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One Result for Superlinear Local Convergence of Gauss-Newton-based BFGS Method with Filter

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A globally convergent filter method constructed on Gauss-Newton-based BFGS method for unconstrained minimization was previously introduced by same authors [*Applied Mathematics and Computation*, **211**, No.2, 2009, 354–362]. This method can also be successfully used in solving systems of symmetric nonlinear equations and its numerical results show reasonably good performance. In this paper local convergence of this method is considered. It is shown that under same assumptions superlinear convergence can not be achieved when infinitely many successive points are added to the filter.

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1 Introduction

Unconstrained minimization problem

$$\min_{x \in R^n} f(x), \tag{1}$$

where $f : R^n \to R$ is two times continuously differentiable function is considered.

Besides all various methods for solving problem (1), see [9, 19], quasi-Newton methods, firstly proposed by Broyden [3] in 1965, are one of the most exploited. They are defined by the iterative formula

$$x_{k+1} = x_k + \lambda_k p_k,$$

where $\lambda_k > 0$ is the step length that is updated by line search and backtracking procedure and $p_k = -B_k^{-1}\nabla f(x_k)$ is the search quasi-Newton direction, where $\nabla f(x)$ is the gradient mapping of $f(x)$. In quasi-Newton methods B_k is an approximation of the Hessian $\nabla^2 f(x_k)$ and it is updated at every iteration with some low-rank matrix. One of the well known update formula is the BFGS formula given by

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k},$$

where $s_k = x_{k+1} - x_k$ and $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$. The BFGS update formula possesses nice properties that are very useful for establishing convergence results, see [4]. For better understanding of quasi-Newton methods, we refer to [2, 8, 18].

In [17], Li and Fukushima proposed a new modified BFGS method called Gauss-Newton-based BFGS method for symmetric nonlinear equations

$$g(x) = 0, \tag{2}$$

where the mapping $g : R^n \to R^n$ is differentiable and the Jacobian $\nabla g(x)$ is symmetric for all $x \in R^n$. By introducing a new line search technique and modifying the BFGS update formula, under suitable conditions, they proved global and superlinear convergence of their method.

In order to avoid small steps induced by line search procedure in Gauss-Newton-based BFGS method as well as to reduce the number of backtracking procedures, in [16] we associate a filter to the Gauss-Newton-based BFGS method. The Gauss-Newton-based BFGS method with filter, [16] is globally convergent under standard assumptions.

If we denote by $g : R^n \to R^n$, the gradient mapping of the function $f(x)$ from (1), then $g(x)$ is differentiable and the Hessian $\nabla^2 f(x)$ is symmetric for all $x \in R^n$. The first order optimality conditions for (1) are given by (2) and therefore the method from [16] is applicable for solving (1).

Filter methods, one of the latest developments in global optimization algorithms, were firstly proposed by Fletcher in 1996, [10]. First filter methods are a kind of alternative to penalty functions used in constrained nonlinear programming optimization algorithms, see [5, 12, 15, 20, 22]. In these methods, filter functions penalize the constraints violations less restrictively then penalty functions. The main purpose of filters is allowing the full Newton step to be taken more often and thus inducing global convergence of the method. Some of filter methods [21, 23] succeeded in transition to superlinear local convergence. Filter methods are extended to solving nonlinear equations, see [6, 13], unconstrained optimization, [14]. For detailed reading about filter methods developments we refer to [11].

In this paper we discuss about local convergence of Gauss-Newton-based BFGS method with filter [16], superlinear local convergence is considered. Section 2 presents the algorithm of the Gauss-Newton-based BFGS with filter method, while in Section 3 assumptions and global convergence results are given. In Section 4 it is shown that under standard assumptions this method does not converge superlinearly when infinitely many successive points are added to the filter.

2 Algorithm

Gauss-Newton-based BFGS method with filter that is used in [16] is combination of Gauss-Newton-based BFGS method from [17] and multidimensional filter that is similarly defined as in [6, 13]. Before speaking of any convergence results let us first explain the concept of multidimensional filter and Gauss-Newton-based BFGS method with filter from [16].

To construct the multidimensional filter, first the equations (2) are partitioned into m sets $\{g_i(x)\}_{i\in I_j}$, $j = 1, ..., m$, with the property $\{1, ..., n\} = I_1 \cup ... \cup I_m$, $I_j \cap I_k = \emptyset$, $j \neq k$ and the *filter functions* are defined as

$$\phi_j(x) \stackrel{def}{=} ||g_{I_j}(x)|| \quad \text{for} \quad j = 1, ..., m \tag{3}$$

where $||\cdot||$ is the Euclidean norm and g_{I_j} is the vector whose components are the components of g indexed by I_j. With this notation x^* satisfies the optimality conditions of (1) if and only if $\phi_j(x^*) = 0$ for all $j = 1, ..., m$. The following abbreviations will be used

$$\phi(x) \stackrel{def}{=} (\phi_1(x), ..., \phi_m(x))^T, \ \phi_k \stackrel{def}{=} \phi(x_k) \text{ and } \phi_{j,k} \stackrel{def}{=} \phi_j(x_k).$$

A *filter* is a list $\mathcal{F}$ of m-tuples of the form $(\phi_{1,k}(x), \phi_{2,k}(x), ..., \phi_{m,k}(x))$ such that

$$\phi_{j,k} < \phi_{j,l} \text{ for at least one } j \in \{1, ..., m\} \text{ and for all } k \neq l.$$

To understand the meaning and usage of the filter a concept of domination is introduced. A point x_1 *dominates* a point x_2 whenever

$$\phi_j(x_1) \leq \phi_j(x_2) \text{ for all } j = 1, ..., m.$$

Therefore we say that the filter keeps all iterates that are not dominated by other iterates in the filter. In [16], we use a filter to construct an additional acceptability condition for a new trial iterate x_k^+. This condition is slightly different than the one in [13]. The first difference is the function δ_1 in (4), where we

use max function while in [13] min is considered. The procedure for removing points from the filter is also different. The reason for these changes is empirical since we realized that the algorithm is more efficient with the rule proposed in this paper.

We say that a new trial point x_k^+ is *acceptable for the filter* $\mathcal{F}$ if

$$\forall \phi_l \in \mathcal{F} \quad \exists j \in \{1, ..., m\} \quad \phi_j(x_k^+) < \phi_{j,l} - \gamma_\phi \delta_1(||\phi_l||, ||\phi_k^+||), \tag{4}$$

where $\gamma_\phi \in (0, 1)$ is a small positive constant and

$$\delta_1(||\phi_l||, ||\phi_k^+||) = \max\{||\phi_l||, ||\phi_k^+||\}.$$

When a trial point is acceptable for the filter, we *add the trial point to the filter* immediately (again different from [13]). In other words add the m-tuple $\phi_k^+ = \phi(x_k^+) = (\phi_1(x_k^+), ..., \phi_m(x_k^+))^T$ to the filter $\mathcal{F}$.

When we add a new trial point x_k^+ to the filter, we remove all points from the filter that are dominated by the trial point x_k^+, which means that we remove m-tuples $(\phi_{1,k_i}, ..., \phi_{m,k_i})^T \in \mathcal{F} \setminus \{\phi_k^+\}$ from the filter if

$$\phi_j(x_k^+) \leq \phi_{j,k_i} \quad j = 1, ..., m.$$

Every trial point which is acceptable for the filter is taken as a new iterate.

The filter algorithm [16] gives its best performance in the case when $m = n$ and $I_j = \{j\}$ - similarly as it is discussed in [11]). In this case we will have $\phi_j(x) = |g_j(x)|$, $\phi(x) = g(x)$, $\phi_k = g(x_k)$ and $\phi_{j,k} = |g_j(x_k)|$.

So, Gauss-Newton based BFGS method with filter [16] acts in following way. Given the gradient mapping g and current iteration $x_k = x_{k-1} + \lambda_{k-1} p_{k-1}$, the search direction p_k is obtained from the following linear system

$$B_k p + \lambda_{k-1}^{-1}(g(x_k + \lambda_{k-1} g(x_k)) - g(x_k)) = 0. \tag{5}$$

The line search rule is governed by a positive sequence $\{\varepsilon_k\}$ satisfying $\sum_k \varepsilon_k < \infty$. For the chosen sequence $\{\varepsilon_k\}$ and fixed parameters $\sigma_1, \sigma_2 > 0$ we take the trial point $x_k^+ = x_k + \lambda_k p_k$ as a new iterate $x_{k+1} = x_k^+$ if

$$||g(x_k^+)||^2 - ||g(x_k)||^2 \leq -\sigma_1 ||\lambda_k g(x_k)||^2 - \sigma_2 ||\lambda_k p_k||^2 + \varepsilon_k ||g(x_k)||^2. \tag{6}$$

Else if x_k^+ is acceptable to the filter then we add x_k^+ to the filter, remove all points from the filter that are dominated by x_k^+ and take $x_{k+1} = x_k^+$ as a new iterate. Backtracking $\lambda_k := r\lambda_k,\ 0 < r < 1$ is enforced only if (6) is not satisfied and a trial point x_k^+ is not acceptable to the filter.

The acceptance rule (6) starts with $\lambda_k = 1$ and is well defined since the inequality has to be satisfied for $\lambda_k > 0$ small enough due to the presence of $\varepsilon_k > 0$ at the right-hand side of (6). After determining x_{k+1} the new matrix B_{k+1} is obtained by (7), where

$$\begin{aligned} s_k &= x_{k+1} - x_k = \lambda_k p_k, \\ \delta_k &= g(x_{k+1}) - g(x_k), \\ y_k &= g(x_k + \delta_k) - g(x_k), \\ B_{k+1} &= B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}. \end{aligned} \tag{7}$$

The usual safe guard condition $y_k^T s_k > 0$ is applied i.e. if $y_k^T s_k \leq 0$ then $B_{k+1} = B_k$.

In computational implementation of the algorithm presented in Section 5 we used BFGS approximation of the inverse Jacobian, $H_k = B_k^{-1}$ and therefore determine the search direction as

$$p_k = -H_k \lambda_{k-1}^{-1}(g(x_k + \lambda_{k-1} g(x_k)) - g(x_k)) \tag{8}$$

with

$$H_{k+1} = (I - \frac{s_k y_k^T}{y_k^T s_k}) H_k (I - \frac{y_k s_k^T}{y_k^T s_k}) + \frac{s_k s_k^T}{y_k^T s_k}. \tag{9}$$

Now the filter algorithm from [16] follows. **ALGORITHM GNbBFGSf.**

Gauss-Newton-based BFGS method with filter

Step 0. Choose an initial point $x_0 \in R^n$, an initial symmetric positive definite matrix $B_0 \in R^{n \times n}$, a positive sequence $\{\varepsilon_k\}$ satisfying $\sum_{k=0}^{\infty} \varepsilon_k < \infty$, and constants $r, \gamma_\phi \in (0,1)$, $\sigma_1, \sigma_2 > 0$, $\lambda_{-1} > 0$. Let $k := 0$.

Step 1. If $g(x_k) = 0$ then Stop. Otherwise, solve the following linear equation to get p_k

$$B_k p + \lambda_{k-1}^{-1}(g(x_k + \lambda_{k-1} g(x_k)) - g(x_k)) = 0. \tag{10}$$

Take $\lambda_k = 1$ and go to Step 2.

Step 2. Let the trial point be $x_k^+ = x_k + \lambda_k p_k$. If

$$||g(x_k^+)||^2 - ||g(x_k)||^2 \leq -\sigma_1 ||\lambda_k g(x_k)||^2 - \sigma_2 ||\lambda_k p_k||^2 + \varepsilon_k ||g(x_k)||^2 \tag{11}$$

then go to Step 3, else if x_k^+ is acceptable to the filter then add x_k^+ to the filter, remove all points from the filter that are dominated by x_k^+ and go to Step 3. Otherwise, put $\lambda_k := r\lambda_k$ and repeat Step 2.

Step 3. Take the next iterate $x_{k+1} = x_k^+$.

Step 4. Put

$$s_k = x_{k+1} - x_k = \lambda_k p_k,$$

$$\delta_k = g(x_{k+1}) - g(x_k),$$

$$y_k = g(x_k + \delta_k) - g(x_k).$$

If $y_k^T s_k \leq 0$, then $B_{k+1} = B_k$ and go to Step 5. Otherwise, update B_k

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k} \tag{12}$$

and go to Step 5.

Step 5. Let $k := k + 1$. Go to Step 1.

At the beginning of Algorithm GNbBFGSf, the filter is initialized to be empty $\mathcal{F} = \emptyset$ or to be $\mathcal{F} = \{\phi(x) : \phi_j(x) \geq \phi_{j\,\max}, j = 1, ..., m\}$ for any $\phi_{j\,\max} > \phi_j(x_0)$, $j = 1, ..., m$ (similarly as in [22]). In practical implementations of this method, the filter is initialized to be empty $\mathcal{F} = \emptyset$ or to be $\mathcal{F} = \{\phi(x_0)\}$.

One should notice that if there are finitely many values of ϕ_k that are added to the filter, then for all k large enough (for all $k \geq k_0$ where ϕ_{k_0} is the last m-tuple added to the filter), Algorithm GNbBFGSf is the same as the one considered in [17].

The Gauss-Newton based BFGS with filter is globally convergent under suitable set of assumptions A0-A3 ([16]), that are listed in this paper in next subsection.

3 Global convergence

For the global convergence of Algorithm GNbBFGSf besides assumptions A1-A3 from [17], in [16] we used following assumption for the values ϕ_k that are added to the filter by Algorithm GNbBFGSf.

Assumption.

A0 There exists a constant $C > 0$ such that for all values ϕ_k that are added to the filter by Algorithm GNbBFGSf the following stands

$$||\phi_k|| \leq C.$$

Under assumption A0, in [16] it is shown that $\{x_k\} \subset \Omega$, where $\{x_k\}$ is a sequence generated by Algorithm GNbBFGSf and Ω is the level set defined by

$$\Omega = \{x : ||g(x)|| \leq e^{\frac{\varepsilon}{2}} \max\{C, ||g(x_0)||\}\}, \tag{13}$$

where ε is a positive constant such that

$$\sum_{k=0}^{\infty} \varepsilon_k < \varepsilon. \tag{14}$$

The following set of assumptions was also necessary for the global convergence analysis of Algorithm GNbBFGSf.

Assumptions.

A1 g is continuously differentiable on an open set Ω_1 containing Ω.

A2 ∇g is symmetric and bounded on Ω_1 i.e. $\nabla g(x)^T = \nabla g(x)$ for every $x \in \Omega_1$ and there exists a positive constant M such that

$$||\nabla g(x)|| \leq M \quad \forall x \in \Omega_1.$$

A3 ∇g is uniformly nonsingular on Ω_1 i.e. there is a constant $m > 0$ such that

$$m||p|| \leq ||\nabla g(x)p|| \quad \forall x \in \Omega_1, p \in R^n.$$

As the mapping g is the gradient of f and f is two times continuously differentiable, A1 and A2 are clearly satisfied while A3 is the standard assumption for global convergence of optimization algorithms.

We need to remark also that assumptions A2 and A3 imply that there exist constants $M \geq m > 0$ such that for all $x \in \Omega_1$ we have

$$m\|x - x^*\| \leq \|g(x)\| \leq M\|x - x^*\|, \tag{15}$$

where x^* is unique solution in Ω_1 for the problem (2).

For discussing the global convergence of Algorithm GNbBFGSf we distinguish two cases. The first one is when there are finitely many values of ϕ_k that are added to the filter by Algorithm GNbBFGSf, and for that case Algorithm GNbBFGSf is the same as the Gauss-Newton based BFGS from [17], which is globally convergent under assumptions A1-A3. Therefore in [16], it is shown that under assumptions A0-A3, when there are infinitely many values of ϕ_k that are added to the filter by Algorithm GNbBFGSf then

$$\lim_{k\to\infty} ||g(x_k)|| = 0. \tag{16}$$

The last result and global convergence of Gauss-Newton based BFGS algorithm from [17] clearly imply the global convergence of GNbBFGSf method under assumptions A0-A3, [16].

4 A local convergence result

As we mentioned before, when there are finitely many values of ϕ_k that are added to the filter by Algorithm GNbBFGSf, this algorithm acts same as the Gauss-Newton based BFGS method from [17], which is superlinearly convergent under assumptions A1-A3 and assumption for Hölder continuity of ∇g at x^*. For discussing superlinear convergence of Gauss-Newton based BFGS with filter method we need to consider case when infinitely many values of ϕ_k are added to the filter by Algorithm GNbBFGSf.

The following lema shows that one can choose set of constants $\sigma_1, \sigma_2, m > 0$ for which when infinitely many successive values of ϕ_k are added to the filter by Algorithm GNbBFGSf, then its superlinear convergence is violated.

Lemma 4.1. *Let assumptions A0-A3 hold and $\{x_k\}$ be generated by Algorithm GNbBFGSf in which constants $\sigma_1, \sigma_2 > 0$ are chosen such that $1 - \sigma_1 - \sigma_2 \frac{2}{m^2} > 0$, where $m > 0$ is the constant from (15). Let assume that there exists k_0 such that for all $k \geq k_0$ iterative points x_k were added to the filter. Then $\{x_k\}$ does not converge to x^* superlinearly.*

Proof. Let $k \geq k_0$ be arbitrary. Then according to (15) we have

$$\|x_k - x^*\| \leq \frac{1}{m}\|g(x_k)\| \text{ and } \|x_k - x^*\| \geq \frac{1}{M}\|g(x_k)\|. \tag{17}$$

Because iterative point $x_{k+1} = x_k + \lambda_k p_k$ was added to the filter we have that inequality in Step 2 of Algorithm GNbBFGSf was violated i.e.

$$\|g(x_{k+1})\|^2 - \|g(x_k)\|^2 > -\sigma_1\|\lambda_k g(x_k)\|^2 - \sigma_2\|\lambda_k p_k\|^2 + \varepsilon_k\|g(x_k)\|^2, \tag{18}$$

from where we have that

$$||g(x_{k+1})||^2 > (1 - \sigma_1\lambda_k^2 + \varepsilon_k)||g(x_k)||^2 - \sigma_2||x_{k+1} - x_k||^2. \tag{19}$$

From (17) we have

$$\begin{aligned} ||x_{k+1} - x_k||^2 &= ||x_{k+1} - x^* + x^* - x_k||^2 \leq \\ &\leq 2(||x_{k+1} - x^*||^2 + ||x_k - x^*||^2) \leq \\ &\leq \tfrac{2}{m^2}\|g(x_{k+1})\|^2 + \tfrac{2}{m^2}\|g(x_k)\|^2. \end{aligned} \tag{20}$$

If we implement (20) in (19) we will have

$$\begin{aligned} ||g(x_{k+1})||^2 &> (1-\sigma_1\lambda_k^2+\varepsilon_k)||g(x_k)||^2 - \sigma_2||x_{k+1}-x_k||^2 \geq \\ &\geq (1-\sigma_1\lambda_k^2+\varepsilon_k)||g(x_k)||^2 - \sigma_2\tfrac{2}{m^2}\|g(x_{k+1})\|^2 - \sigma_2\tfrac{2}{m^2}\|g(x_k)\|^2. \end{aligned} \tag{21}$$

From last inequality, $\lambda_k \leq 1$ and $\varepsilon_k > 0$ we obtain that

$$\begin{aligned} (1+\sigma_2\tfrac{2}{m^2})||g(x_{k+1})||^2 &> (1-\sigma_1\lambda_k^2-\sigma_2\tfrac{2}{m^2}+\varepsilon_k)||g(x_k)||^2 > \\ &> (1-\sigma_1-\sigma_2\tfrac{2}{m^2})||g(x_k)||^2, \end{aligned} \tag{22}$$

and

$$||g(x_{k+1})||^2 > \frac{1-\sigma_1-\sigma_2\frac{2}{m^2}}{1+\sigma_2\frac{2}{m^2}}||g(x_k)||^2. \tag{23}$$

Now from (17) and (23) we have

$$\frac{\|x_{k+1}-x^*\|^2}{\|x_k-x^*\|^2} \geq \frac{\frac{1}{M^2}\|g(x_{k+1})\|^2}{\frac{1}{m^2}\|g(x_k)\|^2} > \frac{m^2}{M^2}\cdot\frac{1-\sigma_1-\sigma_2\frac{2}{m^2}}{1+\sigma_2\frac{2}{m^2}}. \tag{24}$$

So,

$$\frac{\|x_{k+1}-x^*\|^2}{\|x_k-x^*\|^2} > K \; forall \; k \geq k_0, \tag{25}$$

where $K = \frac{m^2}{M^2}\cdot\frac{1-\sigma_1-\sigma_2\frac{2}{m^2}}{1+\sigma_2\frac{2}{m^2}} > 0$ is a positive constant and we conclude that $\{x_k\}$ does not converge to x^* superlinearly. ■

Discussion about superlinear convergence of Gauss-Newton based BFGS with filter method under assumptions for global convergence will be more complete, if we consider one more case, when infinitely many values of ϕ_k are added to the filter by Algorithm GNbBFGSf, but for every k there is k' such that $x_{k'}$ is not added to the filter. This case is still unanswered and under research.

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