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Compactness and Connectedness of the Pareto-optimal Set in Multi-criteria Convex Maximization

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In the present paper we discuss compactness and connectedness of a set of the Pareto-optimal solutions in a multi-criteria convex maximization problem where objective and constraint functions are quasi-concave. Quasi-concavity is not sufficient to prove these topological properties. Some additional conditions are considered to lead to compactness and connectedness. The results are based on the construction of a multifunction from the feasible domain onto the Pareto-optimal set. Using this multifunction it is proved that the Pareto-optimal set is compact and connected.

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1. Introduction

The key idea of every optimization problem is to seek the best design that maximizes or minimizes the objective function by changing design variables while satisfying constraints. During maximization or minimization one often needs to consider several objective functions simultaneously. The objective functions are the mathematical description of the optimization criteria. When more than one objective function is associated, the optimization problem becomes multi-objective or multi-criteria, in which case the usual optimization for a scalar function cannot be used.

The aim of this paper is to present some new facts on compactness and connectedness of the Pareto-optimal set in a multi-criteria convex maximization problem, where objective and constraint functions are quasi-concave. Quasi-concavity is not sufficient to prove these topological properties. Some additional conditions are considered to lead to compactness and connectedness.

Mathematically, the standard form of the multi-criteria maximization is to find a variable $x(x_1, x_2, ..., x_m) \in R^m$, $m \geq 1$, so as to maximize $f(x) = (f_1(x), f_2(x), ..., f_n(x))$ subject to $g(x) = (g_1(x), g_2(x), ..., g_p(x)) \geq 0$ and $x_i \in [a_i, b_i]$ for all $i \in J_m$, where $\{f_i\}_{i=1}^n$ are given continuous objective functions, $n \geq 2$, $\{g_i\}_{i=1}^p$ are given continuous inequality constraint functions, $p \geq 1$, and a_i and b_i are the lower and upper bounds for x_i, $a_i < b_i$, and $J_m = \{1, 2, ..., m\}$ is the index set.

Let the feasible domain denote

$X = \{x(x_1, x_2, ..., x_m) \in R^m | g_i(x) \geq 0 \forall i \in J_p, a_i \leq x_i \leq b_i \forall i \in J_m\}$.

As usual, let us assume that the set X is nonempty.

Definition 1. (a) A point $x \in X$ is called Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $f_k(y) > f_k(x)$ for some $k \in J_n$. The set of the Pareto-optimal solutions of X is denoted by $PO(X, f)$ and is called Pareto-optimal set.

(b) A point $x \in X$ is called strictly Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $x \neq y$. The set of the strictly Pareto-optimal solutions of X is denoted by $SPO(X, f)$ and is called strictly Pareto-optimal set.

(c) A point $x \in X$ is called weakly Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) > f_i(x)$ for all $i \in J_n$. The set of the weakly Pareto-optimal solutions of X is denoted by $WPO(X, f)$ and is called weakly Pareto-optimal set.

(d) A point $x \in X$ is called ideal Pareto-optimal solution if and only if $f_i(x) \geq f_i(y)$ for all $y \in X$ and all $i \in J_n$. The set of the ideal Pareto-optimal solutions of X is denoted by $IPO(X, f)$ and is called ideal Pareto-optimal set.

In our maximization problem, it can be shown that $PO(X, f) \neq \emptyset$, $WPO(X, f)$ is compact, $IPO(X, f) \subset PO(X, f)$ and $SPO(X, f) \subset PO(X, f) \subset WPO(X, f)$ [6] [9].

Note that the statements $PO(X, f) \subset bdX$ and $WPO(X, f) \subset bdX$ are not true in general, but $f(PO(X, f)) \subset bdf(X)$ and $f(WPO(X, f)) \subset bdf(X)$ are true. The images of the Pareto-optimal set and the weakly Pareto-optimal set under the objective function f are called Pareto-front and weakly Pareto-front, respectively. There are $PF(X, f) = f(PO(X, f))$ and $WPF(X, f) = f(WPO(X, f))$ [12] [17].

The well-known open problems in the multi-criteria optimization are the compactness and connectedness of the Pareto-optimal set. Compactness of this set is studied in [12] and [16]. Connectedness of the Pareto-optimal set is considered in [1], [2], [3], [8], [10], [11], [13], [15] and [20].

The paper is organized as follows: In Section 2, some definitions and notions from multi-criteria optimization theory and topological properties of sets are described. In Section 3, an upper semi-continuous multifunction from the feasible domain onto the Pareto-optimal set and a continuous function from the feasible domain onto the Pareto-front set are constructed. Based on this, compactness and connectedness of the Pareto-optimal and Pareto-front sets are discussed.

2. Definitions and notions

Let a function d be a metric on R^m. In a metric space (R^m, d), let τ be the topology induced by d. In a topological space (R^m, τ), for $Y \subset X \subset R^m$ we recall some topological definitions.

Definition 2. (a) The set Y is connected if and only if it is not the union of a pair of nonempty sets of τ, which are disjoint.

(b) The set Y is pathwise connected if and only if for every $x, y \in Y$ there exists a continuous function $p : [0; 1] \longrightarrow Y$ such that $p(0) = x$ and $p(1) = y$.

Of course, compactness and connectedness of the Pareto-optimal set related to compactness and connectedness of the Pareto-front set, respectively.

Remark 1. It is known that convexity implies pathwise connectedness, and pathwise connectedness implies connectedness. However, in general the converse does not hold.

Remark 2. It is known that a subset of R is connected if and only if it is an interval [21].

Additionally, recall some definitions for concave functions.

Definition 3. A real function f on a convex subset $X \subset R^m$ is call to be: (a) quasi-concave on X if and only if for any $x, y \in X$ and $t \in [0; 1]$, then $f(tx + (1-t)y) \geq min(f(x), f(y))$.

(b) semi-strictly quasi-concave on X if and only if f is quasi-concave and for any $x, y \in X$, $f(x) \neq f(y)$ and $t \in (0, 1)$, then $f(tx + (1-t)y) > min(f(x), f(y))$.

(c) strictly quasi-concave on X if and only if for any $x, y \in X$, $x \neq y$ and $t \in (0; 1)$, then $f(tx + (1-t)y) > min(f(x), f(y))$.

(d) concave on X if and only if for any $x, y \in X$ and $t \in [0; 1]$, then $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$.

Remark 3. (a) Note that strictly quasi-concavity implies semi-strictly quasi-concavity and semi-strictly quasi-concavity implies quasi-concavity, but the reverse implications are not true in general.

(b) Let $X = [-2; 2]$ and $f(x) = \begin{cases} 0, x \in [-2, 0] \\ x, x \in (0, 1] \\ 1, x \in (1, 2] \end{cases}$.

In this case f is continuous and quasi-concave on X, but it is not semi-strictly quasi-concave on X.

(c) Let $X = [0; 2]$ and $f(x) = \begin{cases} x, x \in [0, 1] \\ 1, x \in (1, 2] \end{cases}$.

We check at once that f is continuous and semi-strictly quasi-concave on X, but it is not strictly quasi-concave on X.

Remark 4. Let $\{f_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^p$ be all quasi-concave. It is known that:

(a) if one of $\{f_i\}_{i=1}^n$ is strictly quasi-concave, then $SPO(X, f) = PO(X, f)$ [9] [12].

(b) if $\{f_i\}_{i=1}^n$ are all strictly quasi-concave, then $PO(X, f) = WPO(X, f)$, therefore $SPO(X, f) = PO(X, f) = WPO(X, f)$. Thus, under these assumptions the Pareto-optimal set $PO(X, f)$ is compact [9] [12].

3. Main results

First, define a function $s : \Lambda \times X \longrightarrow R$ by $s(\lambda, x) = \sum_{j=1}^n \lambda_j f_j(x)$ for all $x \in X$ and all $\lambda \in \Lambda$, where $\Lambda = \{\lambda(\lambda_1, \lambda_2, ..., \lambda_n) \in R^n_{++} | \sum_{j=1}^n \lambda_j = 1\}$.

Next, define a multifunction $\psi : X \Longrightarrow X$ by $\psi(x) = \{y \in X | f_i(y) \geq f_i(x) \forall i \in J_n\}$ for all $x \in X$.

Remark 5. Let $\{g_i\}_{i=1}^p$ be quasi-concave and $X = [-2; 2]$.

(a) If $f_1(x) = \begin{cases} 0, x \in [-2, 0] \\ x, x \in (0, 2] \end{cases}$ and $f_2(x) = -x$, then we show that $PO(X, f) = \{-2\} \bigcup (0; 2]$ is not compact and not connected. Here, f_1 is quasi-concave and not semi-strictly quasi-concave, and f_2 is strictly quasi-concave.

(b) If $f_1(x) = \begin{cases} 0, x \in [-2, 0] \\ x, x \in (0, 2] \end{cases}$ and $f_2(x) = \begin{cases} -x, x \in [-2, 0] \\ 0, x \in (0, 2] \end{cases}$, then we find that $PO(X, f) = \{-2\} \bigcup \{2\}$ is compact and not connected. In this case, f_1 is quasi-concave and not semi-strictly quasi-concave, and f_2 is semi-strictly quasi-concave and not strictly quasi-concave.

(c) If $f_1(x) = x$ and $f_2(x) = 1 - x$, then we have that $PO(X, f) = [0; 2]$ is compact and connected. Clearly, f_1 and f_2 are strictly quasi-concave. Note that $PO(X, f)$ is convex, see Remark 2.

Now, we will discus the role of the following assumptions in our problem.

Assumption 1. If $\{x_i\}_{i=0}^{\infty} \subset X$ and $lim_{k\to\infty}\ d(x_k, x_0) = 0$, then $lim_{k\to\infty}\ d(y_0, \psi(x_k)) = 0$ for all $y_0 \in \psi(x_0)$.

Assumption 2. $\{f_i\}_{i=1}^{n}$ and $\{g_i\}_{i=1}^{p}$ are all quasi-concave, and $s(\lambda, \cdot)$ is quasi-concave for some $\lambda \in \Lambda$.

These definitions and assumptions allow us the presentation of the basic theorem of this paper.

Theorem 1. *(a) If Assumption 1 holds, then there exists an upper semi-continuous multifunction $r : X \Rightarrow X$ such that $r(X) = PO(X, f)$. In particular, $PO(X, f)$ is compact.*

(b) If Assumptions 1 and 2 hold, then $PO(X, f)$ is connected.

(c) If Assumption 1 holds, then there exists a continuous function $b : X \to PF(X, f)$ such that $b(X) = PF(X, f)$. In particular, $PF(X, f)$ is compact and pathwise connected.

First, let us fix $x \in X$ and $\lambda \in \Lambda$. Consider an optimization problem with single objective function $s(\lambda, \cdot)$ as follows: maximize $s(\lambda, y)$ subject to $y \in \psi(x)$.

We will show that this problem has a solution $\hat{x} \in PO(X, f)$. Thus, a multifunction $\hat{x} = r(x)$ will be constructed.

Lemma 1. *If $x \in X$ and $\lambda \in \Lambda$, then $Argmax(s(\lambda, \cdot), \psi(x)) \subset PO(X, f)$.*

P r o o f. Let us choose $y \in Argmax(s(\lambda, \cdot), \psi(x))$ and assume that $y \notin PO(X, f)$. From the assumption $y \notin PO(X, f)$ it follows that there exists $z \in X$ such that $f_i(z) \geq f_i(y)$ for all $i \in J_n$ and $f_k(z) > f_k(y)$ for some $k \in J_n$. As a result we derive $z \in \psi(x)$ and $s(\lambda, z) > s(\lambda, y)$. This leads to a contradiction, therefore $y \in PO(X, f)$. ■

The lemma is proved.

Using the result of Lemma 1 we are in a position to construct a multifunction $r : X \Rightarrow X$ such that $r(X) \subset PO(X, f)$ and $r(x) = Argmax(s(\lambda, \cdot), \psi(x))$ for all $x \in X$.

Now, define a multifunction $\rho : X \Rightarrow X$ by $\rho(x) = \{y \in X | f(y) = f(x)\}$ for all $x \in X$.

Lemma 2. *If $x \in X$, $x \in PO(X, f)$ is equivalent to $\rho(x) = \psi(x)$.*

P r o o f. Let $x \in PO(X, f)$ and assume that $\rho(x) \neq \psi(x)$. From both conditions $x \in \psi(x)$ and $\rho(x) \neq \psi(x)$, it follows that there exists $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $f_k(y) \neq f_k(x)$ for some $k \in J_n$, i.e. $f_k(y) > f_k(x)$. This leads to a contradiction, therefore $\rho(x) = \psi(x)$.

Conversely, let $\rho(x) = \psi(x)$ and assume that $x \notin PO(X, f)$. From the assumption $x \notin PO(X, f)$, it follows that there exists $y \in X$ such that

$f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $f_k(y) > f_k(x)$ for some $k \in J_n$. Thus we deduce that $y \in \psi(x)$ and $y \notin \rho(x)$, which contradicts the condition $\rho(x) = \psi(x)$. ■

The lemma is proved.

For $x \in X$ applying the previous lemma it follows that if $x \in PO(X, f)$, then $x = r(x)$ and if $x \notin PO(X, f)$, then $x \neq r(x)$.

Lemma 3. $r(X) = PO(X, f)$.

Proof. From Lemma 1 it follows that $r(X) \subset PO(X, f)$. Applying Lemma 2 we deduce $r(PO(X, f)) = PO(X, f)$. This means that $r(X) = PO(X, f)$. ■

Now, we will analyze the multifunction ψ. Using the Maximum Theorem, one of the fundamental results of optimization theory, we will show that the multifunction r is upper semi-continuous on X.

Lemma 4. *ψ is continuous on X.*

Proof. First, we will prove that if $\{x_k\}_{k=1}^{\infty}, \{y_k\}_{k=1}^{\infty} \subset X$, is a pair of sequences such that $lim_{k\to\infty} x_k = x_0 \in X$ and $y_k \in \psi(x_k)$ for all $k \in N$, then there exists a convergent subsequence of $\{y_k\}_{k=1}^{\infty}$ whose limit belongs to $\psi(x_0)$.

The assumption $y_k \in \psi(x_k)$ for all $k \in N$ implies $f_i(y_k) \geq f_i(x_k)$ for all $k \in N$ and all $i \in J_n$. From the condition $\{y_k\}_{k=1}^{\infty} \subset X$ it follows that there exists a convergent subsequence $\{q_k\}_{k=1}^{\infty} \subset \{y_k\}_{k=1}^{\infty}$ such that $lim_{k\to\infty} q_k = y_0 \in X$. Therefore, there exists a convergent subsequence $\{p_k\}_{k=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$ such that $q_k \in \psi(p_k)$ and $lim_{k\to\infty} p_k = x_0$. Thus, we find that $f_i(q_k) \geq f_i(p_k)$ for all $i \in J_n$ and all $k \in N$. Taking the limit as $k \to \infty$ we obtain $f_i(y_0) \geq f_i(x_0)$ for all $i \in J_n$. This implies $y_0 \in \psi(x_0)$.

This means that ψ is upper semi-continuous on X [14].

Second, we will prove that if $\{x_k\}_{k=1}^{\infty} \subset X$ is a convergent sequence to $x_0 \in X$ and $y_0 \in \psi(x_0)$, then there exists a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \psi(x_k)$ for all $k \in N$ and $lim_{k\to\infty} y_k = y_0$.

Denote the distance between the point $y_0 \in X$ and the set $\psi(x_k) \subset X$ by d_k. Clearly, $\psi(x_k)$ is a nonempty and compact set; therefore, if $y_0 \notin \psi(x_k)$, then there exists $\hat{y} \in \psi(x_k)$ such that $d_k = d(\hat{y}, y_k)$. There are two cases as follows: if $y_0 \in \psi(x_k)$, then $d_k = 0$ and let $y_k = y_0$; if $y_0 \notin \psi(x_k)$, then $d_k > 0$ and let $y_k = \hat{y}$. So we get a sequence $\{d_k\}_{k=1}^{\infty} \subset R_+$ and a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \psi(x_k)$ for all $k \in N$ and $d_k = d(y_0, y_k)$. From Assumption 1 we have that if $lim_{k\to\infty} d(x_k, x_0) = 0$, then $lim_{k\to\infty} d(y_0, \psi(x_k)) = 0$; therefore, the sequence $\{d_k\}_{k=1}^{\infty}$ is convergent and $lim_{k\to\infty} d_k = 0$. Finally, we obtain $lim_{k\to\infty} y_k = y_0$.

This means that ψ is lower semi-continuous on X [14].

In summary, the multifunction ψ is continuous on X. ■

The lemma is proved.

Lemma 5 *[19]. Let $S \subset R^n$, $\Theta \subset R^m$, $g : S \times \Theta \to R$ a continuous function, and $D : \Theta \Rightarrow S$ be a compact-valued and continuous multifunction. Then, the function $g^* : \Theta \to R$ defined by $g^*(\theta) = max\{g(x, \theta)|x \in D(\theta)\}$ is continuous on Θ, and the multifunction $D^* : \Theta \Rightarrow S$ defined by $D^*(\theta) = \{x \in D(\theta)|g(x, \theta) = g^*(\theta)\}$ is compact-valued and upper semi-continuous on Θ.*

Lemma 6 *[14]. Let $\Theta \subset R^m$ be compact and $D : \Theta \Rightarrow \Theta$ be an upper semi-continuous multifunction. If $D(x) \neq \emptyset$ is compact for all $x \in \Theta$, then $D(\Theta)$ is compact.*

Lemma 7 *[15]. Let $\Theta \subset R^m$ be connected and $D : \Theta \Rightarrow \Theta$ be an upper semi-continuous multifunction. If $D(x) \neq \emptyset$ is connected for all $x \in \Theta$, then $D(\Theta)$ is connected.*

P r o o f. of Theorem 1. (a) According to Lemmas 1 and 3 we construct the multifunction $r : X \Rightarrow X$ such that $r(X) = PO(X, f)$. Applying Lemma 5 for $S = \Theta = X$, $D = \psi$ and $g = s(\lambda, \cdot)$ we deduce that $r = D^*$ is upper semi-continuous on X.

Of course, X is compact. From Lemma 5 it follows that $r(x) \neq \emptyset$ is compact for all $x \in X$. Applying Lemma 6 we deduce that $r(X) = PO(X, f)$ is compact.

(b) From Assumption 2 it follows that $r(x) \neq \emptyset$ is convex for all $x \in X$; therefore, $r(x)$ is connected. It can be shown that feasible domain X is convex; therefore, is X connected. Applying Lemma 7 we deduce that $r(X) = PO(X, f)$ is connected.

(c) For $x \in X$, $x \in PO(X, f)$ is equivalent to $|f(\psi(X))| = 1$. Consider a function $b : X \to PF(X, f)$ such that $b(x) = f(r(x))$ for all $x \in X$. It is clear to see that b is continuous on X. We have known that X is compact and convex; therefore, it is compact and pathwise connected. Then, $b(X) = PF(X, f)$ is compact and pathwise connected. ■

The theorem is proved.

Corollary 1. *If Assumptions 1 and 2 hold, and $m = 1$, then $PO(X, f)$ is convex.*

P r o o f. From Theorem 1 and Remark 2 it follows that $PO(X, f)$ is closed interval of R, i.e. it is convex. ■

Remark 6. In [11], it is proved that if $\{g_i\}_{i=1}^p$ are quasi-concave, $\{f_i\}_{i=1}^n$ are semi-strictly quasi-concave and one of $\{f_i\}_{i=1}^n$ is strictly quasi-concave, then $PO(X, f)$ is pathwise connected. It is easy to show that $PF(X, f)$ is pathwise connected too.

Remark 7. In [9], it is proved that if $\{g_i\}_{i=1}^p$ are quasi-concave and $\{f_i\}_{i=1}^n$ are concave, then $PO(X, f)$ and $PF(X, f)$ are connected.

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