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Newton-Secant method for solving operator equations *

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We propose and analyze a generalization of the known Newton-Secant iterative method in case of solving system of nonlinear equations, which is essentially of third order. One modification of the presented method is considered, also. Convergence analysis and numerical examples are included.

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1. Introduction

Let consider a operator equation

$$F(x) = 0\,, \tag{1}$$

where F is a Fréchet-differentiable operator defined on an open subset D of a Banach space X with values in a Banach space Y. Finding roots of Eq.(1) is a classical problem arising in many areas of applied mathematics and engineering.

In this study we are concerned with the problem of approximating a locally unique solution α of Eq.(1). Some of the well known methods for this purpose are the following ones:

1.1 Newton's iterative method

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k)\,, \quad for \quad k \geq 0\,, \tag{2}$$

which converges quadratically and requires one evaluation of F' at each step.

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1.2 Classical Secant's method

$$x_{k+1} = x_k - [x_{k-1}, x_k]^{-1} F(x_k)\,, \quad for \quad k \geq 0\,, \tag{3}$$

which starting with two points $x_{-1}\,, x_0 \in D$, and converges super-linearly. Linear operator $[x, y] \in L(X, Y)$ is called *divided difference* of F at the points x and y, and it is characterized by the condition [1]:

$$[x, y](x - y) = F(x) - F(y)\,. \tag{4}$$

Now, let us cite some of the known results about the convergence of Newton's and Secant's methods. For the first one the basic results of the properties of the method are obtained in [1]. In [2, 3, 4] local convergence is obtained under the following Kantorovich-type conditions:

$$\|F'(\alpha)^{-1}(F'(x) - F'(y))\| \leq k_1 \|x - y\|\,, \quad for \;\; all \;\; x, y \in D\,.$$

For the Secant's method, conditions on the *Fréchet* derivative are replaced by conditions on the divided difference [6]:

$$\|[x, y] - [u, v]\| \leq k_2(\|x - u\| + \|y - v\|)\,, \quad for \;\; all \;\; x, y, u, v \in D\, x \neq y, u \neq v\,.$$

See also [7, 8, 9, 11].

2. The New Modifications

We introduce the following two iterative methods:
1.3 Newton-Secant method

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1} F(x_k)\,, \\ x_{k+1} &= x_k - [x_k, y_k]^{-1} F(x_k)\,, \quad for \quad k \geq 0\,, \;\; and \;\; x_0 \in D\,. \end{aligned} \tag{5}$$

This method is a generalization of the known Newton-Secant method for solving a nonlinear equation in one dimensional case (see [10]), and represents a composition of the Newton's method (2) and the Secant's method (3).
1.4 Modified Newton-Secant method

$$\begin{aligned} y_k &= x_k - F'(x_0)^{-1} F(x_k)\,, \\ x_{k+1} &= x_k - [x_k, y_k]^{-1} F(x_k)\,, \quad for \quad k \geq 0\,, \;\; and \;\; x_0 \in D\,. \end{aligned} \tag{6}$$

For the both iterative methods (5) and (6): starting with one point $x_0 \in D$ and execute by one evaluation of divided difference of F at each step. As long as method (5) requires one evaluation of F' at each step, the method (6) requires only one evaluation of F'.

The paper is organized as follows: In section 3 we provide a local convergence analysis for the methods (5) and (6). Some numerical examples are succeeded in section 4, and conclusion in section 5.

3. Local convergence analysis

Here, we present a theoretical result for the local convergence of Newton-Secant method (5) and Modified Newton-Secant method (6).

Let us assume that α is a simple zero of the operator F, at which the Fréchet derivative of F exists and $F'(\alpha)^{-1} \in L(Y, X)$, and there exist nondecreasing functions

$$a, b, c : [0, +\infty) \to [0, +\infty)$$

such that:

$$\begin{aligned} &\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \le a(\|x - \alpha\|)\,, \; for \;\; all \;\; x \in D \\ &\|F'(\alpha)^{-1}([x, y] - [x, \alpha])\| \; \le b(\|y - \alpha\|)\,, \; for \;\; all \;\; x, y \in D \\ &\|F'(\alpha)^{-1}([x, \alpha] - [\alpha, \alpha])\| \le c(\|x - \alpha\|)\,, \;\; for \;\; all \;\; x \in D\,. \end{aligned} \tag{7}$$

Now we can show the following local convergence result for method (5):

Theorem 1. *Let F be a nonlinear Fréchet-differentiable operator defined on a convex open subset D of a Banach space X with values in a Banach space Y. Let us assume that $\alpha \in D$ is a simple zero of the operator F, $F'(\alpha)^{-1} \in L(Y, X)$ and conditions (7) are satisfied. In addition, let us assume that each of equations*

$$a(r) + b(r) = 1 \quad and \quad 2b(r) + c(r) = 1 \tag{8}$$

has a minimum positive zeros r_1 and r_2 respectively. Denote $r = min\{r_1, r_2\}$ and

$$\bar{U}(\alpha, r^*) = \{x \in X : \|x - \alpha\| \le r^*\} \subset D \quad for \quad r^* \in (0, r)\,.$$

Then sequence $\{x_k\}$ $(k \ge 0)$ generated by method (5) is well defined, remains in $\bar{U}(\alpha, r^*)$ for all $k \ge 0$, converges to α provided that $x_0 \in \bar{U}(\alpha, r^*)$. Moreover, the following error bounds hold for all $k \ge 0$:

$$\|x_{k+1} - \alpha\| \le \frac{b(\|y_k - \alpha\|)\|x_k - \alpha\|}{1 - b(\|y_k - \alpha\|) - c(\|x_k - \alpha\|)}\,, \tag{9}$$

where $y_k = x_k - F'(x_k)^{-1}F(x_k)$.

Proof. First we will prove that, if $x \in \bar{U}(\alpha, r^*)$ then $y \in \bar{U}(\alpha, r^*)$, where $y = x - F'(x)^{-1}F(x)$. From the following relations

$$\|I - F'(\alpha)^{-1}F'(x)\| = \|F'(\alpha)^{-1}(F'(\alpha) - F'(x))\| \le a(\|x - \alpha\|) < 1 \tag{10}$$

by the choice of r^* and from the Banach Lemma on invertible operators [1] it follows that $F'(x)$ is invertible and

$$\|F'(x)^{-1}F'(\alpha)\| \leq \frac{1}{1-a(\|x-\alpha\|)}\,. \tag{11}$$

Then using that $F'(x) = [x, x]$ (see [9]), we get

$$\begin{aligned} \|y-\alpha\| &= \|x-\alpha-F'(x)^{-1}F(x)\| = \|F'(x)^{-1}(F'(x)-[x,\alpha])(x-\alpha)\| \\ &= \|F'(x)^{-1}F'(\alpha)F'(\alpha)^{-1}(F'(x)-[x,\alpha])(x-\alpha)\| \\ &\leq \|F'(x)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}(F'(x)-[x,\alpha])\|\,\|x-\alpha\| \end{aligned}$$

and

$$\|y-\alpha\| \;\leq\; \frac{b(\|x-\alpha\|)}{1-a(\|x-\alpha\|)}\|x-\alpha\|\,.$$

Estimate (8), and the choice of r^* imply

$$\|y-\alpha\| < \|x-\alpha\| < r^* \;\Rightarrow\; y \in \bar{U}(\alpha, r^*)\,.$$

Now we will prove that the operator $[x, y]$ is invertible for all $x, y \in \bar{U}(\alpha, r^*)$. Using Eq.(7), we obtain

$$\begin{aligned} \|I - F'(\alpha)^{-1}[x,y]\| &= \|F'(\alpha)^{-1}(F'(\alpha)-[x,y])\| \\ &= \|F'(\alpha)^{-1}(F'(\alpha)-[x,\alpha]+[x,\alpha]-[x,y])\| \\ &\leq b(\|y-\alpha\|)+c(\|x-\alpha\|) \leq b(r^*)+c(r^*) < 1\,, \end{aligned}$$

by the choice of r^*.

It follows from the Banach Lemma on invertible operators that $[x, y]^{-1}$ exists and

$$\|[x,y]^{-1}F'(\alpha)\| \leq \frac{1}{1-b(\|y-\alpha\|)-c(\|x-\alpha\|)}\,,\;\; x,y \in \bar{U}(\alpha, r^*)\,. \tag{12}$$

Let us choice $x_0 \in \bar{U}(\alpha, r^*)$ and let us assume $x_s \in \bar{U}(\alpha, r^*)$, for all $s = 0, 1, \ldots, k$. Then by using relations (5), (7) and (12), we obtain

$$\begin{aligned} \|x_{k+1}-\alpha\| &= \|x_k-\alpha-[x_k,y_k]^{-1}F(x_k)\| = \|[x_k,y_k]^{-1}([x_k,y_k]-[x_k,\alpha])(x_k-\alpha)\| \\ &= \|[x_k,y_k]^{-1}F'(\alpha)F'(\alpha)^{-1}([x_k,y_k]-[x_k,\alpha])(x_k-\alpha)\| \\ &\leq \|[x_k,y_k]^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}([x_k,y_k]-[x_k,\alpha])\|\,\|x_k-\alpha\| \\ &\leq \frac{b(\|y_k-\alpha\|)}{1-b(\|y_k-\alpha\|)-c(\|x_k-\alpha\|)}\,\|x_k-\alpha\|\,, \end{aligned}$$

and the Eq.(9).

Moreover, by definitions of the functions a and b, and the choice of r^*, we get

$$\frac{b(\|y_k-\alpha\|)}{1-b(\|y_k-\alpha\|)-c(\|x_k-\alpha\|)} \le \frac{b(r^*)}{1-b(r^*)-c(r^*)} < 1\,,$$

which imply that there exist $0 < q < 1$, such that

$$\|x_{k+1}-\alpha\| \le q\|x_k-\alpha\| < r^*\,, \quad k \ge 0\,.$$

Hence, we deduce $x_k \in \bar{U}(\alpha, r^*)$ $(k \ge 0)$ and $\lim_{k\to\infty} x_k = \alpha$. Theorem is proved. ■

Remark 1. Under the same assumptions of Theorem 1 and in addition if there exist a constant $N > 0$ such that

$$b(x) \le Nx, \quad x \in D,$$

then the iterative method defined by (5) has third order of convergence.

Theorem 2. *Under the same assumptions of Theorem 1 it follows that the sequence* $\{x_k\}_{k=0}^{\infty}$ *generated by method (6) is well defined, remains in* $\bar{U}(\alpha, r^*)$ *for all* $k \ge 0$*, converges to* α *provided that* $x_0 \in \bar{U}(\alpha, r^*)$*. Moreover, the following error bounds hold for all* $k \ge 0$*:*

$$\|x_{k+1}-\alpha\| \le \frac{b(\|y_k-\alpha\|)\|x_k-\alpha\|}{1-b(\|y_k-\alpha\|)-c(\|x_k-\alpha\|)}\,, \tag{13}$$

where $y_k = x_k - F'(x_0)^{-1}F(x_k)$.

Proof. By analogy of Theorem 1, we will first prof that if $x_0, x \in \bar{U}(\alpha, r^*)$ then $y \in \bar{U}(\alpha, r^*)$, where $y = x - F'(x_0)^{-1}F(x)$.

Using equations (10) and (11), we obtain

$$\|F'(x_0)^{-1}F'(\alpha)\| \le \frac{1}{1-a(\|x_0-\alpha\|)}\,.$$

Furthermore

$$\begin{aligned} \|y-\alpha\| &= \|x-\alpha-F'(x_0)^{-1}F(x)\| = \|F'(x_0)^{-1}(F'(x_0)-[x,\alpha])(x-\alpha)\| \\ &= \|F'(x_0)^{-1}F'(\alpha)F'(\alpha)^{-1}(F'(x_0)-[x,\alpha])(x-\alpha)\| \\ &\le \|F'(x_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}(F'(x_0)-[x,\alpha])\|\,\|x-\alpha\| \end{aligned}$$

and

$$\|y-\alpha\| \;\le\; \frac{b(\|x_0-\alpha\|)}{1-a(\|x_0-\alpha\|)}\|x-\alpha\| < \|x-\alpha\|\,,$$

i.e. $y \in \bar{U}(\alpha, r^*)$.

Let us choice $x_0 \in \bar{U}(\alpha, r^*)$ and assume $x_s \in \bar{U}(\alpha, r^*)$ for all $s = 1, \ldots, k$. Then by using equations (6), (7) and (12), by analogy of the proof of Theorem 1, we obtain the estimate (13). Hence, from the same reasons as in Theorem 1, we deduce $x_k \in \bar{U}(\alpha, r^*)$ $(k \geq 0)$ and $\lim_{k\to\infty} x_k = \alpha$. Theorem is proved. ■

Remark 2. Under the same assumptions of Theorem 2 and in addition if there exist a constant $N > 0$ such that

$$b(x) \leq Nx, \quad x \in D,$$

then the iterative method defined by (6) has second order of convergence.

4. Numerical Examples

To demonstrate the methods given by (5) and (6) numerically, we consider the following examples, where $F(x) = (f_1(x), f_2(x))^T$, $x = (x_1, x_2)$. In Table 1 and Table 2, we can see the advantage of use the presented methods. Displayed are the error $\|x^{(k)} - \alpha\|_\infty$ and number of iterations for Newton method(NM) (2), Newton-Secant method(NS) (5) and Modified Newton-Secant method(MNS) (6).

For divided difference of F, we use the following formula

$$[u, v] = \begin{pmatrix} \frac{f_1(u_1,v_2)-f_1(v_1,v_2)}{u_1-v_1} & \frac{f_1(u_1,u_2)-f_1(u_1,v_2)}{u_2-v_2} \\ \frac{f_2(u_1,v_2)-f_2(v_1,v_2)}{u_1-v_1} & \frac{f_2(u_1,u_2)-f_2(u_1,v_2)}{u_2-v_2} \end{pmatrix},$$

presented by Potra in [6].

Example 1 *Consider Eq. (1), where*

$$\left| \begin{array}{rcc} f_1(x_1, x_2) & = & x_2 - 2 \\ f_2(x_1, x_2) & = & x_1^2 - 2x_2 - 21\,, \end{array} \right.$$

with a solution $\alpha = (5, 2)^T$, *(see [12]).*

Example 2 *Consider Eq. (1), where*

$$\left| \begin{array}{rcl} f_1(x_1, x_2) & = & x_1^2 - x_2 + 1 \\ f_2(x_1, x_2) & = & x_1 - \cos(\frac{\pi}{2}x_2)\,, \end{array} \right.$$

with a solution $\alpha = (0, 1)^T$, *(see [13]).*

Table 1: Number of iterations and error for Example 1.

initial $x^0 = (5.5, 1.5)$				*initial* $x^0 = (6.3, 0.8)$			
iter	NM	MNS	NS	*iter*	NM	MNS	NS
1	8.33e-02	1.16e-02	1.16e-02	1	9.00e-01	2.91e-01	2.91e-01
2	1.66e-03	1.11e-05	9.91e-08	2	1.39e-01	2.74e-02	1.95e-03
3	6.93e-07	1.03e-11	0	3	4.55e-03	2.77e-04	4.64e-10
4	1.20e-13	0		4	5.18e-06	2.89e-08	0
5	0			5	6.71e-12	2.22e-16	
				6	0		

Table 2: Number of iterations and error for Example 2.

initial $x^0 = (0.3, 1.3)$				*initial* $x^0 = (0.5, 1.5)$			
iter	NM	MNS	NS	*iter*	NM	MNS	NS
1	6.00e-02	1.66e-02	1.66e-02	1	1.90e-01	3.40e-02	3.40e-02
2	3.61e-03	2.08e-04	1.06e-05	2	1.20e-02	9.70e-04	8.33e-05
3	2.02e-05	3.51e-08	2.94e-15	3	2.18e-04	9.80e-07	1.42e-12
4	6.46e-10	1.11e-15	6.12e-17	4	7.50e-08	1.00e-12	6.12e-17
5	1.31e-16	1.60e-16		5	8.90e-15	1.72e-16	
6	6.12e-17			6	4.89e-17		

5. Conclusion

We propose two iterative methods as extension of the Newton-Secant method and the Modified Newton-Secant method for operator equations. We prove that the methods converge cubically and quadratically, respectively. From theoretic results and numerical examples, we show that the new methods keep the same good properties as the Newton method. We also refer the readers to Refs. [14, 15] for some results on modifications of Newton-Secant method for simultaneous determination of all roots of polynomial equations.

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