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On a class of Fibonacci polynomials

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This article observes a class of Fibonacci polynomials and some obtained characteristics. It demonstrates that these polynomials are solutions of a linear second-order differential equation with polynomial coefficients and linear third-order differential equation with polynomial coefficients. These polynomials are just defined to satisfy three-term recurrence relation.

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In many areas of practical mathematics close to numerical analysis we meet Fibonacci numbers related to the golden section. There are many different definitions of Fibonacci polynomials in the literature. In this article we will observe a class of Fibonacci polynomials and obtained characteristics.

Definition. Let F_m be the sequence of Fibonacci numbers defined with $F_{m+2} = F_{m+1} + F_m$, $F_0 = F_1 = 1$. The polynomial $N_m(x)$ defined with

$$N_m(x) = \sum_{k=0}^{m} F_k x^k, \tag{1}$$

is called Fibonacci polynomial of the $m-$th degree.

Lemma. *The Fibonacci polynomials* (1) *satisfy the following recurrence-relation:*

$$N_m(x) = x^2 N_{m-2}(x) + x N_{m-1}(x) + 1, \tag{2}$$

Theorem 1. *Let $N_m(x)$ be Fibonacci polynomial of the $m-th$ degree. It will be a particular solution of the linear third-order homogenous differential equation with polynomial coefficients of the following kind:*

$$\begin{aligned} &x^2(x^2+x-1)y''' + x[-2(m-3)x^2-(2m-3)x+2m]y''+\\ &\quad+[(m-1)(m-6)x^2+m(m-3)x-m(m-1)]y'+\\ &\quad+[2m(m-1)x+m(m+1)]y=0. \end{aligned} \tag{3}$$

Theorem 2. *The Fibonacci polynomials $N_m(x)$ are solutions of the linear second-order homogenous differential equation with polynomial coefficients of the following kind:*

$$\begin{aligned} &x(x^2+x-1)[(m+2)x+(m+2)K]y''+\\ &+\{2x(2x+1)[(m+2)x+(m+1)K]-\\ &-(m+1)(x^2+x-1)[(m+2)x+mK]\}y'+\\ &+\{2x[(m+2)x+(m+1)K]-(m+1)(2x+1)[(m+2)x+mK]\}y=0. \end{aligned} \tag{4}$$

or

$$\begin{aligned} &\{(m+2)x^4+[(m+1)K+m+2)]x^3+\\ &+[(m+1)K-(m+2)]x^2-(m+1)Kx\}y''\\ &+\{(m+2)(3-m)x^3+[(4+3m-m^2)K-m^2-\\ &-m+2]x^2(m+1)[(2-m)K+m+2]x+m(m+1)K\}y'\\ &+\{-2m(m+2)x^2+(m+1)[2K(1-m)-m-2]x-m(m+1)K\}y=0. \end{aligned} \tag{5}$$

where

$$K=\frac{F_{m+1}}{F_m}=1+\frac{F_{m-1}}{F_m}.$$

We use generating function of the sequence of Fibonacci numbers

$$G(x)=\frac{1}{1-x-x^2}$$

in the proof of these theorems, where we apply

$$(1-x-x^2)G'(x)-(2x+1)G(x)=0.$$

These equations can be written in a normal kind (self-adjoint expression)

$$\frac{d}{dx}(P(x)\frac{dy}{dx}) + Q(x)y = 0,$$

where

$$P(x) = \frac{(x^2+x-1)^2}{x^m[(m+2)x+(m+1)K]^{m+2}},$$

$$Q(x) = \frac{2Kx-(2mx+m+1)[(m+2)x+mK]}{x^{m+1}[(m+2)x+(m+1)K]^{m+3}}.$$

References

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