}(s)\tau^{\frac{1-p}{p}}(s)|u|^{p-1} \qquad (\sigma(s)|u|^p \le \delta^p).$$

In case $p=\infty$, operator F acts from l_∞ to $l_{q,\tau}$ if and only if

$$(\forall r>0)(\exists a_r \in l_{q,\tau}) : |f(s,u)| \le a_r(s) \qquad (|u| \le r; 0 < r < \infty).$$

Suppose now that there exist a number c such that holds

$$(u-v)(f(s,u)-f(s,v)) \le c(u-v)^2 \qquad (s \in \mathbf{N}, u \in \mathbf{R}). \tag{14}$$

If c_f is the smallest c for which (14) holds, we have

$$\langle Fh^* - Fh^{**}, h^* - h^{**}\rangle = \sum_{s\in\mathbf{N}} [f(s,h^*(s)) - f(s,h^{**}(s))][h^*(s) - h^{**}(s)]\sigma(s) \le$$

$$c_f \sum_{s\in\mathbf{N}} [h^*(s) - h^{**}(s)][h^*(s) - h^{**}(s)]\sigma(s) = c_f\|h^* - h^{**}\|^2.$$

Theorem 1 *Let the operator K, defined by (3), be $\mathbf{P}$-positive. Suppose that the generator $f : \mathbf{N}\times\mathbf{R} \to \mathbf{R}$ of superposition operator F given by (4), satisfies condition (14) for some $c_f > 0$ and $f(s,0)=0$ for all $s \in \mathbf{N}$. If*

$$c_f\mu_K < 1 \tag{15}$$

where μ_K is defined by (12) then, for arbitrary $g \in N(l_{2,\tau})$ the equation $x = KFx + g$ has a solution $\widehat{x} \in N(l_{2,\tau})$. If $g = Nl$ for some $l \in l_{2,\tau}$ then there exists $\widehat{h} \in l_{2,\tau}$ such that $\widehat{x} = N\widehat{h}$, and

$$\|\widehat{h}\| \le \frac{\|l\|}{1 - c_f\mu_K}. \tag{16}$$

Moreover, the solution x is unique in the space $l_{p,\sigma}$.

P r o o f. Put $\Phi h = M^*h - N^*FNh - M^*l$, and consider the operator equation

$$\Phi h = 0 \tag{17}$$

i.e. $M^*h = N^*FNh + M^*l$. If h is a solution of the equation $\Phi h = 0$ (i.e. $M^*\widehat{h} = N^*FN\widehat{h} + M^*l$), then, by the relation (11) we get $M^*(\widehat{h} - C^*FN\widehat{h} + l) = 0$. Since operator M^* has trivial null-space, it follows $\widehat{h} = C^*FN\widehat{h} + l$. Now, by applying the operator N to the last equation we get

$$N\widehat{h} = NC^*FN\widehat{h} + Nl = KFN\widehat{h} + g \tag{18}$$

using the relation (10). From the equations (18) and (5) we conclude that $\widehat{x} = N\widehat{h}$ is a solution of the system (1). In order to prove the existence of a solution of the equation (4), we are going to study the equation (17), under assumptions of the Theorem 1. Operator

$$\Phi h = M^*h - N^*FNh - M^*l \tag{19}$$

is monotone in the Minty-Browder sense (see ([10], [6])), in fact, for any $h_1, h_2 \in l_{2,\tau}$ we have

$$\begin{aligned}
&\langle \Phi h_1 - \Phi h_2, h_1 - h_2\rangle = \langle M^*h_1 - N^*FNh_1 - M^*h_2 + N^*FNh_2, h_1 - h_2\rangle = \\
&= \langle M^*(h_1 - h_2), h_1 - h_2\rangle - \langle N^*FNh_1 - N^*FNh_2, h_1 - h_2\rangle = \\
&= \langle M^*(h_1 - h_2), h_1 - h_2\rangle - \langle FNh_1 - FNh_2, Nh_1 - Nh_2\rangle = \\
&= \|h_1 - h_2\|^2 - \langle FNh_1 - FNh_2, Nh_1 - Nh_2\rangle \geq \\
&\geq \|h_1 - h_2\|^2 - c_f\langle Nh_1 - Nh_2, Nh_1 - Nh_2\rangle \geq \\
&\geq \|h_1 - h_2\|^2 - c_f\|N\|^2\|h_1 - h_2\|^2 = \\
&= (1 - c_f\|N\|^2)\|h_1 - h_2\|^2 \geq (1 - c_f\mu_K)\|h_1 - h_2\|^2.
\end{aligned}$$

So we get

$$\langle \Phi h_1 - \Phi h_2, h_1 - h_2\rangle \geq (1 - c_f\mu_K)\|h_1 - h_2\|^2$$

and on the sphere $S = \{h \in l_{2,\tau} | \|h\| = r\}$ follows estimate

$$\begin{aligned}
\langle \Phi h, h\rangle = \langle \Phi h - \Phi 0, h - 0\rangle + \langle \Phi 0, h\rangle \geq (1 - c_f\mu_K)\|h\|^2 - \|l\| \cdot \|h\| = \\
= (1 - c_f\mu_K)r^2 - \|l\|r.
\end{aligned}$$

Since $F0 = 0$ and $M^*(h - C^*FNh - l) = \Phi h$, taking $h = 0$ we have $\Phi(0) = -l$. If we take a sphere S with

$$r \geq \frac{\|l\|}{1 - c_f \mu_K}$$

then $\langle \Phi h, h \rangle \geq 0$ holds for any $h \in S$. Due to the Minty-Browder existence principle, the equation (17) has the unique solution $\widehat{h} \in S \subset l_{2,\tau}$. On the other hand, $\widehat{x} = N\widehat{h}$ is a solution of the Hammerstein nonlinear system (1). Moreover, if holds (14), $\widehat{x} \in l_{2,\sigma}$ is the unique solution of the system (1). In order to prove it, let us suppose that $\widehat{x}$ and $\widetilde{x}$ are two solutions of the system (1) with $g = Nl$ for some $l \in l_{2,\tau}$. If we put

$$\widehat{h} = C^*F\widehat{x} + l, \quad \widetilde{h} = C^*F\widetilde{x} + l$$

then the elements $\widehat{h}$ i $\widetilde{h}$ belong to $l_{2,\tau}$ and since $\widehat{x} = N\widehat{h}$, $\widetilde{x} = N\widetilde{h}$, we have

$$\widehat{h} = C^*FN\widehat{x} + l, \quad \widetilde{h} = C^*FN\widetilde{x} + l.$$

Let us consider now operator (19) and (given above) relation (9), we get

$$\Phi\widehat{h} = M^*\widehat{h} - M^*C^*FN\widehat{h} - M^*l = M^*(\widehat{h} - C^*FN\widehat{h} - l) = M^*(\widehat{h} - \widehat{h}) = 0$$

and

$$\Phi\widetilde{h} = M^*\widetilde{h} - M^*C^*FN\widetilde{h} - M^*l = M^*(\widetilde{h} - C^*FN\widetilde{h} - l) = M^*(\widetilde{h} - \widetilde{h}) = 0.$$

The equation $\Phi h = 0$ has only one solution, so we conclude $\widehat{h} = \widetilde{h}$ and hence $\widehat{x} = \widetilde{x}$, because the operator N has trivial null-space. ∎

These results can be extended to the case when operator A is not positive defined operator; see [4]. Suppose that operator $A = \frac{1}{2}(K + K^*)$ is a quasi-positive defined i.e. operator A has at most a finite number of the negative eigenvalues of the multiplicity 1. If a matrix-kernel $a(s,t)$ has at most a finite number (for example first n) of the negative eigenvalues each of the multiplicity 1, we can write this kernel in the form ([8])

$$a(s,t) = -\sum_{i=1}^{n} \alpha_i e_i(s)e_i(t) + \sum_{i=n+1}^{\infty} \alpha_i e_i(s)e_i(t) \quad (s,t \in \mathbf{N}) \tag{20}$$

where all $\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1}, ...$ are positive numbers. The relation

$$l(s,t) = a(s,t) + 2\sum_{i=1}^{\infty} \alpha_i e_i(s)e_i(t), \quad (s,t \in \mathbf{N}) \tag{21}$$

gives an important connection between the kernel $a(s,t)$ and the kernel $l(s,t)$ which has all positive eigenvalues $\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1}, ...$ .In this situation we can consider the finite-dimensional orthogonal projection P of the $l_{2,\tau}$ into the subspace of the eigenvalues of A. The operator P acts at the same time in $l_{p,\sigma}$ and $l_{p,\tau}$ and commutes with A. Moreover, in the polar decomposition $A = UL$ mentioned in the previous section we can take

$$A = (I - 2P)L \tag{22}$$

where $L = (I - 2P)A$ is now a positive defined operator. The operator L from (22) can be represented in the form $L = DD^*$, where $D = L^{\frac{1}{2}}$ acts from $l_{2,\tau}$ into $l_{p,\sigma}$ and D^* acts from $l_{p',\tau}$ into $l_{2,\tau}$. We will call the operator K **P**-quasi-positive operator if K satisfies condition (8), and its self-adjoint part is a quasi-positive defined operator.

Lema 2 ([4]) *If the operator K is $\mathbf{P}$-quasi-positive then operator $M = D^{-1}K(D^*)^{-1}$ satisfies $\langle Mh, h\rangle = \|h\|^2 - 2\|Ph\|^2$ for all $h \in l_{2,\tau}$.*

Now using Theorem 1 and number

$$\nu_K = \sup\{\nu | \nu > 0,\ \|Nh\| \geq \sqrt{\nu}\|Ph\| \quad (h \in l_{2,\tau})\}. \tag{23}$$

it is not difficult to prove (see [4]):

Theorem 2 *Let the operator K defined by (K) be P-quasi-positive in $l_{2,\tau}$. Suppose that the generator $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ of the superposition operator F given by (4), satisfies (14) for some $c_f > 0$ and $f(s, 0) = 0$ for all $s \in \mathbf{N}$. If*

$$c_f \nu_K < -1 \tag{24}$$

where ν_K is defined above by (23), then, for arbitrary $g \in N(l_{2,\tau})$, equation

$$x = KFx + g$$

has a solution $\widehat{x} \in N(l_{2,\tau})$. If $g = Nl$ for some $l \in l_{2,\tau}$ then there exists $\widehat{h} \in l_{2,\tau}$ such that $\widehat{x} = N\widehat{h}$, and

$$\|\widehat{h}\| \leq -\frac{\|l\|}{1 + c_f \nu_K}. \tag{25}$$

Moreover, the solution $\widehat{x}$ is unique in the space $l_{p,\sigma}$.

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