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Subdivision of Interpolating Polynomials

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A multi-segment subdivision scheme is established for an arbitrary univariate real polynomial on a finite interval. The method uses Lagrange interpolation in combination with the AIFS (Affine invariant Iterated Function Systems). This makes polynomial geometry submissive to fractal algorithms. Developed methods may be used to realize continuous transition from smooth to fractal functions.

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1. Introduction

The notion of subdivision is present in many areas of applied mathematics. If one speaks on polynomials, then subdivision is mainly relates to Bernstein polynomials and induced Bezier curves or surfaces [10-12]. As it is noted in [4], any polynomial undergoes some subdivision process, and this process is always linear. As it is shown in [5, 10], polynomial subdivision performs by an AIFS, a variant of IFS introduced in [5] and further developed in [6-9, 12]. So, this introduction begins with a reminder concerning IFS/AIFS stuff.

The concept of Iterated Function System (IFS), and its affine invariant counterpart AIFS appear to play a crucial role in constructive theory of fractal sets and in paving the way to have a good modeling tools for such sets. But, if the collection of objects to be modeled, besides fractals contains smooth objects as well (polynomials for ex.), then one needs to revisit classical algorithms for smooth objects generation and to introduce the new one that is capable to create both fractal and smooth forms. In this light, the purpose of this paper is to develop such algorithms for interpolating polynomials.

Let $\{\omega_i, i = 1 \ldots, n\}, n > 1$ be a set of contractive affine mappings defined on the complete Euclidian metric space $(\mathbf{R}^m, d_E)$

$$\omega_i(\mathbf{x}) = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \;\; \mathbf{x} \in \mathbf{R}^m, \; i = 1 \ldots, n, \tag{1.1}$$

where $\mathbf{A}_i$ is an real matrix and $\mathbf{b}_i$ is an m-dimensional real vector. Supposing that the Lipschitz factors $s_i = Lip\{\omega_i\}$, satisfy condition $|s_i| < 1, i = 1, \ldots, n$, the system $\{\mathbf{R}^m; \omega_1, \omega_2, \ldots, \omega_n\}$ is called (hyperbolic) Iterated Function System (IFS). Associated with given IFS, is so called Hutchinson operator $\mathcal{H}(\mathbf{R}^m) \to \mathcal{H}(\mathbf{R}^m)$, defined by

$$B \mapsto \bigcup_{i=1}^{n} \omega_i(B), \quad \forall B \in \mathcal{H}(\mathbf{R}^m).$$

It turns to be a contractive mapping on the complete metric space$(\mathcal{H}(\mathbf{R}^m), h)$ with contractivity factor $s = \max\{s_i\}$. Here,$\mathcal{H}(\mathbf{R}^m)$ is the space of nonempty compact subsets of $\mathbf{R}^m$ and h stands for Hausdorff metric induced by d_E , i.e.

$$h(A, B) = \max\left\{\max_{a\in A}\min_{b\in B} d_E(a, b), \max_{b\in B}\min_{a\in A} d_E(b, a)\right\}, \quad \forall A, B \in \mathcal{H}(\mathbf{R}^m).$$

Let $S_{m+1} = [s_{ij}]_{i,j=1}^{m+1}$ be a row-stochastic real matrix (its rows sum up to 1).

Definition 1.1. We refer to the linear mapping $\mathcal{L} : \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$, such that $\mathcal{L}(\mathbf{r}) = S^T\mathbf{r}$ as the linear mapping associated with S. The corresponding Hutchinson operator is

$$W(B) = \bigcup_{i=1}^{n} \mathcal{L}_i(B), \quad \forall B \in \mathcal{H}(\mathbf{R}^{m+1}).$$

According to the contraction mapping theorem, both Hutchinson operators have the unique fixed point called the attractor of the IFS/AIFS.

Definition 1.2. A (non-degenerate) m-dimensional simplex $\hat{\mathbf{P}}_m$(m-simplex) is the convex hull $\hat{\mathbf{P}}_m = \mathrm{conv}\{\mathbf{P}_m\}$ of a set $\mathbf{P}_m$ of $m+1$ affinely independent points/vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{m+1}$ in Euclidean space of dimension $\geq m$ that will be denoted in matrix form, $\mathbf{P}_m = \left[\mathbf{p}_1^{\mathrm{T}} \; \mathbf{p}_2^{\mathrm{T}} \; \ldots \; \mathbf{p}_{m+1}^{\mathrm{T}}\right]^{\mathrm{T}}$.

Definition 1.3. Let $\hat{\mathbf{P}}$ be a non-degenerate simplex and let $\{\mathbf{S}_i\}_{i=1}^n$ be a set of real square non-singular row-stochastic matrices of order $m+1$. The system $\Omega(\hat{\mathbf{P}}_m) = \left\{\hat{\mathbf{P}}_m; \mathbf{S}_1, \ldots, \mathbf{S}_n\right\}$ is called (hyperbolic) Affine invariant IFS (AIFS), provided that the linear mappings associated with $\mathbf{S}_i$ are contractions in $(\mathbf{R}^m, d_E)$ ([5–7]).

Theorem 1.1. *One eigenvalue of the matrix $\mathbf{S}_i$ is 1, other m eigenvalues coincide with eigenvalues of $\mathbf{A}_i$, the matrix that makes the linear part of the affine mapping ω_i given by (1.1).*

2. Subdivision

Although the notion of subdivision is usually attributed to m-dimensional ($m \geq 1$) continuous parametric mapping $t \mapsto P_n(t),\ t \in [a,b], (a < b)$, so that $P_n(t) \in \mathbf{R}^m$, for the purpose of study the basic properties, it is enough to consider one-dimensional case ($m = 1$). Let

$$P_n(t) = \sum_{k=0}^{n} A_k B_k^n(t), \quad t \in [a,b] \tag{2.1}$$

where A_k are real coefficients and $\mathcal{B}_n(t) = \{B_0^n(t), \ldots, B_n^N(t)\},\ t \in [a,b]$ is some functional basis, it may happened that both A_k and $\mathcal{B}_n(t)$ depend on the interval of definition. To stress this fact, it is suitable to write $A_k[a,b]$ as well as $\mathcal{B}_n[a,b](t)$. Then, the subdivision is defined as follows.

Definition 2.1. The function P_n, defined by (2.1) is said to permit linear subdivision if and only if for each nonempty subinterval $[p,q] \subset [a,b]$, there exists a set of coefficients $\{A_k[p,q]\}_{k=0}^n$ such that

$$\sum_{k=0}^{n} A_k[p,q] B_k^n[p,q](t) = \sum_{k=0}^{n} A_k[a,b] B_k^n[a,b](\varphi(t)) \tag{2.2}$$

for $t \in [a,b]$, where

$$\varphi(t) = \frac{1}{b-a}((q-p)t + bp - aq) \tag{2.3}$$

maps $[a,b]$ into $[p,q]$.

Moreover, restriction on $P_n(t)$ to belong to the set $\mathcal{P}_n$ of algebraic polynomials of $dg \leq n$, allows the subdivision to be linear and only linear.

Theorem 2.1. *(see [4]) The function $P_n(t)$, defined by (2.1) admits linear subdivision if and only if $\mathcal{B}_n(t)$ is a polynomial basis.*

The classic, and best known subdivision phenomena is connected with Bernstein polynomial basis, but subdivision is also possible for monomial, Lagrange, Newton or any other polynomial basis ([1, 4, 10]). Here, the focus will be set on multi-segment subdivision for Lagrange interpolation polynomials.

3. Subdivision for Lagrange interpolant

In geometric modelling and other applications, one often needs to find an analytic, usually polynomial, representation of curve for which no mathematical or only complicated descriptions are known. To construct such a representation, one usually measures a given curve at a number of points and determines an interpolant. Not much interpolating schemas are popular in engineering like the Lagrange polynomial scheme. The reason is multiple. 1. The Lagrange interpolation is analytic tool that assign an algebraic polynomial function to the set of discrete data. Polynomials are easy to process operationally (division, multiplication, differentiation, integration etc), they are free of poles and stable from the numerical point of view; 2. Lagrange polynomial basis (unlike Newton basis, for example) have partition of unity property; 3. The scheme itself is easy to implement in the form of computer program, and simple to use; 4. The interpolation is something that suits the wide range of real problems that are often rigorous in demanding exact matching of the data. There are many examples: Prescribed points where the path of a robotic arm have to pass through, the joints in some mechanical grid construction, the exact temperature should be maintain at exact duration of time and so on.

Here, a parametric scheme will be considered, in the simplest case of data, which will not cause the loss of generality. In fact, similar story is valid for an arbitrary dimension data. So, let the data be given by the plane points, called nodes, that we will identify with vectors

$$\mathbf{P}_i = (x_i, y_i) = [x_i \; y_i]^{\mathrm{T}}, \quad i = 0, \ldots, n \;\; (n \geq 2). \tag{3.1}$$

Let $\tau = [t_0 \; \ldots \; t_n]^{\mathrm{T}}$, $(t_i < t_{i+1})$ be the vector of interpolating knots, usually normalized by setting $t_0 = 0, t_n = 1$. The parametric Lagrange polynomial is given by

$$\mathbf{L}_n(\tau, \mathbf{P}; t) = \sum_{k=0}^{n} l_k^n(\tau, t)\mathbf{P}_k, \tag{3.2}$$

where $l_k^n(\tau, t) = \prod_{i=0}^{n} \dfrac{t - t_i}{t_k - t_i}$, $t \in [t_0, t_n]$, is the set of Lagrange basis polynomials $\mathbf{l}^n(\tau, t) = [l_0^n(t) \; \ldots \; l_n^n(t)]^{\mathrm{T}}$. By setting $\mathbf{P} = [\mathbf{P}_0 \; \ldots \; \mathbf{P}_n]^{\mathrm{T}}$, the vector form of (3.1) is possible

$$\mathbf{L}_n(\tau, \mathbf{P}, t) = \mathbf{P}^{\mathrm{T}} \cdot \mathbf{l}^n(\tau, t), \quad t \in [t_0, t_n] \tag{3.3}$$

Lemma 3.1. *Let* $\theta = [\theta_1 \; \theta_2 \; \ldots \; \theta_n]^{\mathrm{T}}$ *be any set of knots satisfying* $-\infty < \theta_1 < \theta_2 < \ldots < \theta_n < \infty, (n \geq 2)$. *Then, the following expansion holds*

$$\mathbf{L}_n(\tau, \mathbf{P}, t) = \mathbf{P}^{\mathrm{T}} \cdot \mathbf{l}^n(\tau, t), \quad t \in [t_0, t_n] \tag{3.4}$$

P r o o f. The proof of this lemma is given in [4]. ■

Now, let the set of n contractive affine mappings $\{\varphi_1, \dots, \varphi_n\}$ be given so that $\varphi_k : [t_0, t_n] \to [t_{k-1}, t_k]$.

Theorem 3.1. *Let* $\mathbf{L}_n$ *interpolates the points* $\mathbf{P}$ *and let* $\mathbf{Q}^{[k]} = \left[\mathbf{Q}_0^k \ \dots \mathbf{Q}_n^k\right]^{\mathrm{T}}$, *where*

$$\mathbf{Q}_i^k = [L_n^x(\varphi_k(t_i)), L_n^y(\varphi_k(t_i))]^{\mathrm{T}}, \ i = 0, \dots, n. \tag{3.5}$$

Then, the polynomial that interpolates $\mathbf{Q}^{[k]}$ *coincides with* $\mathbf{L}_n$.

P r o o f. Let τ be any set of knots satisfying conditions of Lemma 1, and $\theta = \varphi_k(\tau), k = 1, \dots, n$. The Lagrange basis over θ will be $\{l_k^n(\theta, t)\}_{k=0}^n$. The interpolating polynomial to the points $\mathbf{Q}^{[k]}$ will be

$$\mathbf{L}_n(\theta, \mathbf{Q}^{[k]}, t) = \sum_{j=0}^{n} \left(\sum_{i=0}^{n} l_i^n(\theta, t) l_j^n(\tau, \varphi(t_i)) \right) \mathbf{P}_j$$

From Lemma 1, $\mathbf{L}_n(\theta, \mathbf{Q}^{[k]}, t) = \sum_{j=0}^{n} l_j^n(\tau, t)\mathbf{P}_j = \mathbf{L}_n(\tau, \mathbf{P}, t)$, for all finite t. ■

Theorem 3.2. *The points* $\mathbf{Q}^{[k]}$, $(n \geq 2)$, *given by (3.5) are images of the interpolating nodes* $\mathbf{P}$, *upon the linear mapping* $\mathcal{L}_k(\mathbf{r}) = \mathbf{S}^T\mathbf{r}$, *where*

$$\mathbf{S}_k = \left[l_j^n(\tau; \varphi_k(t_i))\right]_{0 \leq i,j \leq n}, \quad k = 1, \dots, n \tag{3.6}$$

Theorem 3.3. *The graph* G_p *of the polynomial* $P_n(t) \in \mathcal{P}_n$, *defined by the Lagrange interpolation data (3.1), is the attractor of the AIFS* $\{\mathbf{P}; \mathbf{S}_1, \dots, \mathbf{S}_n\}$, *where* $\mathbf{S}_k$ *is given by (3.6).*

P r o o f. Since Lagrange polynomial basis have partition of unity property, $\mathbf{S}_k$ are real row stochastic matrices. Next the hyperbolicity of the AIFS will be shown. The spectrum of the matrix $\mathbf{S}_k$, as it is known from [5–7] is of the form $sp(\mathbf{S}_k) = \{1, \lambda_1^k, \dots, \lambda_n^k\}$, where $\lambda_i^k = \left(\frac{t_k - t_{k-1}}{t_n - t_0}\right)^j$. Since $n \geq 2$, λ for all k, making λ_1^k the second greatest eigenvalue of $\mathbf{S}_k$. So, there exists matrix norm such that $||\mathbf{S}_k|| = \lambda_1^k < 1$ So, the AIFS is hyperbolic, ensuring existence of the unique attractor. Denote it by G_p. Since matrix $\mathbf{S}_k$, that defines the mapping $\mathcal{L}(\mathbf{r}) = \mathbf{S}_k^{\mathrm{T}}\mathbf{r}$, maps $\mathbf{P}_0$ and $\mathbf{P}_n$ into $\mathbf{P}_{k-1}$ and $\mathbf{P}_k$, the image of $\mathbf{P}$ upon Hutchinson operator W contains as a subset. The same is for any power W^m of W. So, $\mathbf{P} \in W^\infty = G_p$, i.e., the interpolating points lie on the attractor. Further, by

Theorem 2, and Theorem 1, the mapping $\mathcal{L}$ maps the graph of the polynomial $L_n(t)$, interpolating the points $\mathbf{P}$, into the segment of the graph of $L_n(t)$, on the subinterval $[L_n^x(\varphi_k(t_0)), L_n^x(\varphi_k(t_0))]$, where $L_n^x(\varphi_1(t_0)) = x_0$, $L_n^x(\varphi_n(t_n)) = x_n$, and $L_n^x(\varphi_{k-1}(t_n)), L_n^x(\varphi_k(t_0))$. The proof follows by the Collage theorem [3]. ■

Example 1. Let $\mathbf{P} = \{(1,0),(2,0),(3,0),(3,1),(2,1),(1,1)\}$, be the interpolation data set in the plane. These data points suggest a parametric polynomial curve of degree 5. Applying Theorem 3, the AIFS subdivision matrices can be obtained.

Since the data are symmetric w.r.t. the line $y = 0.5$, the subdivision matrices are expected to reveal this symmetry in their mutual structure. In fact, it is so. Namely, the matrix $\mathbf{S}_4$ is symmetric to $\mathbf{S}_2$ with respect to "rotation" of elements around the matrix center. The same is for $\mathbf{S}_1$ and $\mathbf{S}_5$. The attractor is obtained by random algorithm [2]. If one uses the deterministic algorithm (also [1]), the polygonal approximation (in fact it is interpolation) be obtained instead. The starting figure (Initiator) is the polygonal line connecting the data set P.

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