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A Method for Solving Families of Quartic Thue Inequalities

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In this paper we give a method for solving families of quartic Thue inequalities. We use the generalizations of the classical results of Legendre and Fatou concerning Diophantine approximations of the form $|\alpha - \frac{a}{b}| < \frac{1}{2b^2}$ and $|\alpha - \frac{a}{b}| < \frac{1}{b^2}$, to the approximations of the form

$$\left|\alpha - \frac{a}{b}\right| < \frac{k}{b^2}$$

for a positive real number k, due to Worley, Dujella and Ibrahimpašić.

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1. Introduction

Let $f \in \mathbb{Z}[X,Y]$ be a homogeneous irreducible polynomial of degree $n \geq 3$ and $\mu \neq 0$ fixed integer. Then the Diophantine equation

$$f(x,y) = \mu \tag{1.1}$$

is called Thue equation. In 1909, Thue [21] proved that equation (1.1) has only finitely many solutions $x, y \in \mathbb{Z}$. In 1968, Baker [1] gave an upper bound for the solutions of Thue equation, based on the theory of linear forms in logarithms of algebraic numbers.

Starting with Thomas [20] in 1990, parametrized families of Thue equations have been considered (see [12], [13] for references).

Tzanakis [22] considered Thue equations of the above mentioned form, where f is a quartic form which corresponding quartic field $\mathbb{K}$ is the compositum of two real quadratic fields. He showed that solving the mentioned equation reduces to solving a system of Pellian equations.

The applications of above mentioned method of Tzanakis for solving Thue equations of the special type has several advantages (see [22], [9], [10]). This method has been applied to several parametric families of quartic Thue equations and inequalities ([5], [8], [9], [10], [14], [24]).

2. Tzanakis method

Tzanakis [22] considered the equation

$$f(x,y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_2xy^3 + a_4y^4 \in \mathbb{Z}[x,y], \quad a_0 > 0, \quad (2.1)$$

where corresponding quartic field $\mathbb{K}$ of f is Galois and non-cyclic. To this equation we assign the cubic equation

$$4\rho^3 - g_2\rho - g_3 = 0 \quad (2.2)$$

where

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 \in \frac{1}{12}\mathbb{Z},$$

and

$$g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \in \frac{1}{432}\mathbb{Z}.$$

The roots of equation (2.2) are opposite to the roots of the cubic resolvent of the quartic equation $f(x,1) = 0$. By [17, Theorem 1], this condition on the field $\mathbb{K}$ is equivalent with $\mathbb{K}$ having three quadratic subfields, which happens when the cubic equation (2.2) has three distinct rational roots ρ_1, ρ_2, ρ_3 and

$$\frac{a_1^2}{a_0} - a_2 \geq \max\{\rho_1, \rho_2, \rho_3\}.$$

Let $H(x,y)$ and $G(x,y)$ be the quartic and sextic covariants of $f(x,y)$ respectively [16], with

$$H(x,y) = -\frac{1}{144}\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix},$$

$$G(x,y) = -\frac{1}{8}\begin{vmatrix} f_x & f_y \\ H_x & H_y \end{vmatrix},$$

where $H(x,y) \in \frac{1}{48}\mathbb{Z}[x,y]$ and $G(x,y) \in \frac{1}{96}\mathbb{Z}[x,y]$. Then we have $4H^3 - g_2Hf^2 - g_3f^3 = G^2$. Tzanakis proved that f and H are coprime in $\mathbb{Q}[x,y]$. If we put

$$H_0 = 48H, \qquad G_0 = 96G, \qquad r_i = 12\rho_i \quad (i = 1,2,3),$$

we have $H_0, G_0 \in \mathbb{Z}[x,y]$ and $r_i \in \mathbb{Z}, (i = 1,2,3)$, and we obtain

$$(H_0 - 4r_1 f)(H_0 - 4r_2 f)(H_0 - 4r_3 f) = 3G_0^2.$$

Since f and H are coprime in $\mathbb{Q}[x,y]$, then the factors on the left side are in pairs coprime in $\mathbb{Z}[x,y]$. There exist positive square–free integers k_1, k_2, k_3 and quadratic forms $G_1, G_2, G_3 \in \mathbb{Z}[x,y]$ such that

$$H_0 - 4r_i f = k_i G_i^2, \quad (i = 1,2,3) \tag{2.3}$$

and

$$k_1 k_2 k_3 (G_1 G_2 G_3)^2 = 3G_0^2.$$

If $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ is solutions of (1.1), where the form of f is (2.1), then from (2.3) we have

$$k_2 G_2^2 - k_1 G_1^2 = 4(r_1 - r_2)\mu, \tag{2.4}$$

$$k_3 G_3^2 - k_1 G_1^2 = 4(r_1 - r_3)\mu. \tag{2.5}$$

In this way, solving the Thue equation (1.1) reduces to solving the system of Pellian equations (2.4) and (2.5) with one common unknown.

In results of that type, the authors were able to solve completely the corresponding system(s) of Pellian equations, and from these solutions it is straightforward to find all solutions of the Thue equation (inequality).

Note that the system and the original Thue equation are not equivalent. The authors [8] do not have to solve the system of Pellian equations completely, as some solutions to the system do not lead to solutions of the Thue inequality.

We consider the parametric family of Thue inequalities

$$|f(x,y,c)| \le \mu(c),$$

for an positive integer c.

We apply the above mentioned method of Tzanakis to mentioned inequalities, i.e. to the equations

$$f(x,y,c) = m, \quad |m| \le \mu(c).$$

We obtain the system(s) of Pellian equations

$$\varphi_1(c)U^2 - \varphi_2(c)V^2 = \psi_1(m), \tag{2.6}$$

$$\varphi_3(c)U^2 - \varphi_4(c)Z^2 = \psi_2(m), \tag{2.7}$$

and we consider separately both equations of the mentioned system.

3. Diophantine approximations

We use the generalizations of the classical results of Legendre and Fatou concerning Diophantine approximations of the form $|\alpha - \frac{a}{b}| < \frac{1}{2b^2}$ and $|\alpha - \frac{a}{b}| < \frac{1}{b^2}$, to the approximations of the form $|\alpha - \frac{a}{b}| < \frac{k}{b^2}$, for a positive real number k, due to Worley, Dujella and Ibrahimpašić (see [23], [6], [7]).

Theorem 3.1. *(Worley [23], Dujella [6]) Let α be a real number and let a and b be coprime nonzero integers, satisfying the inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{k}{b^2}, \tag{3.1}$$

where k is a positive real number. Then $(a,b) = (rp_{m+1} \pm sp_m, rq_{m+1} \pm sq_m)$, for some $m \geq -1$ and nonnegative integers r and s such that $rs < 2k$, where $\frac{p_m}{q_m}$ denotes the $m-$th convergent of the continued fraction expansion of α.

Dujella and Ibrahimpašić [7] showed that this theorem is sharp, in the sence that the condition $rs < 2k$ cannot be replaced by $rs < (2-\varepsilon)k$ for any $\varepsilon > 0$.

Worley [23] gave explicit version of his result for $k = 2$. Dujella and Ibrahimpašić [7] extended Worley's work and gave explicit and sharp versions of mentioned theorem for $k = 3, 4, 5, \ldots, 12$. They gave the pairs (r, s) which appear in the expression of solutions to (3.1) in the form

$$(a,b) = (rp_{m+1} \pm sp_m, rq_{m+1} \pm sq_m).$$

Recently, Ibrahimpašić [15] extended this result to $0 \leq k \leq 13$.

4. The method

We consider the equation (2.6)

$$\varphi_1(c)U^2 - \varphi_2(c)V^2 = \psi_1(m)$$

of the mentioned system of Pellian equations, and we obtain the continued expansion of the corresponding quadratic irrationals

$$\sqrt{\frac{\varphi_1(c)}{\varphi_2(c)}}.$$

The simple continued fraction expansion of a quadratic irrational $\alpha = \frac{e+\sqrt{d}}{f}$ is periodic. This expansion can be obtained using the following algorithm (see Chapter 7.7 in [18]). Multiplying the numerator and the denominator by f, if necessary, we may assume that $f|(d-e^2)$. Let $s_0 = e$, $t_0 = f$ and

$$a_n = \left\lfloor \frac{s_n+\sqrt{d}}{t_n} \right\rfloor, \quad s_{n+1} = a_n t_n - s_n, \quad t_{n+1} = \frac{d-s_{n+1}^2}{t_n} \quad \text{for } n \geq 0.$$

If $(s_j, t_j) = (s_k, t_k)$ for $j < k$, then

$$\alpha = [a_0; \ldots, a_{j-1}, \overline{a_j, \ldots, a_{k-1}}].$$

In order to determine the values of m for which mentioned equation (2.6) has a solution, we use the following result (see Lemma 1 in [10]).

Lemma 4.1. *Let $\alpha\beta$ be a positive integer which is not a perfect square, and let p_k/q_k denotes the k-th convergent of the continued fraction expansion of $\sqrt{\frac{\alpha}{\beta}}$. Let the sequences (σ_k) and (τ_k) be defined by $\sigma_0 = 0$, $\tau_0 = \beta$ and*

$$a_k = \left\lfloor \frac{\sigma_k+\sqrt{\alpha\beta}}{\tau_k} \right\rfloor, \quad \sigma_{k+1} = a_k \tau_k - \sigma_k, \quad \tau_{k+1} = \frac{\alpha\beta-\sigma_{k+1}^2}{\tau_k} \quad \textit{for } k \geq 0.$$

Then

$$\alpha(rq_{k+1} + sq_k)^2 - \beta(rp_{k+1} + sp_k)^2 = (-1)^k(s^2\tau_{k+1} + 2rs\sigma_{k+2} - r^2\tau_{k+2}). \tag{4.1}$$

Now we use results from Dujella and Ibrahimpašić [7] and we inserting all possibilities for r and s in formula (4.1). In this way we obtain sets M_c^1 of the values of m for which the first equation (2.6) of the system has a solution. In the same way, we obtain the sets M_c^2 of the values of m for which the second equation (2.7) of the system has a solution. Finally, comparing these sets M_c^1 with sets M_c^2, we obtain the finite set of the values of m for which mentioned system(s) (2.6) and (2.7) has a solution.

From the comparison of a lower bound for solutions of mentioned system, obtained using the congruence method introduced in [11], and an upper bound obtained from the following theorem of Bennett [4] on simultaneous approximations of algebraic numbers, we obtained results for $c \geq W$, where W is the some positive integer.

Theorem 4.1. *If a_i, p_i, q and N are integers for $0 \leq i \leq 2$, with $a_0 < a_1 < a_2$ and $a_j = 0$ for some $0 \leq j \leq 2$, $q \geq 1$ and $N > M^9$, where*

$$M = \max_{0\leq i\leq 2} \{|a_i|\} \geq 3,$$

then we have

$$\max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}}-\frac{p_i}{q}\right|\right\} > (130N\Upsilon)^{-1}\,q^{-\lambda},$$

where

$$\lambda = 1+\frac{\log(32.04N\Upsilon)}{\log\left(1.68N^2\prod_{0\le i<j\le 2}(a_i-a_j)^{-2}\right)}$$

and

$$\Upsilon = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} & , \quad a_2-a_1\ge a_1-a_0 \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0} & , \quad a_2-a_1< a_1-a_0\ . \end{cases}$$

For $c<W$ we use a version of the reduction procedure due to Baker and Davenport [2] and the following theorem of Baker and Wüstholz [3].

Theorem 4.2. *For a linear form* $\Lambda = b_1\log\alpha_1+\cdots+b_l\log\alpha_l\neq 0$ *in logarithms of* l *algebraic numbers* $\alpha_1,\ldots,\alpha_l$ *with rational integer coefficients* $b_1,\ldots,b_l$ *we have*

$$\log\Lambda \ge -18\,(l+1)!\,l^{l+1}\,(32d)^{l+2}\,h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B,$$

where $B=\max\{|b_j| : 1\le j\le l\}$, *and where* d *is the degree of the number field generated by* $\alpha_1,\ldots,\alpha_l$.

In the special cases (for some values of m) we can use the Thue equation solver in PARI/GP [19] to solve directly corresponding Thue equations.

5. Application of the method

In [10], Dujella and Jadrijević considered the family of quartic Thue inequalities

$$\left|x^4-4cx^3y+(6c+2)x^2y^2+4cxy^3+y^4\right|\le 6c+4,$$

where $c\ge 0$ is an integer. They obtained set $M_c^1=\{1,-2c,2c+1,-6c+1,6c+4\}$. Analysing the corresponding system (2.6) and (2.7) and using the properties of the convergents of the corresponding quadratic irrationals, authors were able to show that the system has no solutions for $m=-2c,2c+1,-6c+1$. In [7], Dujella and Ibrahimpašić comparing the sets M_c^1 and M_c^2 showed that the system has solutions only for $m=1$ and $m=6c+4$.

In [14], Ibrahimpašić considered the family of quartic Thue inequalities

$$\left|x^4-2cx^3y+2x^2y^2+2cxy^3+y^4\right|\le 6c+4,$$

where $c \geq 0$ is an integer. Comparing the sets M_c^1 and M_c^2 the author showed that the system has solutions only for $m = 1, 4$ and $m = -12c + 25$ for $c = 3, 4$. Using above mentioned method the author obtained all trivial and nontrivial solutions of the family.

In [8], Dujella, Ibrahimpašić and Jadrijević considered the family of quartic Thue inequalities

$$\left| x^4 + 2(1 - c^2)x^2y^2 + y^4 \right| \leq 2c + 3,$$

where $c \geq 0$ is an integer. The system and the original Thue equation are not equivalent. Each solution of the Thue equation induces a solution of the system, but not vice–versa. An illustration of these phenomena is the mentioned family of Thue inequalities.

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