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Some Properties Of The Bellman Gamma Distribution

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The Bellman gamma distribution is a matrix variate distribution, which is a generalization of the Wishart distribution. The exact distributions of determinants and quotient of determinants of some submatrices of Bellman gamma distributed random matrices are obtained. The method, considered in this paper, gives the possibility to be derived the distribution of products and quotient of products of principal minors of a Bellman gamma matrix.

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1. Introduction

The Bellman gamma distribution is a matrix variate distribution, which is a generalization of the Wishart and the matrix gamma distributions (see [1]). In practice it arises as a distribution of the empirical normal covariance matrix for samples with monotone missing data (see [3]).

Definitions of the Bellman gamma type I and II distributions are given in the next section. Section 2 contains also some notations and preliminary notes. The main results are given in Section 3. The exact distributions of determinants and quotient of determinants of some submatrices of Bellman gamma distributed random matrices are obtained. The method, considered in this paper, gives the possibility to be derived the distribution of products and quotient of products of principal minors of a Bellman gamma matrix.

2. Preliminary notes

Let A be a real square matrix of order n. Let α and β be nonempty subsets of the set $N_n = \{1, \ldots, n\}$. By A$[\alpha, \beta]$ we denote the submatrix of A, composed of the rows with numbers from α and the columns with numbers from β. When $\beta \equiv \alpha$, $\mathrm{A}[\alpha,\alpha]$ is denoted simply by $\mathrm{A}[\alpha]$. Let $i,j \in N_n$ and $i,j \notin \alpha$. Suppose that in the submatrix $\mathrm{A}[\alpha \cup \{i\}, \alpha \cup \{j\}]$ of the matrix $\mathrm{A} = (a_{i,j})$ we replace the element $a_{i,j}$ with 0. We shall denote the obtained matrix by $\mathrm{A}[\alpha \cup \{i\}, \alpha \cup \{j\}]^0$.

The next definitions of Bellman gamma type I and II distributions are given in [1]. By $\Gamma_n^*(a_1,\ldots,a_n)$ is denoted the generalized multivariate gamma function, $\Gamma_n^*(a_1,\ldots,a_n) = \pi^{n(n-1)/4}\prod\limits_{j=1}^{n}\Gamma\Big(a_j-(j-1)/2\Big)$, $a_j > (j-1)/2, j = 1,\ldots,n$.

Definition 2.1. A random positive definite matrix $\mathbf{U}$ $(n\times n)$ is said to follow *Bellman gamma type I distribution*, denoted by $\mathbf{U} \sim BG_n^I(a_1,\ldots,a_n;\mathrm{C})$, if its probability density function is given by

$$f_{\mathbf{U}}(\mathrm{U}) = \frac{\left(\prod\limits_{i=1}^{n}(\det \mathrm{C}[\{i,\ldots,n\}])^{a_i-a_{i-1}}\right)(\det \mathrm{U})^{a_n-(n+1)/2}}{\Gamma_n^*(a_1,\ldots,a_n)\prod\limits_{i=2}^{n}(\det \mathrm{U}[\{1,\ldots,i-1\}])^{a_i-a_{i-1}}}e^{-tr(\mathrm{CU})}, \qquad (2.1)$$

where C $(n\times n)$ is a positive definite constant matrix, $a_0 = 0$ and $a_j > (j-1)/2$, $j = 1,\ldots,n$.

Definition 2.2. A random positive definite matrix $\mathbf{U}$ $(n\times n)$ is said to follow *Bellman gamma type II distribution*, denoted by $\mathbf{U} \sim BG_n^{II}(b_1,\ldots,b_n;\mathrm{B})$, if its probability density function is given by

$$f_{\mathbf{U}}(\mathrm{U}) = \frac{\left(\prod\limits_{i=1}^{n}(\det \mathrm{B}[\{1,\ldots,i\}])^{b_{n-i+1}-b_{n-i}}\right)(\det \mathrm{U})^{b_n-(n+1)/2}}{\Gamma_n^*(b_1,\ldots,b_n)\prod\limits_{i=1}^{n-1}(\det \mathrm{U}[\{i+1,\ldots,n\}])^{b_{n-i+1}-b_{n-i}}}e^{-tr(\mathrm{BU})},$$

where B $(n\times n)$ is a positive definite constant matrix, $b_0 = 0$ and $b_j > (j-1)/2$, $j = 1,\ldots,n$.

The next five Propositions are proved in [4]. We shall denote by $\tilde{\mathrm{I}}_n$ the square matrix of size n with units on the anti-diagonal and zeros elsewhere.

Proposition 2.1. *Let* $\mathbf{U} \sim BG_n^I(a_1,\ldots,a_n;\mathrm{C})$. *Then the matrix* $\mathbf{V} = \tilde{\mathrm{I}}_n\mathbf{U}\,\tilde{\mathrm{I}}_n$ *is Bellman gamma type II distributed* $BG_n^{II}(a_1,\ldots,a_n;\mathrm{B})$, $\mathrm{B} = \tilde{\mathrm{I}}_n\mathrm{C}\tilde{\mathrm{I}}_n$.

Proposition 2.2. *Let* $\mathbf{U} \sim BG_n^I(a_1,\ldots,a_n;\mathrm{C})$ *and* L *be an arbitrary lower triangular constant matrix of size* n. *Then the matrix* $\mathbf{W} = \mathrm{LUL}^t$ *has distribution* $BG_n^I(a_1,\ldots,a_n;(\mathrm{L}^t)^{-1}\mathrm{CL}^{-1})$.

For an arbitrary positive definite matrix U there exist a unique lower triangular matrix V with positive diagonal elements, such that $\mathrm{U} = \mathrm{V}\,\mathrm{V}^{\mathrm{t}}$. The matrix V is called the Cholesky triangle.

Proposition 2.3. *Let* $\mathrm{U} = (u_{i,j})$ *be an arbitrary positive definite matrix of size* n. *Then* $\mathrm{U} = \mathrm{V}\,\mathrm{V}^{\mathrm{t}}$, *where* $\mathrm{V} = (v_{i,j})$ *is a lower triangular matrix,*

$$v_{j,i} = \frac{\det \mathrm{U}[\{1,\ldots,i\},\{1,\ldots,i-1,j\}]}{v_{i,i}\det \mathrm{U}[\{1,\ldots,i-1\}]},\ 2 \le i < j \le n,$$

$$v_{1,1} = \sqrt{u_{1,1}},\ \ v_{j,j} = \sqrt{\frac{\det \mathrm{U}[\{1,\ldots,j\}]}{\det \mathrm{U}[\{1,\ldots,j-1\}]}},\ v_{j,1} = \frac{u_{1,j}}{v_{1,1}},\ \ j = 2,\ldots,n.$$

Proposition 2.4. *Let* $\mathrm{C} = (c_{i,j})$ *be an arbitrary positive definite matrix of size* n. *Then* $\mathrm{C} = \mathrm{D}\mathrm{D}^{\mathrm{t}}$, *where* $\mathrm{D} = (d_{i,j})$ *is an upper triangular matrix,*

$$d_{i,j} = \frac{\det \mathrm{C}[\{i,j+1,\ldots,n\},\{j,\ldots,n\}]}{d_{j,j}\det \mathrm{C}[\{j+1,\ldots,n\}]},\ 1 \le i < j \le n-1,$$

$$d_{n,n} = \sqrt{c_{n,n}},\ \ d_{i,i} = \sqrt{\frac{\det \mathrm{C}[\{i,\ldots,n\}]}{\det \mathrm{C}[\{i+1,\ldots,n\}]}},\ d_{i,n} = \frac{c_{i,n}}{d_{n,n}},\ \ i = 1,\ldots,n-1.$$

Proposition 2.5. *Let* $\mathbf{U} \sim BG^{I}_{n}(a_1,\ldots,a_n;\mathrm{I}_n)$ *and* $\mathbf{U} = \mathbf{V}\,\mathbf{V}^{t}$, *where* $\mathbf{V} = (V_{i,j})$ *is a lower triangular random matrix with* $V_{i,i} > 0$. *Then* $V_{i,j}$, $1 \le j \le i \le n$ *are independently distributed,* $V^2_{i,i} \sim G\Big(a_i - (i-1)/2, 1\Big)$, $i = 1,\ldots,n$ *and* $\sqrt{2}V_{i,j} \sim N(0,1)$, $1 \le j < i \le n$.

The next properties of the Beta and Gamma distributions can be easily checked.

Proposition 2.6. *If* $\zeta \sim Beta(a,a,-1,1)$ *then* $1-\zeta^2 \sim Beta(a,1/2,0,1)$.

Proposition 2.7. *Let* ζ_1 *and* ζ_2 *be independent random variables,* $\zeta_1 \sim Beta(a,b,0,1)$, $\zeta_2 \sim Beta(a+b,c,0,1)$. *Then the product* $\zeta_1\zeta_2$ *has distribution* $Beta(a,b+c,0,1)$.

Proposition 2.8. *Let* ζ_1 *and* ζ_2 *be independent random variables,* ζ_1 *is Gamma distributed* $G(\alpha,\lambda)$ *and* $\zeta_2 \sim Beta(\alpha-\delta,\delta,0,1)$. *Then* $\zeta_1\zeta_2 \sim G(\alpha-\delta,\lambda)$.

3. Main results

From Proposition 2.1 it follows that the properties of a Bellman gamma type I distributed random matrix can be reformulated for Bellman gamma type II matrices.

Using Proposition 2.2, the properties of $BG_n^I(a_1, \dots, a_n; \mathrm{I}_n)$ distribution can be generalized for $BG_n^I(a_1, \dots, a_n; \mathrm{C})$, where C is an arbitrary positive definite matrix.

Theorem 3.1. *Let* $\mathbf{U} \sim BG_n^I(a_1, \dots, a_n; \mathrm{C})$ *and* η_i*,* $i = 1, \dots, n$ *be the random variables*

$$\eta_i = \frac{\det \mathbf{U}[\{1, \dots, i\}]}{\det \mathbf{U}[\{1, \dots, i-1\}]} \frac{\det \mathrm{C}[\{i, \dots, n\}]}{\det \mathrm{C}[\{i+1, \dots, n\}]}, \ i = 2, \dots, n-1,$$

$$\eta_1 = \det \mathbf{U}[\{1\}] \frac{\det \mathrm{C}}{\det \mathrm{C}[\{2, \dots, n\}]}, \ \eta_n = \frac{\det \mathbf{U}}{\det \mathbf{U}[\{1, \dots, n-1\}]} \det \mathrm{C}[\{n\}].$$

Then η_i*,* $i = 1, \dots, n$ *are mutually independent and* η_i *is gamma distributed* $G\Big(a_i - (i-1)/2, 1\Big)$*,* $i = 1, \dots, n$*.*

Proof. Suppose at first that $\mathrm{C} = \mathrm{I}_n$. Let $\mathbf{V} = (V_{i,j})$ be the Cholesky triangle of $\mathbf{U}$. According to Propositions 2.3, $\eta_1 = V_{1,1}^2$, $\eta_i = V_{i,i}^2$, $i = 2, \dots, n$. Therefore from Proposition 2.5 the Theorem follows.

Let now C be an arbitrary $n \times n$ positive definite matrix. Let D be the upper triangular matrix, defined by Proposition 2.4. Then $\mathrm{D}\mathrm{D}^t = \mathrm{C}$ and according to Proposition 2.2, the matrix $\mathbf{W} = \mathrm{D}^t \mathbf{U} \mathrm{D}$ has distribution $BG_n^I(a_1, \dots, a_n; \mathrm{I}_n)$. Since $\mathbf{W}[\{1, ..., i\}] = \mathrm{D}^t[\{1, ..., i\}]\mathbf{U}[\{1, ..., i\}]\mathrm{D}[\{1, ..., i\}]$ for $i = 1, \dots, n$, it can be seen that

$$\eta_1 = \det \mathbf{W}[\{1\}], \eta_i = \frac{\det \mathbf{W}[\{1, \dots, i\}]}{\det \mathbf{W}[\{1, \dots, i-1\}]}, i = 2, \dots, n.$$

Hence, according to the first part of the proof, the Theorem follows. ∎

Corollary 3.1 *Let* $\mathbf{U} \sim BG_n^I(a_1, \dots, a_n; \mathrm{C})$*. Then the random variable* $\det \mathbf{U} \det \mathrm{C}$ *is distributed as the product* $\eta_1 \dots \eta_n$*, where* $\eta_1, \dots, \eta_n$ *are mutually independent random variables,* $\eta_i \sim G\Big(a_i - (i-1)/2, 1\Big)$*,* $i = 1, \dots, n$*.*

Theorem 3.2. *Let* $\mathbf{U} \sim BG_n^I(a_1, \dots, a_n; \mathrm{I}_n)$ *and* i *be an integer,* $1 < i < n$*. Then for all integers* j*,* $i < j \leq n$ *the random variable*

$$\det \mathbf{U}[\{1, \dots, i\}, \{1, \dots, i-1, j\}] \tag{3.1}$$

is distributed as the product $\nu \eta_1 \dots \eta_{i-1} \sqrt{\eta_i}$*, where* ν*,* $\eta_1, \dots, \eta_i$ *are mutually independent,* $\sqrt{2}\nu \sim N(0,1)$*,* $\eta_k \sim G\Big(a_k - (k-1)/2, 1\Big)$*,* $k = 1, \dots, i$*.*

P r o o f. Let $\mathbf{V} = (V_{i,j})$ be the Cholesky triangle of $\mathbf{U}$. Let us consider the random variables $\nu = V_{j,i}$, $\eta_1 = V_{1,1}^2$, $\eta_k = V_{k,k}^2$, $k = 2, \ldots, i$. According to Proposition 2.3, for $i < j \leq n$ the random variable (3.1) is equal to $\nu\,\eta_1 \ldots \eta_{i-1}\sqrt{\eta_i}$. Now, using Proposition 2.5 we complete the proof. ∎

Corollary 3.2 *Let* $\mathbf{U} \sim BG_n^I(a_1, \ldots, a_n; \mathrm{I}_n)$ *and* i *be an integer,* $1 < i < n$. *Then for all integers* j, $i < j \leq n$

$$\frac{\det \mathbf{U}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}]}{\det \mathbf{U}[\{1, \ldots, i-1\}]} \sim \nu\sqrt{\eta}, \tag{3.2}$$

$$\frac{\det \mathbf{U}[\{1, \ldots, i\}, \{1, \ldots, i-1, j\}]}{\det \mathbf{U}[\{1, \ldots, i\}]} \sim \frac{\nu}{\sqrt{\eta}}, \tag{3.3}$$

where ν *and* η *are independent,* $\sqrt{2}\nu \sim N(0,1)$ *and* $\eta \sim G\,(a_i - (i-1)/2, 1)$.

Let $P(n, \Re)$ be the set of all real, symmetric, positive definite matrices of order n. Let us denote by $D(n, \Re)$ the set of all real, symmetric matrices of order n, with positive diagonal elements, which off-diagonal elements are in the interval (-1,1). There exist a bijection (one-to-one correspondence) $\tilde{h} : D(n, \Re) \to P(n, \Re)$, considered in [2]. The image of an arbitrary matrix $\mathrm{X} = (x_{i,j})$ from $D(n, \Re)$ by the bijection $\tilde{h}$, is a matrix $\mathrm{Y} = (y_{i,j})$ from $P(n, \Re)$, such that

$$y_{i,i} = x_{i,i}, \quad i = 1, \ldots, n, \tag{3.4}$$

$$y_{i,i+1} = x_{i,i+1}\sqrt{y_{i,i}y_{i+1,i+1}}, \quad i = 1, \ldots, n-1, \tag{3.5}$$

$$\begin{aligned} y_{i,j} = \Big\{ &(-)^{j-i} \det \mathrm{Y}[\{i, \ldots, j-1\}, \{i+1, \ldots, j\}]^0 \\ &+ x_{i,j}\sqrt{\det \mathrm{Y}[\{i, \ldots, j-1\}] \det \mathrm{Y}[\{i+1, \ldots, j\}]} \Big\} / \det \mathrm{Y}[\{i+1, \ldots, j-1\}], \\ &1 < i+1 < j \leq n. \end{aligned} \tag{3.6}$$

It is shown in [2] that

$$1 - x_{i,j}^2 = \frac{\det \mathrm{Y}[\{i, \ldots, j\}] \det \mathrm{Y}[\{i+1, \ldots, j-1\}]}{\det \mathrm{Y}[\{i, \ldots, j-1\}] \det \mathrm{Y}[\{i+1, \ldots, j\}]}, \; 2 \leq i+1 < j \leq n, \tag{3.7}$$

$$1 - x_{i,i+1}^2 = \frac{\det \mathrm{Y}[\{i, i+1\}]}{y_{i,i}y_{i+1,i+1}}, \; i = 1, \ldots, n-1, \tag{3.8}$$

$$\det \mathrm{Y}[\{i, \ldots, j\}] = x_{i,i} \ldots x_{j,j} \left(\prod_{i \leq s < t \leq j} (1 - x_{s,t}^2) \right), \; 1 \leq i < j \leq n. \tag{3.9}$$

The Jacobian of the transformation from $(x_{i,j})$ to $(y_{i,j})$ is

$$\det \mathrm{J} = \left[\sqrt{y_{1,1} y_{n,n}} \left(\prod_{j=2}^{n-1} \sqrt{\det \mathrm{Y}[\{1,\ldots,j\}] \det \mathrm{Y}[\{j,\ldots,n\}]} \right) \right]^{-1}. \quad (3.10)$$

Theorem 3.3. *Let $a_1, \ldots, a_n$ be real numbers, such that $a_i > (i-1)/2$, $i = 1,\ldots,n$. Let $\xi = (\xi_{i,j})$ be a symmetric $n \times n$ random matrix. Suppose that $\xi_{i,j}$, $1 \le i \le j \le n$ are mutually independent, $\xi_{i,j} \sim Beta\Big(a_j - (j-i)/2, a_j - (j-i)/2, -1, 1\Big)$, $1 \le i < j \le n$ and $\xi_{i,i} \sim G(a_i, 1)$, $i = 1,\ldots,n$. Then the matrix $\mathbf{U} = \tilde{h}(\xi)$ has Bellman gamma type I distribution $BG_n^I(a_1,\ldots,a_n;\mathrm{I}_n)$.*

Proof. The joint density function of $\xi_{i,j}$, $1 \le i \le j \le n$ has the form

$$f(x_{i,j}, 1 \le i \le j \le n) = K \left(\prod_{i=1}^{n} x_{i,i}^{a_i - 1} e^{-x_{i,i}} \right) \left(\prod_{1 \le i < j \le n} (1 - x_{i,j}^2)^{a_j - \frac{(j-i)}{2} - 1} \right),$$

$x_{i,i} > 0$, $i = 1,\ldots,n$, $x_{i,j} \in (-1,1)$, $1 \le i < j \le n$. Using the well-known property of the gamma function $\Gamma(2x) = \pi^{-1/2} 2^{2x-1} \Gamma(x)\Gamma(x+1/2)$, it can be seen that $K^{-1} = \Gamma_n^*(a_1,\ldots,a_n)$. The new variables are the elements $U_{i,j}$, $1 \le i \le j \le n$ of the matrix $\mathbf{U}$. Using (3.4), (3.7), (3.8) and (3.10) we obtain that the joint density of $U_{i,j}$, $1 \le i \le j \le n$ is equal to the right hand side of (2.1) with $\mathrm{C} = \mathrm{I}_n$. ■

Corollary 3.3 *Let $\mathbf{U} \sim BG_n^I(a_1,\ldots,a_n;\mathrm{I}_n)$ and p, q be integers, $1 \le p \le q \le n$. Then the matrix $\mathbf{U}[\{p,\ldots,q\}]$ has distribution $BG_{q-p+1}^I(a_p,\ldots,a_q; \mathrm{I}_{q-p+1})$.*

Proof. According to Theorem 3.3, $\mathbf{U}$ can be considered as an image $\mathbf{U} = \tilde{h}(\xi)$. From formulas (3.4) – (3.6) it can be seen that if $\mathrm{Y} = \tilde{h}(\mathrm{X})$ and p, q be integers, $1 \le p \le q \le n$ then

$$\mathrm{Y}[\{p,\ldots,q\}] = \tilde{h}(\mathrm{X}[\{p,\ldots,q\}]). \quad (3.11)$$

Applying again Theorem 3.3 we complete the proof. ■

Corollary 3.4 *Let $\mathbf{U} \sim BG_n^I(a_1,\ldots,a_n;\mathrm{I}_n)$ and p be an integer, $1 \le p \le n$. Then the random matrices $\mathbf{U}[\{1,\ldots,p\}]$ and $\mathbf{U}[\{p+1,\ldots,n\}]$ are independent.*

Proof. Using (3.11) we have $\mathbf{U}[\{1,\ldots,p\}] = \tilde{h}(\xi[\{1,\ldots,p\}])$, $\mathbf{U}[\{p+1,\ldots,n\}] = \tilde{h}(\xi[\{p+1,\ldots,n\}])$. Since $\xi_{i,j}$, $1 \le i \le j \le n$ are mutually independent, the Corollary follows. ■

Theorem 3.4. *Let* $\mathbf{U} \sim BG_n^I(a_1,\dots,a_n;\mathrm{I}_n)$ *and* $\mathbf{U}$ *be partitioned with submatrices* $\mathbf{U}_{i,j}$*,* $i,j=1,\dots,k$*, where* $\mathbf{U}_{i,i}$ *are square matrices of size* n_i*,* $i=1,\dots,k$*. Then*

$$\frac{\det \mathbf{U}}{\det \mathbf{U}_{1,1}\dots\det \mathbf{U}_{k,k}} \sim \beta_{n_1+1}\dots\beta_n, \tag{3.12}$$

where β_j*,* $j=n_1+1,\dots,n$ *are mutually independent,* $\beta_j \sim Beta\Big(a_j-(j-1)/2, (n_1+\cdots n_{r_j})/2,0,1\Big)$*;* r_j *is the greatest integer such that* $n_1+\cdots+n_{r_j}<j$*,* $j=n_1+1,\dots,n$*.*

Proof. According to Theorem 3.3, $\mathbf{U}$ can be considered as an image $\mathbf{U}=\tilde{h}(\xi)$. Applying (3.9) to $\det\mathbf{U}$ and $\det\mathbf{U}_{i,i}$, $i=1,\dots,k$, we obtain that the left hand side of (3.12) equals to $\prod\limits_{j=n_1+1}^{n}\prod\limits_{s=1}^{n_1+\cdots+n_{r_j}}(1-\xi_{s,j}^2)$. Let us substitute $\beta_j=\prod\limits_{s=1}^{n_1+\cdots+n_{r_j}}(1-\xi_{s,j}^2)$, $j=n_1+1,\dots,n$. Since $\xi_{s,j}$, $1\le s\le j\le n$ are mutually independent, β_{n_1+1}, ..., β_n are also independent. Using Propositions 2.6 and 2.7, we obtain that for $1\le u\le v<j\le n$

$$(1-\xi_{u,j}^2)\dots(1-\xi_{v,j}^2)\sim Beta\Big(a_j-(j-u)/2,(v-u+1)/2,0,1\Big). \tag{3.13}$$

Now, applying (3.13) we find the distribution of β_j and complete the proof. ■

Theorem 3.5. *Let* $\mathbf{U} \sim BG_n^I(a_1,\dots,a_n;\mathrm{I}_n)$ *and* p*,* q *be integers,* $1<p<q<n$*. Then*

$$\frac{\det\mathbf{U}[\{1,\dots,q\}]\det\mathbf{U}[\{p,\dots,n\}]}{\det\mathbf{U}[\{p,\dots,q\}]}\sim\eta_1\dots\eta_n, \tag{3.14}$$

where η_i*,* $i=1,\dots,n$ *are mutually independent and* $\eta_i{\sim}G\Big(a_i-(i-1)/2,1\Big)$*,* $i=1,\dots,q$*,* $\eta_i{\sim}G\Big(a_i-(i-p)/2,1\Big)$*,* $i=q+1,\dots,n$*.*

Proof. Applying (3.9) to $\det\mathbf{U}[\{1,\dots,q\}]$, $\det\mathbf{U}[\{p,\dots,n\}]$ and $\det\mathbf{U}[\{p,\dots,q\}]$ we obtain that the left hand side of (3.14) equals to

$$\xi_{1,1}\dots\xi_{n,n}\left(\prod_{t=2}^{q}\prod_{s=1}^{t-1}(1-\xi_{s,t}^2)\right)\left(\prod_{t=q+1}^{n}\prod_{s=p}^{t-1}(1-\xi_{s,t}^2)\right).$$

Let us substitute $\beta_t=\prod\limits_{s=1}^{t-1}(1-\xi_{s,t}^2)$, $t=2,\dots,q$, $\beta_t=\prod\limits_{s=p}^{t-1}(1-\xi_{s,t}^2)$, $t=q+1,\dots,n$. From (3.13) we have that $\beta_t\sim Beta\Big(a_t-(t-1)/2,(t-1)/2,0,1\Big)$,

$t = 2, \ldots, q$, $\beta_t \sim Beta\Big(a_t - (t-p)/2, (t-p)/2, 0, 1\Big)$, $t = q+1, \ldots, n$. Let $\eta_1 = \xi_{1,1}$, $\eta_i = \xi_{i,i}\beta_i$, $i = 2, \ldots, n$. Since $\xi_{i,i} \sim G(a_i, 1)$, $i = 1, \ldots, n$, from Proposition 2.8 the Theorem folows. ■

Theorem 3.6. *Let* $\mathbf{U} \sim BG_n^I(a_1, \ldots, a_n; \mathrm{I}_n)$ *and* p, q *be integers,* $1 < p < q < n$. *Then*

$$\frac{\det \mathbf{U} \det \mathbf{U}[\{p, \ldots, q\}]}{\det \mathbf{U}[\{1, \ldots, q\}] \det \mathbf{U}[\{p, \ldots, n\}]} \sim \beta_{q+1} \ldots \beta_n, \tag{3.15}$$

where $\beta_{q+1}, \ldots, \beta_n$ *are mutually independent and* $\beta_i \sim Beta\Big(a_i - (i-1)/2, (p-1)/2, 0, 1\Big)$, $i = q+1, \ldots, n$.

Proof. Using (3.9) we obtain that the left hand side of (3.15) equals to $\prod_{t=q+1}^{n} \prod_{s=1}^{p-1} (1 - \xi_{s,t}^2)$. Let us substitute $\beta_t = \prod_{s=1}^{p-1} (1 - \xi_{s,t}^2)$, $t = q+1, \ldots, n$. Since $\xi_{s,t}$, $1 \le s \le t \le n$ are mutually independent, $\beta_{q+1}, \ldots, \beta_n$ are also independent. Finally, applying (3.13) we complete the proof. ■

The method of approach, used in the proofs of Theorems 3.4–3.6, can be applied for the derivation of the distribution of products and quotient of products of principal minors of a kind $\mathbf{U}[\{i, \ldots, j\}]$ of a Bellman gamma matrix $\mathbf{U}$.

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